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DIFFRACTION OF ELECTROMAGNETIC WAVES ON TWO BODIES

by Ye. A. Ivanov

"Nauka i Tekhnika" Press, Minsk, 1968

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • APRIL 1970

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By Ye. A. Ivanov

Translation of "Difraktsiya Elektromagnitnykh Voln na Dvukh Telakh"
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ABSTRACT

This book deals with problems of electromagnetic wave diffraction on barriers composed of two disconnected objects of complex geometrical shape. The fundamental electromagnetic field equations are described, and problems pertaining to the separation of the Maxwell equations in various orthogonal curvilinear coordinate systems are considered. The reduction of vector electrodynamic problems to scalar (with respect to the auxiliary potentials) problems is explained. Special attention is given to fundamental properties of special wave functions, formulas for expanding primary fields in series with respect to wave functions, and theorems for summing wave functions in clyindrical, elliptic spherical, and spherodial coordinate systems. The solution to a number of problems involving electromagnetic wave diffraction on circular and elliptic cylinders, spheres, and disks are obtained by the classical method of separation of variables generalized for the case of two bodies. The numerical solutions of infinite systems of linear equations for the coefficients of the expansion of the unknown functions in wave eigenfunctions of the corresponding body are given.

ANNOTATION

The basic equations for an electromagnetic field are described riefly in this book, and the problems associated with the separaion of Maxwell equations are studied in the various orthogonal urvilinear coordinate systems as are those which are associated ith the reduction of electrodynamic vector problems to scalar prolems relative to the auxiliary potentials. A discussion is given f the basic properties of special wave functions; formulas are erived for expansion of the primary fields into series of wave unctions as are addition theorems for the wave functions in cylinrical, eliptical, spherical, and spheroidal coordinate systems.

The solution to a number of problems is found here on the difraction of electromagnetic waves on circular and elliptical cylnders, spheres, spheroids (discs), obtained by the classical method of separation of variables generalized to the case of two bodies. Questions are studied of the numerical solution to infinite systems of linear equations for the expansion coefficient of the unknown unctions for the wave eigenfunctions of the corresponding body.

This book is intended for students, engineers and other persons involved with the theory of the diffraction of electromagnetic waves and its application.

66 figures, 192 bibliographic citations.

Editor, S.I. Gayduk,
Candidate Physical-Mathematical
Sciences

FOREWORD

The diffraction of electromagnetic waves on barriers of an /6
arbitrary shape involves a number of well-studied phenomena in the sense that the differential equations which describe the physical process are known, as are the boundary conditions which must satisfy the functions sought. In principle, for a complete determination of the diffracted field it is necessary only to find a solution to the equation which satisfies the boundary conditions that represent a given specific source of the primary field and the shape of the barrier producing the diffraction.

In mathematical formulation, the electrodynamic problem of diffraction consists of integrating the system of Maxwell equations relative to the vectors of the electrical and magnetic intensities E and H of the diffracted field which satisfy certain boundary conditions on the surface of the barrier and the condition of radiation also on the edge (if such exists).

As yet there is no general method for a strict solution to diffraction problems for the shape of the surfaces actually used because of the great mathematic difficulties involved. Their solution is extremely complex even in the case of single barriers under simple and idealized conditions.

From the mathematical point of view, the analytical simplicity of the diffraction problem is associated in first order with those properties of the barrier and the electromagnetic field incident on it which determine the possibility of reducing the electrodynamic vector problem to a scalar wave equation for auxiliary scalar potential functions (Hertz, Debye, etc., potentials), that satisfy the simple boundary conditions on the barrier surface. In particular, the surface shape is simple if it agrees with one of the complete coordinate surfaces in the chosen system of coordinates, allowing separation of the Maxwell equations.

Strict solutions for the problem of diffraction of electromagnetic waves exist for two cases of a circular infinite cylinder, an infinite elliptical cylinder, a sphere, a spheroid, an infinitely thin circular disk and for several other bodies, whose surfaces agree with the surface of the coordinate constant in the respective coordinate system. They are found by the classical method of separation of variables. It is obvious that the number of problems which /7

can be solved by this method is rather limited since the scalar wave function of Helmholtz $\Delta u+k^2u = 0$ may be separated only in eleven coordinate systems. But, in these systems, computer calculation of the values characterizing the diffraction field have been delayed because of the difficulties which occur in tabulating special functions in wide ranges of variation in the arguments. However, in spite of the incompleteness of such investigations, calculations of the solutions obtained and the ideas based on them are extremely valuable for developing a diffraction theory and for expanding our knowledge about diffraction.

Involved in the more complex problems of diffraction theory, which is of great practical and theoretical significance, are the problems associated with the interaction of two (or more) neighboring reflections of waves. These types of problems, which require careful development, have been studied to a significantly smaller degree than have problems of the diffraction of waves on single barriers.

In this book problems are studied of the diffraction of electromagnetic waves on complex barriers composed of two mutually unconnected bodies of different geometric configuration: on two infinitely long parallel circular cylinders, two infinite elliptical cylinders (bands) with parallel longitudinal axes, two spheres, two ellipsoids of rotation (extended and flattened spheroids) with a common axis of rotation and finally two coaxial infinitely thin ideally conducting circular disks, located in a finite uniform isotropic space with electrical and magnetic permeabilities, ε, μ and with a conductivity $\sigma = 0$. The bodies are first excited by a plane electromagnetic wave or by the field of the electric (magnetic) dipole placed at a given point near the diffracting system or on the surface of one of the bodies. If the proper coordinate systems are chosen the vector problems are reduced to scalar boundary problems for the Helmholtz wave equation relative to the scalar potentials. Their strict solutions are found by the method of separation of variables. This method (in the cases when it is applicable) is an effective means of finding the numerical (approximate) values of the quantities defining the desired fields, with the aid of computers for a rather wide range of variation in the parameters of the problem. However, just as in the case of a single barrier, the computational possibilities are also limited here by the difficulties which occur in tabulating the wave functions. In practice, a numerical solution to the problems of the interaction of two bodies may be accomplished (with an allowable degree of accuracy in calculation) for all those values of the parameters for which solution is possible of the respective diffraction problems on a single barrier (in particular, in the range of long waves and waves that are comparable with the dimensions of the barrier).

The first five chapters of the book, written on the basis of [1-44] are of an auxiliary nature. The basic equations of the electromagnetic field are described briefly in them and problems

are studied that are associated with the separation of equations of the Maxwell system in different orthogonal curvilinear coordinate systems and with reduction of the electrodynamic vector boundary problems to scalar ones relative to the auxiliary potentials. The Helmholtz wave equation is studied in cylindrical, spherical, elliptical and spheroidal coordinates; several properties of special wave functions are described; formulas are derived for expansion of the primary fields into series of wave functions and summation theorems are given for them. Certain of the results, in particular those pertaining to the summation theorems for wave functions, belong to the author (for summation theorems see also [1, 24, 25, 27, 32, 35, 40, 115, 131, 138, 154, 156, 175]).

The diffraction problems are formulated and solved in Chapters 6 through 9, which include the results of the author [76-104] pertaining to a strict solution of the diffraction problems on two bodies. In each chapter the problems of diffraction on two bodies are preceded by problems of scattering on a single barrier of the corresponding shape. They have been studied in many sources (a bibliography of a large number of papers devoted to the diffraction of waves on single barriers of the simplest shape is contained in, for example, [46-48, 50-56, 118, 124, 145, 152, 158, 159]), however we felt it would be useful to give a brief discussion of their solutions here.

Let us note that several problems of the theory of diffraction of waves on two bodies have been studied in [112, 117, 119, 129, 130, 149] (parallel circular cylinders of infinite length), [106, 107, 132-136, 148, 186] (parallel elliptical cylinders of infinite length), [109, 110, 115, 138] (spheres, a discussion of several problems of diffraction of waves on two spheres can be found also in [108, 111, 137-139, 147, 164]), [146, 160-162] (coaxial disks, circular openings in an infinitely thin plane screen). The diffractions of waves on several barriers of arbitrary shape have been studied in references [119-123, 163, 191].

The author expresses his appreciation to Professor V.P. Shestopalov and V.A. Borovikov for reviewing the book in manuscript and for a number of useful comments.

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PART ONE

CHAPTER 1

BASIC CONCEPTS OF THE ELECTROMAGNETIC FIELD. SYSTEMS OF ORTHOGONAL CURVILINEAR COORDINATES. BOUNDARY PROBLEMS OF THE MATHEMATICAL THEORY OF DIFFRACTION.

§1. Equations of the Electromagnetic Field. Boundary Conditions.

By the electromagnetic field we mean a set of four vectors E, H, D, B, which are termed, respectively, vectors of the intensity of the electrical and the magnetic fields (E, H) and vectors of the electrical and magnetic inductions (D, B). These vectors are assumed to be finte in the entire field and are continuous functions at each space-time point (x, y, z, t), in the vicinity of which the electromagnetic properties of the medium are continuous. Breaks in the continuity of the field vectors or their derivatives may be encountered only at points of the surface which separate media with different electromagnetic properties. The electromagnetic properties of a material medium is characterized by the quantities[1] ε, μ, σ, where ε is the dielectric constant, μ is the magnetic permeability and σ is the conduction of the medium. /9*

The sources of the electromagnetic field are currents and charges, which are defined as functions of the point and time by means of the charge density ρ and the vector J of the current density.

At each point of the medium which is fixed relative to the coordinate axes, the field vectors satisfy the complete system of Maxwell equations, having the form[2]

$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\cdot\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\,\mathbf{J} + \frac{4\pi}{c}\,\mathbf{J}^{(\mathrm{ext})},$$

$$\operatorname{rot}\mathbf{E} = -\frac{1}{c}\cdot\frac{\partial \mathbf{B}}{\partial t}, \qquad (1.1.1)$$

$$\operatorname{div}\mathbf{B} = 0, \quad \operatorname{div}\mathbf{D} = 4\pi\rho$$

---

[1] For the cases studied in this book.

[2] We have used a gaussian system of measurements.

* Numbers in the margin indicate pagination in the foreign text.

(J is the density of the conduction currents; $J^{(ext)}$ is the density of currents originating from the effect of the extraneous emf; c is the speed of light in a vacuum). Equations (1.1.1) have physical sense only under the condition that they are considered along with the so-called material equations of the field

$$\begin{aligned} \mathbf{D} &= \varepsilon \mathbf{E}, \\ \mathbf{B} &= \mu \mathbf{H}, \\ \mathbf{J} &= \sigma \mathbf{E}. \end{aligned} \tag{1.1.2}$$

In a uniform isotropic medium (a medium whose electromagnetic properties remain constant with transition from point to point and are independent of the direction of the transition), the quantities ε, μ, σ are scalar constants (in a vacuum $\varepsilon = \mu = 1$, $\sigma = 0$). In such a medium, system (1.1.1), on the basis of (1.1.2), assumes the form

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= \frac{\varepsilon}{c} \cdot \frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi\sigma}{c} \mathbf{E} + \frac{4\pi}{c} \mathbf{J}^{(\mathrm{ext})}, \\ \operatorname{rot} \mathbf{E} &= -\frac{\mu}{c} \cdot \frac{\partial \mathbf{H}}{\partial t}, \\ \operatorname{div} \mathbf{H} &= 0, \quad \operatorname{div} \mathbf{E} = \frac{4\pi\rho}{\varepsilon}. \end{aligned} \tag{1.1.3}$$

If the electromagnetic field is created by currents and charges that depend sinusoidally on the time t and oscillate at a circular frequency ω, then on the assumption that the dependence on time is described by the complex function $e^{-i\omega t}$, it is feasible to assume[3] that the field vectors are real parts of expressions such as $\mathrm{F}e^{-i\omega t}$, where F is a complex vector (complex amplitude) that is independent of the time t. The Maxwell equations for the complex amplitudes, if we retain for them the designations of the field vectors representing them, have the form

$$\begin{aligned} \operatorname{rot} \mathbf{H} &= \frac{4\pi\sigma - i\omega\varepsilon}{c} \mathbf{E} + \frac{4\pi}{c} \mathbf{J}^{(\mathrm{ext})}, \\ \operatorname{rot} \mathbf{E} &= \frac{i\omega\mu}{c} \mathbf{H}, \\ \operatorname{div} \mathbf{H} &= 0, \quad \operatorname{div} \mathbf{E} = \frac{4\pi\rho}{\varepsilon}. \end{aligned} \tag{1.1.4}$$

[3] The dependence on time can be taken also in the form $e^{i\omega t}$. Then in order to convert from the formulas obtained with the function $e^{-i\omega t}$, $-i$ must be substituted by i in the formulas of interest to us.

If interfaces of the different media exist in space, it follows to adjoin the boundary conditions to the Maxwell equations. For two media with finite conductivity ($\sigma \neq \infty$), they consist of the tangential components of the intensity of the electrical and magnetic fields remaining continuous functions during passage through the interface of the media. In vector form, the boundary conditions are described by the equations, /11

$$[n(E_2 - E_1)] = 0, \quad [n(H_2 - H_1)] = 0, \tag{1.1.5}$$

where E_1, H_1 are vectors in medium *I* with constants ε_1, μ_1, σ_1; E_2, H_2 are vectors in medium *II* with constants ε_2, μ_2, σ_2, and n is a unit normal to the surface of the medium interface, directed from medium *I* to medium *II* (Fig. 1). In radio engineering, it it usually assumed that metal bodies have an infinitely large conductivity ($\sigma = \infty$). Such conductors are called ideally conducting. The electromagnetic field does not penetrate inside an ideal conductor. Therefore, in the case if, for example, $\sigma_1 = \infty$, the vectors E_1, H_1 in medium *I* are identically equal to zero. On the surface of an ideal conductor the boundary conditions are described by the equations

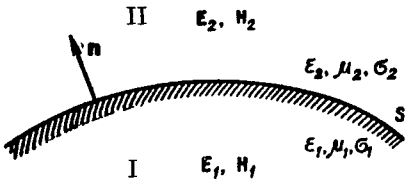

Fig. 1. For Determination of the Boundary Conditions.

$$[nE_2] = 0, \tag{1.1.6}$$

$$[nH_2] = \frac{4\pi}{c} j, \tag{1.1.7}$$

where j is the vector of the density of the surface currents, induced on the surface of the ideal conductor by the field H_2. Since the vector j does not enter into the Maxwell equation, then equation (1.1.7) does not impose any restrictions on its solution. Therefore, on the surface of an ideal conductor, only condition (1.1.6) is used; this condition requires that the tangential components of the vector of the electrical intensity E_2 disappear. Usually the surface current is determined from equation (1.1.7) after solution /12 to the Maxwell equations is found.

The Maxwell equations, along with the boundary conditions, permit uniquely finding the electromagnetic field in space under certain additional conditions for infinity. If the medium possesses a conductivity, even a minimal one, these conditions require the field from any system of emitters completely lying inside a certain finite region to become infinite. If this same conductivity of the medium is equal to zero, they then are exchanged by the so-called condition of radiation (Sommerfeld condition) which requires, at

infinity, that there exist only waves which drift into infinity, but do not leave. The analytical sign that the condition of radiation exists is that the boundary equations

$$\lim_{r\to\infty} r\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad \lim_{r\to\infty} u = 0, \tag{1.1.8}$$

are satisfied; by u we mean any of the component vectors E, H. For a two-dimensional space the condition of radiation is written in the form

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u}{\partial r} - iku\right) = 0, \quad \lim_{r\to\infty} u = 0. \tag{1.1.9}$$

In (1.1.8) and (1.1.9), k is the wave number defined in the next section.

§2. Wave Equations for the Vectors E, H. Potentials of the Electromagnetic Field.

From (1.1.4) we can obtain equations for each of the vectors E, H separately. Thus, applying to the first of the equations of the system the operation rot, we find

$$\operatorname{rot}\operatorname{rot}\mathbf{H} = \frac{4\pi\sigma - i\omega\varepsilon}{c}\operatorname{rot}\mathbf{E} + \frac{4\pi}{c}\operatorname{rot}\mathbf{j}^{(\mathrm{ext})}.$$

If we drop rot E from here and use the second equation of the system and assume that in rectangular Cartesian coordinates for any vector F the following equation is valid:

$$\operatorname{rot}\operatorname{rot}\mathbf{F} = \operatorname{grad}\operatorname{div}\mathbf{F} - \Delta\mathbf{F}, \tag{1.2.1}$$

where by definition ΔF is a vector whose rectangular components are /1
equal to ΔF_x, ΔF_y, ΔF_z, and where Δ is a three-dimensional LaPlace operator, we find (div H = 0)

$$\Delta\mathbf{H} + k^2\mathbf{H} = -\frac{4\pi}{c}\operatorname{rot}\mathbf{J}^{(\mathrm{ext})} \tag{1.2.2}$$

Here

$$k^2 = \frac{\omega^2\mu\varepsilon + i\,4\pi\sigma\omega\mu}{c^2} \tag{1.2.3}$$

(in a vacuum, $k = \frac{\omega}{c} = \frac{2\pi}{\lambda_0} = k_0$, λ_0 is the wave length). In the same manner, if we drop the vector H from system (1.1.4), we obtain an

quation for the vector E:

$$\Delta \mathbf{E} + k^2 \mathbf{E} = -\frac{4\pi i k_0 \mu}{c} \mathbf{j}^{(\text{ext})} + \frac{4\pi}{\varepsilon} \operatorname{grad} \rho. \tag{1.2.4}$$

oth equations (1.2.2) and (1.2.4) are written in rectangular coordinates. In particular, if u is one of the rectangular components of the vector E or of the vector H, then

$$\Delta u + k^2 u = f, \tag{1.2.5}$$

here f is a known function.

An equation of the type,

$$\Delta \mathbf{F} + k^2 \mathbf{F} = \mathbf{P}, \tag{1.2.6}$$

here P is a known vector and F is an unknown vector, is termed a onhomogeneous (homogeneous, if P ≡ 0) Helmholtz vector wave equation. An equation such as (1.2.5), where u and f are scalar functions, is called a nonhomogeneous (homogeneous, if $f \equiv 0$) Helmholtz calar wave equation. Usually equations (1.2.5) and (1.2.6) are imply called Helmholtz scalar and vector nonhomogeneous (homogeneous) quations.

Equations (1.2.2) and (1.2.4) are not at all equivalent to ystem (1.1.4), since the vectors E and H are related among themelves by the equations in (1.1.4), on the strength of which they o not appear to be independent solutions to equations (1.2.2) and 1.2.4). Therefore, in fact, only one of the Helmholtz equations s used, for example, for the vector E, after determination of which he vector H is found from the second equation of system (1.1.4).

From (1.2.2) and (1.2.4) it is clear that the nonhomogeneity f the Helmholtz equations is due to the presence of extraneous irrents and charges in the region where system (1.1.4) is determined. In fact nonhomogeneous equations for the fields are usually ot satisfied, since for determination of the field created by these urrents and charges it is more convenient to proceed directly from /14
ystem (1.1.4) or it is more feasible to introduce into consideration certain auxiliary quantities.

From (1.1.4) it follows that, for determination of the electroagnetic field, it is necessary to solve six scalar equations for he six component vectors E and H. However, the field equations ay be reduced to a form in which the number of equations for the uantities determining the field will be less than six. One of the ethods of such a reduction is the introduction into consideration f the auxiliary quantities, i.e., the field potentials. Let us

establish them after first assuming that in (1.1.4) $J^{(ext)} \equiv 0$ and $\rho \equiv 0$.

From the solenoidality of the vector H (div H = 0) it follows that it can be written with the help of another vector A in the form

$$\mathbf{H} = \frac{1}{\mu} \operatorname{rot} \mathbf{A} \tag{1.2.7}$$

(μ = const; A is an arbitrary vector). Then on the basis of (1.2.7) from the second equation of system (1.1.4) we find

$$\operatorname{rot} (\mathbf{E} - ik_0 \mathbf{A}) = 0,$$

i.e., the vector $E - ik_0A$ is a potential vector, and therefore may be written in the form of a gradient of a certain scalar function ϕ:

$$\mathbf{E} = ik_0 \mathbf{A} - \operatorname{grad} \varphi. \tag{1.2.8}$$

After substituting (1.2.7) and (1.2.8) into the still unused first equation of system (1.1.4), we find after several transformations and using equation (1.2.1) that in satisfying the condition[4]

$$\operatorname{div} \mathbf{A} + \frac{k^2}{ik_0} \varphi = 0 \tag{1.2.9}$$

the vector A and the scalar function ϕ will be solutions to the homogeneous equations (1.2.6) and (1.2.5), respectively. Since these equations for A and ϕ have been derived from the basic equations of the electromagnetic field, this field will then also satisfy, the vectors E and H, found through A and ϕ from the formulas in (1.2.7) and (1.2.8). Thus using the vector A and the scalar function ϕ, introduced as described, seeking the vectors E and H is reduced to finding the four scalar quantities, i.e., the function ϕ and three components of the vector A. The functions A and ϕ are called, respectively, the vector and scalar electrodynamic potentials.

The existence of a relationship (1.2.9) between the potentials A and ϕ permits expressing them, and in the same way the vectors E and H as well, through one auxiliary vector and as a result reducing the problem of finding the vectors E and H to seeking only

---

[4] This condition, sometimes termed the Lorentz condition, removes the indeterminancy in determining A and ϕ.

three scalar quantities, i.e., the components of this vector. Let us denote it by $\vec{\Pi}$ and set

$$\varphi = -\operatorname{div}\vec{\Pi}. \tag{1.2.10}$$

Then condition (1.2.9) will be satisfied if

$$\mathbf{A} = \frac{k^2}{ik_0}\vec{\Pi}, \tag{1.2.11}$$

where $\vec{\Pi}$ satisfies the homogeneous equation

$$\Delta\vec{\Pi} + k^2\vec{\Pi} = 0. \tag{1.2.12}$$

Equation (1.2.12) is also a consequence of the basic equations of the electromagnetic field, therefore it will also satisfy the vectors E and H, found through $\vec{\Pi}$ from the formulas

$$\begin{aligned} \mathbf{E} &= \operatorname{grad}\operatorname{div}\vec{\Pi} + k^2\vec{\Pi} = \operatorname{rot}\operatorname{rot}\vec{\Pi}, \\ \mathbf{H} &= \frac{k^2}{ik_0\mu}\operatorname{rot}\vec{\Pi}, \end{aligned} \tag{1.2.13}$$

found from (1.2.7) and (1.2.8) by substitution of (1.2.10) and (1.2.11). In other words, any solution to the vector equation (1.2.12) will determine a certain electromagnetic field by means of the expressions in (1.2.13). The vector $\vec{\Pi}$, through which the particular electromagnetic field is found from the formulas in (1.2.3), is called a Hertz electric vector, and the field determined by them is called an electrical type field.

Since in a space that is free of charges, the vector E is also solenoidal (div E = 0), then in a similar manner another solution to (1.1.4) can be constructed, having the form

$$\begin{aligned} \mathbf{E} &= \frac{1}{\varepsilon}\operatorname{rot}\mathbf{A}^*, \\ \mathbf{H} &= \frac{k^2}{ik_0\mu\varepsilon}\mathbf{A}^* - \operatorname{grad}\varphi^*, \end{aligned} \tag{1.2.14}$$

where under the condition

$$\operatorname{div}\mathbf{A}^* + ik_0\mu\varepsilon\varphi^* = 0 \tag{1.2.15}$$

the electrodynamic potentials A^* and ϕ^* are solutions to the homogeneous equations (1.2.6) and (1.2.5), respectively. After writing

$$\varphi^* = -\operatorname{div}\vec{\Pi}^*, \quad \mathbf{A}^* = ik_0\mu\varepsilon\vec{\Pi}^*, \tag{1.2.16}$$

we find from (1.2.14), for solution to system (1.1.4), the expression

$$\mathbf{E} = ik_0\mu\operatorname{rot}\vec{\Pi}^*, \tag{1.2.17}$$
$$\mathbf{H} = \operatorname{grad}\operatorname{div}\vec{\Pi}^* + k^2\vec{\Pi}^* = \operatorname{rot}\operatorname{rot}\vec{\Pi}^*,$$

where the vector $\vec{\Pi}^*$ is any solution to equation (1.2.12).

The vector $\vec{\Pi}^*$, through which the particular electromagnetic field is found from formulas in (1.2.17), is called a Hertz magnetic vector, and the field determined by it is called a magnetic type field.

Consequently, in the region in which ε, μ and σ are constants, and $J^{(ext)} = 0$, $\rho = 0$, the electromagnetic field may be separated into two components of the field, one of which is determined by the Hertz electric vector by the formulas in (1.2.13) and the second by the Hertz magnetic vector by the formulas in (1.2.17). The combination of expressions (1.2.13) and (1.2.17) gives the most common representation for the vectors E and H of the electromagnetic field, namely,

$$\mathbf{E} = \operatorname{rot}\operatorname{rot}\vec{\Pi} + ik_0\mu\operatorname{rot}\vec{\Pi}^*,$$
$$\mathbf{H} = \operatorname{rot}\operatorname{rot}\vec{\Pi}^* - \frac{ik^2}{k_0\mu}\operatorname{rot}\vec{\Pi}, \tag{1.2.18}$$

where $\vec{\Pi}$ and $\vec{\Pi}^*$ are the solutions to equation (1.2.12).

In system (1.1.4), if we make the substitution,

$$\mathbf{E}^* = \pm\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{H}, \quad \mathbf{H}^* = \mp\sqrt{\frac{\varepsilon}{\mu}}\,\mathbf{E}, \tag{1.2.19}$$

as a result then we find the relationships

$$\operatorname{rot}\mathbf{E}^* = \frac{ik^2}{k_0\varepsilon}\mathbf{H}^* \pm \frac{4\pi}{c}\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{J}^{(ext)},$$
$$\operatorname{rot}\mathbf{H}^* = -ik_0\varepsilon\mathbf{E}^*, \tag{1.2.20}$$
$$\operatorname{div}\mathbf{H}^* = \mp\frac{4\pi}{\mu}\sqrt{\frac{\mu}{\varepsilon}}\,\rho, \quad \operatorname{div}\mathbf{E}^* = 0,$$

which may be assumed to be a system of Maxwell equations for the field E^*, H^*, created by the imaginary "magnetic" current $\pm\sqrt{\frac{\mu}{\varepsilon}}\,\mathbf{j}^{(\text{ext})}$ and the imaginary "magnetic" charges with the distribution density /17 $\mp\sqrt{\frac{\mu}{\varepsilon}}\,\rho$. If we introduce into the system of Maxwell equations the imaginary magnetic currents and charges, we then obtain a generalized system of Maxwell equations (the asterisks are omitted)

$$\begin{aligned}
\operatorname{rot}\mathbf{H} &= -\frac{ik^2}{k_0\mu}\,\mathbf{E} + \frac{4\pi}{c}\,\mathbf{J}_e,\\
\operatorname{rot}\mathbf{E} &= ik_0\mu\,\mathbf{H} - \frac{4\pi}{c}\,\mathbf{J}_m,\\
\operatorname{div}\mathbf{E} &= \frac{4\pi}{\varepsilon}\,\rho_e, \quad \operatorname{div}\mathbf{H} = \frac{4\pi}{\mu}\,\rho_m,
\end{aligned} \tag{1.2.21}$$

in which J_e is the extraneous electrical currents; ρ_e is the distribution density of the electrical charges; J_m is the imaginary extraneous magnetic currents and ρ_m is the distribution density of the imaginary magnetic charges. From (1.2.21), in the same way as before, we can establish that in the region where $J_m^{(\text{ext})} = 0$, $\rho_m = 0$, but $J_e \neq 0$, $\rho_e \neq 0$, the vectors E, H of the electrical type field will be determined through the potentials in the following manner:

$$\begin{aligned}
\mathbf{E} &= -\operatorname{grad}\varphi + ik_0\mathbf{A},\\
\mathbf{H} &= \frac{1}{\mu}\operatorname{rot}\mathbf{A},\\
\Delta\mathbf{A} + k^2\mathbf{A} &= -\frac{4\pi\mu}{c}\,\mathbf{J}_e^{(\text{ext})}
\end{aligned} \tag{1.2.22}$$

where

$$\begin{aligned}
\Delta\varphi + k^2\varphi &= -\frac{4\pi}{\varepsilon}\,\rho_e,\\
\operatorname{div}\mathbf{A} - \frac{ik^2}{k_0}\,\varphi &= 0,
\end{aligned} \tag{1.2.23}$$

and through the Hertz electric vector by the relationships

$$\mathbf{E} = \operatorname{grad}\operatorname{div}\vec{\Pi} + k^2\vec{\Pi} = \operatorname{rot}\operatorname{rot}\vec{\Pi} + \Delta\vec{\Pi} + k^2\vec{\Pi},$$

$$\mathbf{H} = -\frac{ik^2}{\mu k_0}\operatorname{rot}\vec{\Pi}, \tag{1.2.24}$$

where /18

$$\Delta\vec{\Pi} + k^2\vec{\Pi} = \frac{4\pi\mu k_0}{ik^2c}\mathbf{J}_e^{(\text{ext})} \tag{1.2.25}$$

In the region under study when $\mathbf{J}_e^{(\text{ext})} = 0$; $\rho_e = 0$, but $\mathbf{J}_m^{(\text{ext})} \neq 0$, $\rho_m \neq 0$, the vectors E, H of the magnetic type field are determined through the potentials and the Hertz magnetic vector by the formulas

$$\mathbf{E} = -\frac{1}{\varepsilon}\operatorname{rot}\mathbf{A}^*,$$

$$\mathbf{H} = -\operatorname{grad}\varphi^* + \frac{ik^2}{k_0\varepsilon\mu}\mathbf{A}^*, \tag{1.2.26}$$

where

$$\Delta\mathbf{A}^* + k^2\mathbf{A}^* = -\frac{4\pi\varepsilon}{c}\mathbf{J}_m^{(\text{ext})} \tag{1.2.27}$$

and

$$\Delta\varphi^* + k^2\varphi^* = -\frac{4\pi}{\mu}\rho_m,$$

$$\operatorname{div}\mathbf{A}^* - ik_0\varepsilon\mu\varphi^* = 0, \tag{1.2.28}$$

$$\mathbf{H} = \operatorname{grad}\operatorname{div}\vec{\Pi}^* + k^2\vec{\Pi}^* = \operatorname{rot}\operatorname{rot}\vec{\Pi}^* + \Delta\vec{\Pi}^* + k^2\vec{\Pi}^*,$$

where

$$\mathbf{E} = \frac{ik^2}{k_0\varepsilon}\operatorname{rot}\vec{\Pi},$$

$$\Delta\vec{\Pi}^* + k^2\vec{\Pi} = \frac{4\pi\varepsilon k_0}{ik^2c}\mathbf{J}_m. \tag{1.2.29}$$

If the electromagnetic field is created both by the electrical and by the magnetic currents and charges, then it will be represented through A, ϕ and A^*, ϕ^* in the form of an overlapping of the particular

fields of the electrical (1.2.22) and the magnetic (1.2.26) types, and through the vectors $\vec{\Pi}$ and $\vec{\Pi}^*$ in the form of an overlapping of fields (1.2.24) and (1.2.28).

In solving the specific problem of overlapping of an electromagnetic field, satisfying certain boundary conditions, depending on the type of boundary conditions we may find it to be feasible to use only the electrical potentials alone or the magnetic potentials alone (or the Hertz vectors representing them), or these or others together. In this case, according to the relationships which couple, for example, the Hertz vectors $\vec{\Pi}$ and $\vec{\Pi}^*$ with the field vectors E, H, the boundary conditions are first determined for the components of the vectors $\vec{\Pi}$ and $\vec{\Pi}^*$, which guarantee that the boundary conditions will be satisfied for the desired field E, H, then the boundary problems are solved by seeking $\vec{\Pi}$ and $\vec{\Pi}^*$, after which the field E, H, is found through $\vec{\Pi}$ and $\vec{\Pi}^*$. /19

§3. Plane Waves in a Homogeneous Unbounded Medium. Electrical and Magnetic Dipoles.

In solving problems of the diffraction of electromagnetic waves, it is usually assumed that the source of primary excitation of bodies producing diffraction is the field of a plane linearly polarized electromagnetic wave or the field of the point source.

Let us look at system (1.1.4) in an unbounded medium in which there are no extraneous currents and charges ($J^{(ext)} = 0$, $\rho = 0$). The simplest solution will be one such as

$$\mathbf{H} = \mathbf{H}(\xi) \quad (\xi = r(\xi)). \tag{1.3.1}$$

which depends only on a single space coordinate ξ, figured along a certain arbitrary axis directed along the unit vector n, forming the angles α, β, θ with coordinate axes of a fixed frame of reference $Oxyz$ ($n_x = \cos\alpha$, $n_y = \cos\beta$, $n_z = \cos\theta$; $n_x^2 + n_y^2 + n_z^2 = 1$). Functions (1.3.1), being complex amplitude expressions $\mathbf{H}(\xi)e^{-i\omega t}$, $\mathbf{E}(\xi)e^{-i\omega t}$ are customarily called a plane electromagnetic wave.

Since

$$\frac{\partial}{\partial x} = n_x \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial y} = n_y \frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial z} = n_z \frac{\partial}{\partial \xi},$$

from (1.2.2) and (1.2.4) it obviously then follows that (1.3.1) will satisfy the homogeneous Helmholtz vector equation,

$$\frac{d^2\mathbf{F}}{d\xi^2} + k^2\mathbf{F} = 0, \tag{1.3.2}$$

where by F we mean either the vector H or the vector E. By integrating (1.3.2) we can find the explicit form of the plane electromagnetic wave

$$\mathbf{H} = \mathbf{H}^0 e^{\pm ik\xi} \quad (\mathbf{E} = \mathbf{E}^0 e^{\pm ik\xi}), \tag{1.3.3}$$

which is propagated, respectively, in the positive and the negative directions of the axis ξ. Here H^0 and E^0 are arbitrary complex constants. By using the equations in system (1.1.4) we can establish that in any rectangular Cartesian system of coordinates ξ, ζ, η, in which the origin coincides with the origin of the system *Oxyz*, but the axis ξ remains, as before, oriented along n, both vectors E and H are found (at any moment of time) in the plane ξ = (nR), *normal* to the axis ξ(E_ξ = H_ξ = 0), and that at each point of a field such as (1.3.3) the vectors E and H (at any moment of time) are orthogonal to one another. /2

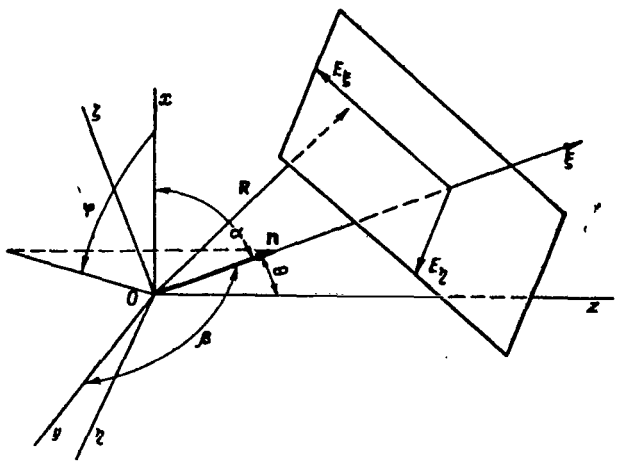

Fig. 2. Determination of a Plane Wave Propagating in the Direction of the Unit Vector n. The Point of Observation has the Radius-Vector R.

Then the plane wave, propagating in the direction of the vector n, can be written in the form

$$\mathbf{H} = \mathbf{H}^0 e^{ik(\mathbf{nR})} \quad (\mathbf{E} = \mathbf{E}^0 e^{ik(\mathbf{nR})}) \tag{1.3.4}$$

or, after inserting the time factor $e^{-i\omega t}$, it can be written in the form,

$$\mathbf{H} = \mathbf{H}^0 e^{ik(\mathbf{nR})-i\omega t} \quad (\mathbf{E} = \mathbf{E}^0 e^{ik(\mathbf{nR})-i\omega t}). \tag{1.3.5}$$

If the wave is propagated in a medium for which $\sigma \neq 0$, then $k = p + iq$, where p and q are real numbers ($p > 0$) and then the factor $e^{-q(nR)}$ contained in (1.3.4) indicates that in an absorbing medium, in proportion to the increase in (nR) in the direction of the vector n, the wave is damped exponentially.

Since

$$(\mathbf{nR}) = n_x x + n_y y + n_z z, \tag{1.3.6}$$

but the direction cosines of the axis ξ are expressed through the polar angles θ and ϕ (Fig. 2) by the formulas

$$n_x = \sin\theta\cos\varphi, \quad n_y = \sin\theta\sin\varphi, \quad n_z = \cos\theta, \qquad /21$$

then

$$(\mathbf{nR}) = x\sin\theta\cos\varphi + y\sin\theta\sin\varphi + z\cos\theta, \tag{1.3.7}$$

but

$$\mathbf{H} = \mathbf{H}^0\psi \quad (\mathbf{E} = \mathbf{E}^0\psi), \tag{1.3.8}$$

where

$$\psi = e^{ik(x\sin\theta\cos\varphi + y\sin\theta\sin\varphi + z\cos\theta)}.$$

By varying the values of θ and ϕ, we can assign any direction to the axis of propagation of the plane wave.

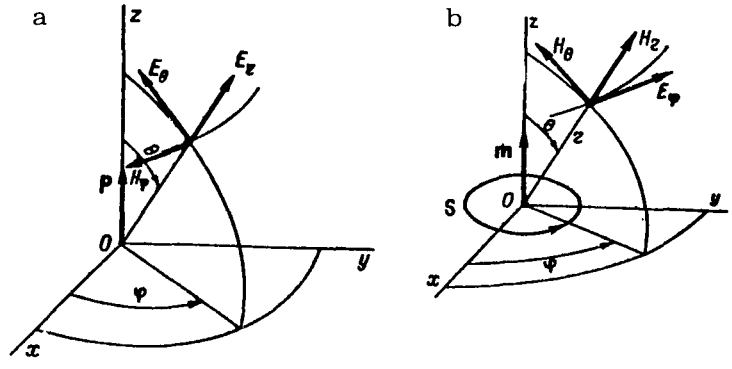

Fig. 3. Field of the Electrical (a) and Magnetic (b) Dipoles in a Spherical Coordinate System.

The vectors E, H in a plane electromagnetic wave at each moment of time are oriented in a given manner in space and at each point their ends, with the passage of time, will describe several curves (travel-time curves) as a function of which type one or another polarization of the wave will be involved. In the plane waves which we analyzed the travel-time curves of the vectors E and H have the same form and lie in one and the same plane, perpendicular to the vector n. Therefore, for them, the concepts of polarization associated with the direction of fluctuations of the vector E and with the direction of the fluctuations of the vector H, are of equal strength. For determinancy, the polarization is usually determined along the vector E. In the general case, in a polarized wave, the end of the vector E describes an ellipse. It is called an elliptically polarized wave. An important particular case in that when the ellipse degenerates into a straight line. Then the vectors E at all points of space at any moment of time are parallel to one another. We then say that such a wave is linearly polarized.

As an example of point sources of waves we can speak of elementary electrical or magnetic fluctuating dipoles (Fig. 3). The electrical (magnetic) dipole is customarily called a source whose /22
electromagnetic field is fully defined by giving only one vector, i.e., the electrical p (magnetic m) dipole moment. The dipole moments are associated with the electrical and magnetic Hertz vectors by the simply relationships:

$$\vec{\Pi} = \mathbf{p}\frac{e^{ikR}}{R}, \qquad (1.3.9)$$

$$\vec{\Pi}^* = \mathbf{m}\frac{e^{ikR}}{R}, \qquad (1.3.10)$$

where R is the distance from the point of observation to the source.

A model of an electrical dipole is a system of two charges, equal in value, opposite in sign and variable in time, $+q$ and $-q$, located at a distance l from one another ($l \ll \lambda_0$). If $q(t) = qe^{-i\omega t}$, so that the force of the current in the dipole $I = -i\omega q$, then $\mathbf{p} = k_0^2\mu I\mathbf{l}/i\omega k^2$, where the vector $\mathbf{l}$ ($|\mathbf{l}| = l$) is directed along the dipole axis from the negative charge toward the positive. If we assume the electrical dipole to be a linear current of infinitely small length dl, then $d\mathbf{p} = k_0^2\mu I d\mathbf{l}/i\omega k^2$.

Any real antenna, in which the primary role is played by the charge and the electrical current of the charge (an antenna of electrical type), may be expanded to elementary electrical dipoles with the respective dipole moments.

A model of a magnetic dipole is a system of two magnetic "charges" $+m$ and $-m$, located at a distance l from one another. Then $\mathbf{m} = m\mathbf{l}$,

where the vector 1 is directed from the south pole toward the north. In radio engineering the concept of a magnetic dipole is associated with loop antennas, in which the primary role is played by the electrical current and the magnetic moment generated by it (a magnetic type antenna). Thus, for example, a loop with current I, flowing in a loop around the area element S, bounded by the loop, may be interpreted (under the condition that $S \ll \lambda_0$) as an elementary magnetic dipole with the moment $\mathbf{m} = \frac{1}{c}(IS)\,\mathbf{n}$, where $\mathbf{n}$ is the unit normal to S (Fig. 3,b).

In practice, each real antenna for medium and long waves may be assumed as a point source of electromagnetic waves of either electrical or magnetic dipole type.

In the case of a point source of the electromagnetic field the vectors $\vec{\Pi}$ and $\vec{\Pi}^*$ from (1.3.9) and (1.3.10) are solutions to the nonhomogeneous Helmholtz equation such as

$$\Delta \mathbf{F} + k^2 \mathbf{F} = \mathbf{a}\,\delta(\mathbf{r} - \mathbf{r}_0), \tag{1.3.11}$$

where F denotes $\vec{\Pi}$ or $\vec{\Pi}^*$; $\delta(\mathbf{r} - \mathbf{r}_0)$ is a three-dimensional Dirac δ-function, and $\mathbf{r}$ and $\mathbf{r}_0$ are the radius-vectors connecting the origin of the system with the observation point (x, y, z) and the point (x_0, y_0, z_0), at which the source is located, respectively. Here $\mathbf{a}$ is a vector quantity determined through the dipole moments. /23

If V is a certain body containing within itself the source of the field, then the cundamental property determining the δ-function is the formal operator equation

$$\iiint_V \delta(\mathbf{r} - \mathbf{r}_0) f(\mathbf{r})\, d\mathbf{r} = \begin{cases} f(\mathbf{r}_0), & \text{if } \mathbf{r}_0 \in V, \\ 0, & \text{if } \mathbf{r}_0 \overline{\in} V, \end{cases} \tag{1.3.12}$$

where $f(\mathbf{r})$ is an arbitrary continuous function of the point $\mathbf{r}$. Sometimes the δ-function is determined formally also with the aid of the relationships

$$\left.\begin{aligned} \delta(\mathbf{r} - \mathbf{r}_0) &= 0, \quad \text{if } \mathbf{r} \neq \mathbf{r}_0, \\ \delta(\mathbf{r} - \mathbf{r}_0) &= \infty, \quad \text{if } \mathbf{r} = \mathbf{r}_0, \end{aligned}\right\}$$

$$\iiint_V \delta(\mathbf{r} - \mathbf{r}_0)\, d\mathbf{r} = \begin{cases} 1, & \text{if } \mathbf{r}_0 \in V, \\ 0, & \text{if } \mathbf{r}_0 \overline{\in} V. \end{cases}$$

The three-dimensional Dirac δ-function is expressed through one-dimensional δ-functions by the equation

$$\delta(\mathbf{r} - \mathbf{r}_0) = \delta(x - x_0)\,\delta(y - y_0)\,\delta(z - z_0).$$

For the electrical (magnetic) dipole with a moment, whose axis is directed parallel to one of the coordinate axes of the system $Oxyz$, for example, the axis Oz ($dl = dz$), in (1.3.11) we have $a_x = a_y = 0$, and

$$a_z = 4\pi I dz_0/i\omega\varepsilon \quad (a_z = 4\pi K dz_0/i\omega\mu),$$

where I is a given electrical (K is the magnetic) linear current in the dipole.

Let us mention one property which is important in the applications and which is possessed by the electromagnetic fields created by the dipole sources. Let there be in the space two electrical dispoles with the moments $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$, placed respectively at the point O_1 and O_2. Then, regardless of the orientation of the dipole moments as well as of the properties of the intermediate medium (provided that in the immediate vicinity of both dipoles the medium possesses identical electrical properties), the fields $\mathbf{E}^{(1)}$ and $\mathbf{E}^{(2)}$, created by them, are coupled to $\mathbf{p}^{(1)}$ and $\mathbf{p}^{(2)}$ by the equation

$$\mathbf{p}^{(1)}\mathbf{E}^{(2)}(O_1) = \mathbf{p}^{(2)}\mathbf{E}^{(1)}(O_2) \qquad (1.3.13)$$

It expresses the basic content of the so-called reciprocal theorem of two electrical dipoles. If, for example, $\mathbf{p}^{(1)} = \mathbf{p}^{(2)}$, then formula (1.3.13) means that a dipole with the moment $\mathbf{p}^{(1)}$, located at the point O_1, creates at the point O_2, where the other dipole is located with the moment $\mathbf{p}^{(2)}$, a field of the same intensity $\mathbf{E}^{(1)}$ as the intensity $\mathbf{E}^{(2)}$ of the field created at the point O_1 by the dipole with the moment $\mathbf{p}^{(2)}$, located at the point O_2 (Fig. 4). /24

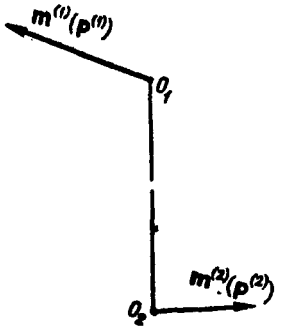

Fig. 4. Reciprocal Theorem for Two Dipoles.

The reciprocal theorem is also valid for magnetic dipoles:

$$\mathbf{m}^{(1)}\mathbf{H}^{(2)}(O_1) = \mathbf{m}^{(2)}\mathbf{H}^{(1)}(O_2). \qquad (1.3.14)$$

Formulas (1.3.13) and (1.3.14) are valid also for a space which contains within itself bodies with another medium, in particular when interfaces of the media exist in it.

Finally, let us mention that the field of a plane electromagnetic wave may in general be assumed to be a field of dipole source, removed to infinity.

§4. Quadratic Field Values. Scattering Cross Sections. Radiational Resistance. Radiational Conduction.

An electromagnetic field possesses energy that is localized in space. Furthermore, a transfer of energy from certain regions in space to others is associated with any electromagnetic wave process. The amount of energy, transferred through a surface unit, perpendicular to the direction of propagation of the energy per unit of time, is determined by the expression /25

$$\mathbf{S} = \frac{c}{4\pi} [\mathbf{EH}], \tag{1.4.1}$$

which is called the Poynting vector. The Poynting vector gives an instantaneous value for the current density of the power through the unit area in the direction of the normal to the wave front. The volume densities of the electromagnetic energy (density of field energy per unit volume) are determined by the formula

$$W = W_e + W_m, \tag{1.4.2}$$

where $W_e = \frac{\varepsilon}{8\pi} E^2$, $W_m = \frac{\mu}{8\pi} H^2$ are the densities of the energy of the electrical and magnetic fields, respectively. In studying the Poynting vector and the volume density of the energy, as is clear from (1.4.1), we are involved with the products and squares of the components of the electromagnetic field, with respect to which they are usually called quadratic field values. In formulas (1.4.1) and (1.4.2), E and H are true values of the vectors of the electrical and magnetic field intensities. The nonlinearity of S and W leads to a situation where, in computing S and W using formulas (1.4.1) and (1.4.2), instead of the true real vectors E, H, it is impossible to use in them the complex vectors corresponding to them as was done in the Maxwell equations. In practice, in studying an electromagnetic field which is harmonic in time, instead of the instantaneous values of the Poynting vector and the volume density of the energy, the mean values of these quantities for the period of the fluctuations are usually employed. The mean value of the Poynting vector is equal to the real part of the complex vector $\frac{c}{8\pi} [EH^*]$ (the asterisk here denotes conversion to the complex conjugate value). If the mean value of the vector S is denoted through $\overline{S}$, then

$$\overline{\mathbf{S}} = \frac{c}{8\pi} \operatorname{Re} [\mathbf{EH}^*] = \frac{c}{16\pi} \{[\mathbf{EH}^*] + [\mathbf{E}^*\mathbf{H}]\}. \tag{1.4.3}$$

The mean value of the volume density of the electrical energy is $\overline{W}_e = \frac{\varepsilon}{16\pi} |E|^2$, and that of the magnetic energy is $\overline{W}_m = \frac{\mu}{16\pi} |H|^2$, so that

$$\overline{W} = \frac{\varepsilon}{16\pi} |\mathbf{E}|^2 + \frac{\mu}{16\pi} |\mathbf{H}|^2, \tag{1.4.4}$$

where $\overline{W}$ is the mean value of the density of energy in an electromagnetic field. The average power $\overline{P}$, radiated through any surface S (radiational power) is computed through the complex field amplitudes from the formula ($\mathbf{n}$ is the unit vector outside S in the normal direction), /2

$$\overline{P} = \frac{c}{8\pi} \operatorname{Re} \int\int_S ([\mathbf{E}\mathbf{H}^*]\mathbf{n})\, dS.$$

The intensity I of the electromagnetic wave is associated with the Poynting vector and is a quantity that is numerically equal to the energy transmitted by the wave per unit of time through a unit surface area normal to the direction of the wave propagation:

$$I = |\overline{\mathbf{S}}| = \frac{1}{T} \int_0^T |\mathbf{S}|\, dt,$$

where T is the period of the fluctuations.

In problems of diffraction of a plane electromagnetic wave on barriers, along with knowledge of the scattered field, the so-called scattering cross sections are of considerable interest; they are quantities which characterize the ability of the barrier to re-emit, absorb and admit energy of the plane wave incident on it. The scattering cross section is denoted by σ and is computed from the formula [46-48, 151],

$$\sigma = \lim_{r \to \infty} 4\pi r^2 \frac{|\overline{S}_r^s(\theta, \varphi)|}{|\overline{S}^i(\theta_i, \varphi_i)|}, \tag{1.4.5}$$

where r, θ, ϕ are spherical coordinates of the system, at the origin of which is found the diffracting body (Fig. 5), $\overline{S}_r^s(\theta, \phi)$ is the value of the radial component of the Poynting vector, averaged in time and corresponding to the radial component of the Poynting vector, averaged in time and corresponding to the scattering in the

directions θ, ϕ in the wave band[5] at a distance r from the barrier; $\overline{S}^i(\theta_i\ \phi_i)$ is the value of the Poynting vector of the plane wave, averaged in time and incident on the body from the direction θi, ϕi. If the quantity σ is computed in the direction to the source of the plane wave ($\theta = \theta i$, $\phi = \phi i$), then it is called a cross section of inverse scattering and is denoted by σ_B:

$$\sigma_B = \lim_{r\to\infty} 4\pi r^2 \frac{|\overline{S}_r^s(\theta_i,\ \varphi_i)|}{|\overline{S}^i(\theta_i,\ \varphi_i)|}. \qquad (1.4.6)$$ /27

For any other direction, σ is called a two-position scattering cross section (sometimes it is called a differential scattering cross section). The number

$$\sigma_s = \frac{\overline{P}^s}{|\overline{S}^i|}, \qquad (1.4.7)$$

obtained from (1.4.5) by integration over the sphere S of radius r, with the center at the origin, is called a complete scattering cross section. Here $\overline{P}^s$, just as before, is the average power (complete scattered or re-emitted power) through the surface S:

$$\overline{P}^s = \frac{c}{8\pi}\,\mathrm{Re}\iint_S (|\mathbf{E}^s\mathbf{H}^{*s}|\,\mathbf{r})\,dS = \qquad (1.4.8)$$

$$= \frac{c}{8\pi}\,\mathrm{Re}\iint_S |E_\theta^s H_\varphi^{*s} - E_\varphi^s H_\theta^{*s}|\,dS,$$

where $\mathbf{r}$ is the unit vector to S in the radial direction. The vectors of the scattered electromagnetic field are denoted by E^s, H^s. /28

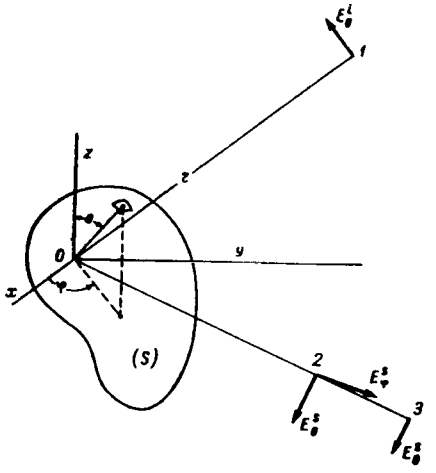

Fig. 5. Determination of the Scattering Cross Section: 1. Field of the Incident Wave with Linear Polarization; 2. Field of the Scattered Wave with Elliptical Polarization; 3. Field at Reception Site.

The plane wave, incident on the barrier and with linear polarization, usually creates an elliptically polarized scattered wave, containing both E_θ^s, and E_ϕ^s. The

[5] The wave (remote) band is that region of space that remains away from the radiating system at distances which significantly exceed the dimensions of the radiating system and the length of the waves radiated by it. In the wave band the scattered wave is spherical and the Poynting vector has only a radial component.

vectors $E^s_\theta e_\theta$ and $E^s_\phi e_\phi$ may be, respectively, parallel or perpendicular to the field of the vector E^i of the incident wave. The radar instruments which emit and receive the radio waves are usually sensitive to that part of the wave scattered on the barrier that is polarized the same as the plane wave. In this case the two-position scattering cross section is computed from the formula

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|E^s|^2}{|E^i|^2}, \qquad (1.4.9)$$

where E^s is the component of the scattering field at the reception site in the direction θ, ϕ, polarized the same as the field E^i of the plane wave, incident on the barrier from the direction θ_i, ϕ_i; r is the distance from the barrier to the receiver, located in the wave band of the field. When $\theta = \theta_i$, $\phi = \phi_i$ in formula (1.4.9) (the case of inverse scattering), the value of σ is called a radar cross section. The radar cross section is a measure of the ability of the barrier to scatter energy in a given direction (a measure of the visibility of the target).

In two-dimensional problems, for example in the case of cylindrical barriers, an analog to σ is the linear scattering cross section

$$\sigma_s = \frac{\bar{P}^s}{|\bar{S}^i|}, \qquad (1.4.10)$$

where $\bar{P}^s$ is the complete linear averaged power, per unit of length, and the integration is carried out over a circle of radius r, which includes within itself the cross section of the barrier. Instead of (1.4.9) we will have

$$\sigma = \lim_{r\to\infty} 2\pi r \frac{|E^s|^2}{|E^i|^2} \qquad (1.4.11)$$

etc. If the diffracted barrier possesses a finite conductivity, then the value

$$\sigma = \frac{\bar{P}^a}{|\bar{S}^i|}, \qquad (1.4.12)$$

where $\bar{P}^a$ is the complete power absorbed by the body, is termed the complete absorption cross section. If the barrier is a nontransparent plane screen with an opening, and $\bar{P}^t$ is the complete power passing through it, then the value

$$\sigma = \frac{\bar{P}^t}{|\bar{S}^i|} \qquad (1.4.13)$$

s called the transmission cross section.

In problems on the diffraction of the field of a dipole emitter ocated near the reflecting surface or on the surface of the body oducing the diffraction, an important parameter characterizing the operties of the radiating system, is the resistance of the radi- ion, i.e., the value

$$R = \frac{2\bar{P}}{|I_0|^2}, \tag{1.4.14}$$

here I_0 is a complex current on the terminals of the emitter (cur- ent at the dipole), and

$$\bar{P} = \frac{c}{8\pi} \operatorname{Re} \iint_S ([\mathbf{EH}^*]\,\mathbf{n})\,dS \tag{1.4.15}$$

nd the integration is carried out over the surface S, which encom- asses the radiating system, $\mathbf{n}$ is the unit normal outside of S, and , H are the vectors of the complete electromagnetic field). We lso analyze the quantity

$$\Gamma = \frac{R}{R_0}, \tag{1.4.16}$$

here R_0 is the resistance of the radiation from the field of the ipole in free space, and R is the same in the presence of a re- lector in it.

If the emitter is a slit, located on the surface of the body, hen its radiating properties are characterized by the value

$$G = \frac{2\bar{P}}{|V_0|^2}, \tag{1.4.17}$$

alled the radiational conduction. Here V_0 is a given complex volt- ge on the slit.

For quantitative characteristics of the properties of the irected effect of the radiating system we can introduce a dimension- ess value equal to the ratio of power emitted per unit of solid ngle in the direction θ, ϕ to the mean power per unit of solid angle. f we denote this by $g(\theta, \phi)$, then

$$g(\theta, \varphi) = \frac{4\pi\Psi(\theta, \varphi)}{\int_0^{2\pi}\int_0^{\pi}\Psi(\theta, \varphi)\sin\theta\,d\theta\,d\varphi}, \tag{1.4.18}$$

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where in approximation of a wave band

$$\Psi(\theta, \varphi) = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{8\pi r^2 \overline{S}_r^s}{c} = (|E_\theta^s|^2 + |E_\varphi^s|^2) r^2. \qquad (1.4.19)$$

The value $g(\theta, \phi)$ is usually called a coefficient of the directed effect of the radiating system (antenna) and its maximal value is the absolute coefficient of the directed effect of the antenna.

§5. Systems of Orthogonal Curvilinear Coordinates. Field Equations in Orthogonal Curvilinear Coordinates.

In the case of bodies having the simplest geometrical shape (circular and elliptical cylinders, spheres, spheroids), the diffraction problems can be conveniently solved in orthogonal curvilinear coordinates, in which the surface of the body agrees with one of the complete coordinate surfaces of the system, since in this case it is possible to give the simplest analytical description of the boundry condition for the desired solution to the problem.

The curvilinear coordinates of the point P, having the numbers x, y, z as the rectangular coordinates, are denoted by three numbers q_1, q_2, q_3, defined as values that are single-valued and continuous in the region of functions of the system being analyzed,

$$q_1 = \varphi_1(x, y, z),\ q_2 = \varphi_2(x, y, z),\ q_3 = \varphi_3(x, y, z), \qquad (1.5.1)$$

solvable relative to x, y, z:

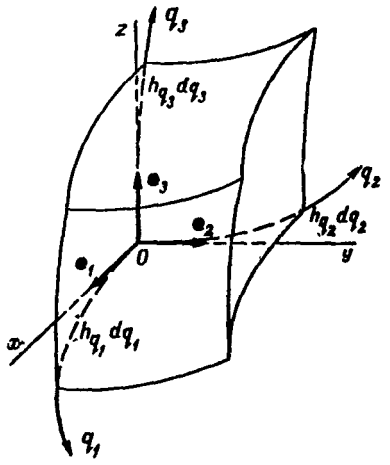
$$x = \psi_1(q_1, q_2, q_3),\ y = \psi_2(q_1, q_2, q_3),$$
$$z = \psi_3(q_1, q_2, q_3) \qquad (1.5.2)$$

(the functions ψ_i, $i = 1, 2, 3$, are also single-valued and continuous in the region of analysis). From a given point $P(x, y, z)$ from (1.5.1), we can find the unique values q_1, q_2, q_3 corresponding to them and, on the contrary, from the assigned numbers q_1, q_2, q_3, we can find the point $P(x, y, z)$ corresponding to it, after determining its coordinates from (1.5.2). Therefore, on a par with the denotation $P(x, y, z)$ the point P can also be denoted by $P(q_1, q_2, q_3)$.

If we substitute into (1.5.1) $q_i = C_i$, where C_i ($i = 1, 2, 3$) are constant numbers, then as a result we find three families of surfaces, given by the equations

$$\varphi_1(x, y, z) = C_1,\ \varphi_2(x, y, z) = C_2,\ \varphi_3(x, y, z) = C_3, \qquad (1.5.3)$$

ch of the families (1.5.3), because of the uniqueness of the functions (1.5.1), is characteristic in the sense that any two of its surfaces corresponding to the different values of the parameter C do not intersect themselves, whereas any of the surfaces $C_i = \phi_i(x, y, z)$ do intersect with any of the surfaces $C_j = \phi_j(x, y, z)$ $(i \neq j;\ i, j = 1, 2, 3)$.



ig. 6. Curvilinear rthogonal Coordinates.

The surfaces $q_i = C_i$ $(i = 1, 2, 3)$ are called coordinate surfaces in the coordinate system q_1, q_2, q_3. The line of intersection of two coordinate surfaces $q_i = C_i$ and $q_j = C_j$ $(i \neq j;\ i, j = 1, 2, 3)$ (in the general case, the curve along which only the value $q_k, k = 1, 2, 3, k \neq i, k \neq j$ varies) is called the coordinate line in the coordinate system q_1, q_2, q_3. The coordinate system q_1, q_2, q_3 is called a system of curvilinear orthogonal coordinates if any two coordinate surfaces (lines) of the system at each oint of the region intersect at a right angle (Fig. 6).

The condition of reciprocity of the orthogonality of coordinate ines is expressed by the equation

$$\sum_{k=1}^{3} \frac{\partial \psi_k}{\partial q_i} \cdot \frac{\partial \psi_k}{\partial q_j} = 0 \quad (i \neq j;\ i, j = 1, 2, 3). \tag{1.5.4}$$

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n the system of orthogonal curvilinear coordinates q_1, q_2, q_3 the lement of length dS of the arbitrary curve is equal to

$$(dS)^2 = (dx)^2 + (dy)^2 + (dz)^2 = \sum_{k=1}^{3} \left(\sum_{i=1}^{3} \frac{\partial \psi_k}{\partial q_i}\, dq_i \right)^2,$$

›r, if we take into account (1.5.4),

$$(dS)^2 = \sum_{i=1}^{3} (h_{q_i} dq_i)^2, \tag{1.5.5}$$

where the values

$$h_{q_i} = \sqrt{\sum_{k=1}^{3} \left(\frac{\partial \psi_k}{\partial q_i} \right)^2} \tag{1.5.6}$$

are called metric coefficients of the system q_1, q_2, q_3 or Lame coefficients of the system q_1, q_2, q_3. The systems of orthogonal curvilinear coordinates are fully characterized by these values. Thus, an element of length dl_i of the coordinate line q_i is equal to

$$dl_i = h_{q_i} dq_i \quad (i = 1, 2, 3), \tag{1.5.7}$$

the elements dS of the coordinate surfaces are equal to

$$dS_i = h_{q_j} h_{q_k} dq_j dq_k \quad (i \neq j,\ i \neq k,\ j \neq k;\ i, j, k = 1, 2, 3), \tag{1.5.8}$$

and the volume element is equal to

$$dv = dl_1 dl_2 dl_3 = h_{q_1} h_{q_2} h_{q_3} dq_1 dq_2 dq_3. \tag{1.5.9}$$

Using (1.5.6), the unit vectors $\mathbf{e}_{q_i}$ (i = 1, 2, 3) of the system of orthogonal curvilinear coordinates are expressed through the unit vectors i, j, k of the rectangular coordinate system by the relationships

$$\mathbf{e}_{q_i} = \frac{1}{h_{q_i}} \left\{ \frac{\partial x}{\partial q_i} \mathbf{i} + \frac{\partial y}{\partial q_i} \mathbf{j} + \frac{\partial z}{\partial q_i} \mathbf{k} \right\} \quad (i = 1, 2, 3). \tag{1.5.10}$$

Therefore if A is a vector function of the point, A_x, A_y, A_z are the projections of the vector A on the axis of the rectangular system of coordinates, but A_{q1}, A_{q2}, A_{q3} are the projections of it on the tangents to the curvilinear axes, then

$$\begin{aligned} h_{q_1} A_{q_1} &= A_x \frac{\partial x}{\partial q_1} + A_y \frac{\partial y}{\partial q_1} + A_z \frac{\partial z}{\partial q_1}, \\ h_{q_2} A_{q_2} &= A_x \frac{\partial x}{\partial q_2} + A_y \frac{\partial y}{\partial q_2} + A_z \frac{\partial z}{\partial q_2}, \\ h_{q_3} A_{q_3} &= A_x \frac{\partial x}{\partial q_3} + A_y \frac{\partial y}{\partial q_3} + A_z \frac{\partial z}{\partial q_3}. \end{aligned} \tag{1.5.11}$$

The differential operators grad, div, Δ and rot in the system /33 of orthogonal curvilinear coordinates are expressed by the formulas:

$$\operatorname{grad} \psi = \sum_{i=1}^{3} \frac{1}{h_{q_i}} \cdot \frac{\partial \psi}{\partial q_i} \mathbf{e}_{q_i}, \tag{1.5.12}$$

$$\operatorname{div} \mathbf{A} = \frac{1}{h_{q_1} h_{q_2} h_{q_3}} \left[\frac{\partial}{\partial q_1} (h_{q_2} h_{q_3} A_{q_1}) + \right.$$
$$\left. + \frac{\partial}{\partial q_2} (h_{q_1} h_{q_3} A_{q_2}) + \frac{\partial}{\partial q_3} (h_{q_1} h_{q_2} A_{q_3}) \right], \qquad (1.5.13)$$

$$\Delta \psi = \frac{1}{h_{q_1} h_{q_2} h_{q_3}} \left[\frac{\partial}{\partial q_1} \left(\frac{h_{q_2} h_{q_3}}{h_{q_1}} \cdot \frac{\partial \psi}{\partial q_1} \right) + \right.$$
$$+ \frac{\partial}{\partial q_2} \left(\frac{h_{q_1} h_{q_3}}{h_{q_2}} \cdot \frac{\partial \psi}{\partial q_2} \right) +$$
$$\left. + \frac{\partial}{\partial q_3} \left(\frac{h_{q_1} h_{q_2}}{h_{q_3}} \cdot \frac{\partial \psi}{\partial q_3} \right) \right], \qquad (1.5.14)$$

$$\operatorname{rot} \mathbf{A} = \frac{1}{h_{q_1} h_{q_2} h_{q_3}} \begin{vmatrix} h_{q_1} \mathbf{e}_{q_1} & h_{q_2} \mathbf{e}_{q_2} & h_{q_3} \mathbf{e}_{q_3} \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_{q_1} A_{q_1} & h_{q_2} A_{q_2} & h_{q_3} A_{q_3} \end{vmatrix} \qquad (1.5.15)$$

System of Cylindrical Coordinates (Fig. 7). The so-called variable quantities $q_1 = \rho$, $q_2 = \phi$, $q_3 = z$, are coupled with the Cartesian rectangular coordinates x, y, z by the conversion formulas

$$x = \rho \cos \varphi, \; y = \rho \sin \varphi, \; z = z, \qquad (1.5.16)$$

where $0 \leq \rho < \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < z < \infty$. In this system the coordinate surfaces are the surfaces of the coaxial circular cylinders $x^2 + y^2 = \rho_0^2$ (ρ_0^2 = const) with the axis of rotation Oz; the half-planes ϕ_0 = const, passing through the axis Oz and forming the angle ϕ_0 with the plane xOz, are also determined from the equations /34
$x = \rho_0 \cos\phi_0$, $y = \rho_0 \sin\phi_0$, as are the planes z = const, perpendicular to the axis Oz.

The Lame coefficients of the system are equal to

$$h_\rho = 1, \; h_\varphi = \rho, \; h_z = 1. \qquad (1.5.17)$$

If $\mathbf{e}_\rho$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ are unit vectors of the cylindrical system of coordinates, then formulas (1.5.10) - (1.5.15), on the basis of (1.5.17), in this system have the form:

$$\left. \begin{aligned} \mathbf{e}_\rho &= \cos \varphi \, \mathbf{i} + \sin \varphi \, \mathbf{j}, \\ \mathbf{e}_\varphi &= -\sin \varphi \, \mathbf{i} + \cos \varphi \, \mathbf{j}, \\ \mathbf{e}_z &= \mathbf{k}, \end{aligned} \right\} \qquad (1.5.18)$$

$$\left.\begin{aligned} A_\rho &= A_x \cos\varphi + A_y \sin\varphi, \\ A_\varphi &= -A_x \sin\varphi + A_y \cos\varphi, \\ A_z &= A_z, \end{aligned}\right\} \qquad (1.5.19)$$

$$\operatorname{grad}\psi = \frac{\partial\psi}{\partial\rho}\,\mathbf{e}_\rho + \frac{1}{\rho}\cdot\frac{\partial\psi}{\partial\varphi}\,\mathbf{e}_\varphi + \frac{\partial\psi}{\partial z}\,\mathbf{e}_z, \qquad (1.5.20)$$

$$\operatorname{div}\mathbf{A} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{\partial}{\partial\varphi}(A_\varphi) + \rho\frac{\partial}{\partial z}(A_z)\right], \qquad (1.5.21)$$

$$\Delta\psi = \frac{1}{\rho}\cdot\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\cdot\frac{\partial^2\psi}{\partial\varphi^2} + \frac{\partial^2\psi}{\partial z^2}, \qquad (1.5.22)$$

$$\operatorname{rot}\mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{e}_\rho & \rho\,\mathbf{e}_\varphi & \mathbf{e}_z \\ \dfrac{\partial}{\partial\rho} & \dfrac{\partial}{\partial\varphi} & \dfrac{\partial}{\partial z} \\ A_\rho & \rho A_\varphi & A_z \end{vmatrix}. \qquad (1.5.23)$$

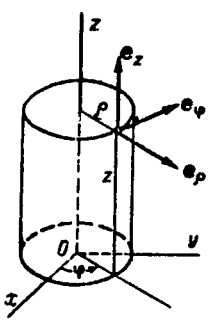
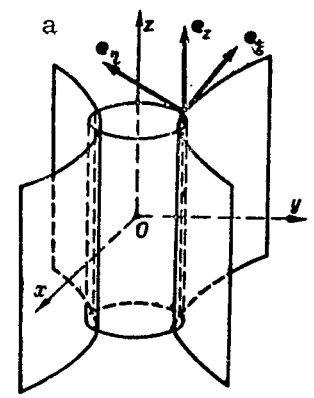


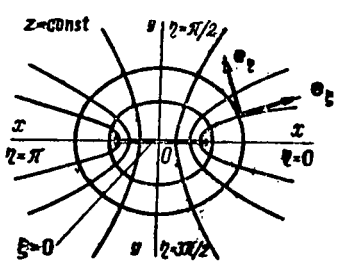
Fig. 7. System of Cylindrical Coordinates.

System of Elliptical Cylindrical Coordinates (Fig. 8). The so-called variable quantities $q_1 = \xi$, $q_2 = \eta$, $q_3 = z$, are coupled with the Cartesian rectangular coordinates x, y, z by the conversion formulas, /35

$$x = f\,\mathrm{ch}\,\xi\cos\eta,\quad y = f\,\mathrm{sh}\,\xi\sin\eta,\quad z = z, \qquad (1.5.24)$$

where $0 \le \xi < \infty$, $0 \le \eta \le 2\pi$, $-\infty < z < \infty$. In this system the coordinate surfaces are the





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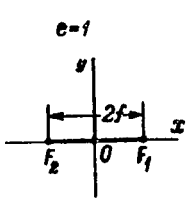

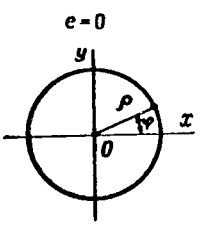

Fig. 8. System of Elliptical Cylindrical Coordinates (a) and System of Elliptical Coordinates (b) on the Plane z = const.

surfaces determined by the equations,

$$\frac{x^2}{f^2 \operatorname{ch}^2 \xi} + \frac{y^2}{f^2 \operatorname{sh}^2 \xi} = 1$$

(ξ is a constant),

$$\frac{x^2}{f^2 \cos^2 \eta} - \frac{y^2}{f^2 \sin^2 \eta} = 1$$

(η is a constant),

$$z = \text{const.}$$

The first of these equations in an equation of the family of elliptical cylinders with interfocal distances equal to $2f$. The second equation is an equation of the family of cofocal two-sheet hyperbolic cylinders. /36

The Lame coefficients of the system are equal to

$$h_\xi = h_\eta = f\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}, \quad h_z = 1. \tag{1.5.25}$$

If e_ξ, e_η, e_z are the unit vectors of the elliptical cylindrical system of coordinates, then, according to formulas (1.5.10) - (1.5.15), we obtain:

$$\left.\begin{aligned} \mathbf{e}_\xi &= \frac{\operatorname{sh}\xi \cos\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}\,\mathbf{i} + \frac{\operatorname{ch}\xi \sin\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}\,\mathbf{j}, \\ \mathbf{e}_\eta &= -\frac{\operatorname{ch}\xi \sin\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}\,\mathbf{i} + \frac{\operatorname{ch}\xi \cos\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}\,\mathbf{j}, \\ \mathbf{e}_z &= \mathbf{k}, \end{aligned}\right\} \tag{1.5.26}$$

$$\left.\begin{aligned} A_\xi &= A_x \frac{\operatorname{sh}\xi \cos\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}} + A_y \frac{\operatorname{ch}\xi \sin\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}, \\ A_\eta &= -A_x \frac{\operatorname{ch}\xi \sin\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}} + A_y \frac{\operatorname{sh}\xi \cos\eta}{\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}}, \\ A_z &= A_z, \end{aligned}\right\} \tag{1.5.27}$$

$$\operatorname{grad}\psi = \frac{1}{f\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}} \cdot \frac{\partial \psi}{\partial \xi}\,\mathbf{e}_\xi + \frac{1}{f\sqrt{\operatorname{ch}^2 \xi - \cos^2 \eta}} \cdot \frac{\partial \psi}{\partial \eta}\,\mathbf{e}_\eta + \frac{\partial \psi}{\partial z}\,\mathbf{e}_z, \tag{1.5.28}$$

$$\operatorname{div}\mathbf{A}=\frac{1}{f(\operatorname{ch}^2\xi-\cos^2\eta)}\left[\frac{\partial}{\partial\xi}\left(\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,A_\xi\right)+\right.$$
$$\left.+\frac{\partial}{\partial\eta}\left(\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,A_\eta\right)\right]+\frac{\partial}{\partial z}A_z, \qquad (1.5.29)$$

$$\Delta\psi=\frac{1}{f^2(\operatorname{ch}^2\xi-\cos^2\eta)}\left[\frac{\partial^2\varphi}{\partial\xi^2}+\frac{\partial^2\varphi}{\partial\eta^2}\right]+\frac{\partial^2\varphi}{\partial z^2}, \qquad (1.5.30)$$

$$\operatorname{rot}\mathbf{A}=\frac{1}{f^2(\operatorname{ch}^2\xi-\cos^2\eta)}\times$$
$$\times\begin{vmatrix} f\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,\mathbf{e}_\xi & f\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,\mathbf{e}_\eta & \mathbf{e}_z \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial z} \\ f\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,A_\xi & f\sqrt{\operatorname{ch}^2\xi-\cos^2\eta}\,A_\eta & A_z \end{vmatrix} \qquad (1.5.31)$$

Since the major and minor semiaxes of the coordinate elliptical cylinders are equal, respectively, to $a = f\operatorname{ch}\xi$ and $b = f\operatorname{sh}\xi$, then the eccentricity of the family of elliptical coordinate cylinders will be determined by the relationship $e = 1/\operatorname{ch}\xi$. Consequently, if $e \to 1$ in the fixed coordinate grid in the plane z = const, then $\xi \to 0$ and $a \to f$, and the elliptical cylinder degenerates into an infinitely long band with a width $2f$ (Fig. 8,b). If this same distance between focal points of the cylinders remains constant, but $a \to \infty$, then $\xi \to \infty$, but $e \to 0$. As a result, the family of confocal elliptical cylinders changes at the limit to a family of coaxial circular cylinders with the axis Oz and the radius $\rho = fe^{\xi}/2$.

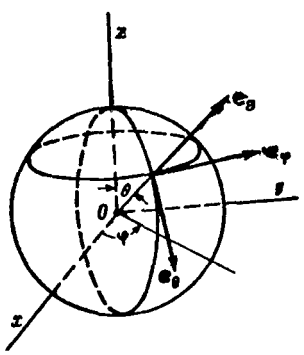

Fig. 9. System of Spherical Coordinates.

In the case when the major semiaxis a of the coordinate elliptical cylinders remains constant, but $e \to 0$ (the coordinate grid in the plane z = const varies), then $\xi \to \infty$ and $f \to 0$, so that $f\operatorname{ch}\xi = f\operatorname{sh}\xi \to \rho = a$. As a result, the system of elliptical cylindrical coordinates changes to a system of cylindrical coordinates with polar coordinates $\rho = fe^{\xi}/2$ and $\eta = \phi$.

System of Spherical Coordinates (Fig. 9). The so-called variable quantities $q_1 = r$, $q_2 = \theta$, $q_3 = \phi$, are coupled with the Cartesian rectangular coordinates x, y, z by the conversion formulas

$$x = r\sin\theta\cos\varphi,\ y = r\sin\theta\sin\varphi,\ z = r\cos\theta, \qquad (1.5.32)$$

where $0 \le r \le \infty$, $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$. The coordinate surfaces here are the concentric spheres r = const with centers at the origin, the family of cosines θ = const with the vertex at the center of the sphere and the family of meridional planes ϕ = const.

The Lame coefficients of the system are equal to /38

$$h_r = 1,\ h_\theta = r,\ h_\varphi = r\sin\theta, \tag{1.5.33}$$

therefore if $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_\phi$ are the unit vectors of the system of spherical coordinates, then according to formulas (1.5.10) - (1.5.15), we obtain:

$$\left.\begin{aligned} \mathbf{e}_r &= \sin\theta\cos\varphi\cdot\mathbf{i} + \sin\theta\sin\varphi\cdot\mathbf{j} + \cos\theta\cdot\mathbf{k},\\ \mathbf{e}_\theta &= \cos\theta\cos\varphi\cdot\mathbf{i} + \cos\theta\sin\varphi\cdot\mathbf{j} - \sin\theta\cdot\mathbf{k},\\ \mathbf{e}_\varphi &= -\sin\varphi\cdot\mathbf{i} + \cos\varphi\cdot\mathbf{j}, \end{aligned}\right\} \tag{1.5.34}$$

$$\left.\begin{aligned} A_r &= A_x\sin\theta\cos\varphi + A_y\sin\theta\sin\varphi + A_z\cos\theta,\\ A_\theta &= A_x\cos\theta\cos\varphi + A_y\cos\theta\sin\varphi - A_z\sin\theta,\\ A_\varphi &= -A_x\sin\varphi + A_y\cos\varphi, \end{aligned}\right\} \tag{1.5.35}$$

$$\operatorname{grad}\psi = \frac{\partial\psi}{\partial r}\mathbf{e}_r + \frac{1}{r}\cdot\frac{\partial\psi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\times \times\frac{\partial\psi}{\partial\varphi}\mathbf{e}_\varphi, \tag{1.5.36}$$

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\varphi}(A_\varphi), \tag{1.5.37}$$

$$\Delta\psi = \frac{1}{r^2}\cdot\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\times \times\frac{\partial}{\partial\theta}\left(\sin\theta\cdot\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\psi}{\partial\varphi^2}, \tag{1.5.38}$$

$$\operatorname{rot}\mathbf{A} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \mathbf{e}_r & r\mathbf{e}_\theta & r\sin\theta\cdot\mathbf{e}_\varphi\\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi}\\ A_r & rA_\theta & r\sin\theta A_\varphi \end{vmatrix} \tag{1.5.39}$$

System of Prolate Spheroidal Coordinates (Fig. 10). The so-called variable quantities $q_1 = \xi$, $q_2 = \eta$, $q_3 = \phi$, are connected with the rectangular Cartesian coordinates x, y, z by the conversion formulas, /39

$$\left.\begin{aligned} x &= f[(\xi^2-1)(1-\eta^2)]^{1/2}\cos\psi,\\ y &= f[(\xi^2-1)(1-\eta^2)]^{1/2}\sin\varphi,\ z = f\xi\eta, \end{aligned}\right\} \tag{1.5.40}$$

where $1 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, $0 \leq \phi \leq 2\pi$. The coordinate surfaces here are the family of prolate spheroids (ellipsoids of rotation) ξ = const with the major axis equal to $2f\xi$, and the minor axis equal to $2f(\xi^2-1)^{1/2}$ ($2f$ is the interfocal distance). The degenerate coordinate surface $\xi = 1$ is a segment of the z axis of length $2f$ (extending from $z = -f$ to $z = f$). The family of coordinate surfaces η = const represents a family of two-sheet hyperboloids of rotation with an asymptotic cone, whose generatrices pass through the origin and comprise an angle α = arc cos η with the axis z. The degenerate coordinate surface $|\eta| = 1$ is the part of the axis z for which $|z| > f$. The third family of coordinate surfaces is a family of planes ϕ = const, passing through the axis z at an angle ϕ to the plane Oxz.

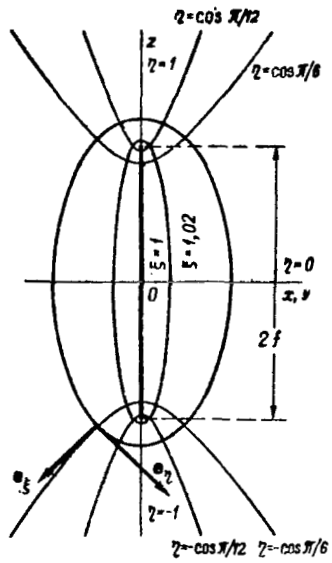

Fig. 10. System of Prolate Spheroidal Coordinates.

The Lame coefficients of the system are equal to

$$h_\xi = f\sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}, \quad h_\eta = f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}, \quad h_\varphi = f\sqrt{(\xi^2-1)(1-\eta^2)}, \tag{1.5.41}$$

therefore, if $\mathbf{e}_\xi$, $\mathbf{e}_\eta$, $\mathbf{e}_\phi$ are unit vectors of the system of prolate spheroidal coordinates, then according to formulas (1.5.10) - (1.5.15) we obtain: /40

$$\left.\begin{aligned}
\mathbf{e}_\xi &= \sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\left\{\xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\cos\varphi\cdot\mathbf{i} + \right.\\
&\left. + \xi\sqrt{\frac{1-\eta^2}{\xi^2-1}}\sin\varphi\cdot\mathbf{j} + \eta\cdot\mathbf{k}\right\},\\
\mathbf{e}_\eta &= \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\left\{-\eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\cos\varphi\cdot\mathbf{i} - \right.\\
&\left. - \eta\sqrt{\frac{\xi^2-1}{1-\eta^2}}\sin\varphi\cdot\mathbf{j} + \xi\cdot\mathbf{k}\right\},\\
\mathbf{e}_\varphi &= -\sin\varphi\cdot\mathbf{i} + \cos\varphi\cdot\mathbf{j},
\end{aligned}\right\} \tag{1.5.42}$$

$$\left.\begin{aligned}
A_\xi &= A_x\xi\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\cos\varphi + \\
&+ A_y\xi\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\sin\varphi + A_z\eta\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}},\\
A_\eta &= -A_x\eta\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\cos\varphi - \\
&- A_y\eta\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\sin\varphi + A_z\xi\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}},\\
A_\varphi &= -A_x\sin\varphi + A_y\cos\varphi,
\end{aligned}\right\} \tag{1.5.43}$$

$$\operatorname{grad}\psi = \frac{1}{f}\sqrt{\frac{\xi^2-1}{\xi^2-\eta^2}}\,\frac{\partial\psi}{\partial\xi}\,\mathbf{e}_\xi + \frac{1}{f}\sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}}\,\frac{\partial\psi}{\partial\eta}\,\mathbf{e}_\eta + \\ + f\sqrt{\frac{1}{(1-\eta^2)(\xi^2-1)}}\,\frac{\partial\psi}{\partial\varphi}\,\mathbf{e}_\varphi, \qquad (1.5.44)$$

$$\operatorname{div}\mathbf{A} = \frac{1}{f(\xi^2-\eta^2)}\left[\frac{\partial}{\partial\xi}\left(A_\xi\sqrt{(\xi^2-\eta^2)(\xi^2-1)}\right) + \right. \\ \left. + \frac{\partial}{\partial\eta}\left(A_\eta\sqrt{(\xi^2-\eta^2)(1-\eta^2)}\right) + \frac{\partial}{\partial\varphi}\left(A_\varphi\frac{\xi^2-\eta^2}{\sqrt{(1-\eta^2)(\xi^2-1)}}\right)\right], \qquad (1.5.45)$$

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$$\Delta\psi = \frac{1}{f^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\psi}{\partial\xi}\right] + \right. \\ \left. + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right]\right\} + \frac{1}{f^2(\xi^2-1)(1-\eta^2)}\cdot\frac{\partial^2\psi}{\partial\varphi^2}, \qquad (1.5.46)$$

$$\operatorname{rot}\mathbf{A} = \frac{1}{f(\xi^2-\eta^2)}\times \\ \times\begin{vmatrix} \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}\,\mathbf{e}_\xi & \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}\,\mathbf{e}_\eta & \sqrt{(1-\eta^2)(\xi^2-1)}\,\mathbf{e}_\varphi \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} & \frac{\partial}{\partial\varphi} \\ \sqrt{\frac{\xi^2-\eta^2}{\xi^2-1}}\,A_\xi & \sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}\,A_\eta & \sqrt{(1-\eta^2)(\xi^2-1)}\,A_\varphi \end{vmatrix} \qquad (1.5.47)$$

At the limit, when the focal distance $f \to 0$, the spheroidal coordinate system changes to a spherical system of coordinates. For f = const and $\xi \to \infty$, the surface ξ = const in each case becomes a sphere. Here $f\xi \to r$, but $\eta \to \cos\theta$, where r and θ are spherical coordinates.

System of Oblate Spheroidal Coordinates (Fig. 11). The so-called variable quantities $q_1 = \xi$, $q_2 = \eta$, $q_3 = \phi$ are connected with the rectangular Cartesian coordinates x, y, z by the conversion formulas,

$$x = f\left[(\xi^2+1)(1-\eta^2)\right]^{1/2}\cos\varphi, \\ y = f\left[(\xi^2+1)(1-\eta^2)\right]^{1/2}\sin\varphi,\ z = f\xi\eta, \qquad (1.5.48$$

where $0 \leq \xi < \infty$, $-1 \leq \eta \leq 1$, $0 \leq \phi \leq 2\pi$. In an oblate spheroidal system of coordinates, the family of coordinate surfaces ξ = const represents a family of oblate spheroids with the major axis equal to $2f(\xi^2 + 1)^{1/2}$ and the minor axis equal to $2f\xi$ ($2f$ is the interfocal distance). The degenerate surface $\xi = 0$ is a circular disk of radius $r = f$, lying on the plane Oxy with the center at the origin. The family of coordinate surfaces $|\eta|$ = const < 1 is a

family of one-sheet hyperboloids of rotation with an asymptotic cone, whose generatrices pass through the origin and form an angle $\alpha = \arccos \eta$ with the z axis. The degenerate surface $|\eta| = 1$ is the z axis. The surface $\eta = 0$ is the plane Oxy, with the exception of the disk $\xi = 0$. The third family of coordinate surfaces is the planes $\phi = \text{const}$, which pass through the z axis at an angle ϕ to the plane Oxz. /42

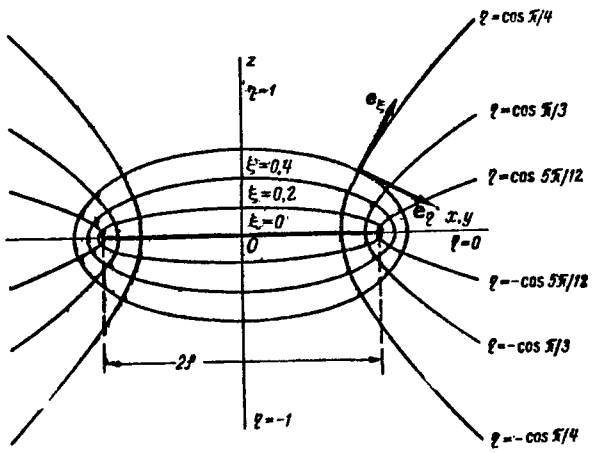

Fig. 11. System of Oblate Spheroidal Coordinates.

The Lame coefficients of the system are equal to:

$$h_\xi = f\sqrt{\frac{\xi^2+\eta^2}{\xi^2+1}}, \quad h_\eta = f\sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}}, \quad h_\varphi = f\sqrt{(\xi^2+1)(1-\eta^2)}, \qquad (1.5.49)$$

therefore, if e_ξ, e_η, e_ϕ are unit vectors of the system of oblate spheroidal coordinates, then according to formulas (1.5.10) - (1.5.15),

$$\left.\begin{aligned} \mathbf{e}_\xi &= \sqrt{\frac{1+\xi^2}{\xi^2+\eta^2}}\left\{\xi\sqrt{\frac{1-\eta^2}{1+\xi^2}}\cos\varphi\cdot\mathbf{i} + \right. \\ &\left. + \xi\sqrt{\frac{1-\eta^2}{\xi^2+1}}\sin\varphi\cdot\mathbf{j} + \eta\cdot\mathbf{k}\right\}, \\ \mathbf{e}_\eta &= \sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\left\{-\eta\sqrt{\frac{\xi^2+1}{1-\eta^2}}\cos\varphi\cdot\mathbf{i} - \right. \\ &\left. - \eta\sqrt{\frac{\xi^2+1}{1-\eta^2}}\sin\varphi\cdot\mathbf{j} + \xi\cdot\mathbf{k}\right\}, \\ \mathbf{e}_\varphi &= -\sin\varphi\cdot\mathbf{i} + \cos\varphi\cdot\mathbf{j}, \end{aligned}\right\} \qquad (1.5.50)$$

$$\left.\begin{aligned} A_\xi &= A_x\xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\cos\varphi + A_y\xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\sin\varphi + \\ &+ A_z\eta\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}, \\ A_\eta &= -A_x\eta\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\cos\varphi - A_y\eta\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\sin\varphi + \\ &+ A_z\xi\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}, \\ A_\varphi &= -A_x\sin\varphi + A_y\cos\varphi, \end{aligned}\right\} \qquad (1.5.51) \qquad /43$$

$$\operatorname{grad}\psi = \frac{1}{f}\sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}}\,\frac{\partial\psi}{\partial\xi}\,\mathbf{e}_\xi + \frac{1}{f}\sqrt{\frac{1-\eta^2}{\xi^2+\eta^2}}\,\frac{\partial\psi}{\partial\eta}\,\mathbf{e}_\eta + \\ + \frac{1}{f\sqrt{(1-\eta^2)(\xi^2+1)}}\cdot\frac{\partial\psi}{\partial\varphi}\,\mathbf{e}_\varphi, \qquad (1.5.52)$$

$$\operatorname{div}\mathbf{A} = \frac{1}{f(\xi^2+\eta^2)}\left[\frac{\partial}{\partial\xi}\left(A_\xi\sqrt{(\xi^2+\eta^2)(1+\xi^2)}\right) + \right. \\ + \frac{\partial}{\partial\eta}\left(A_\eta\sqrt{(\xi^2+\eta^2)(1-\eta^2)}\right) + \\ \left. + \frac{\partial}{\partial\varphi}\left(A_\varphi\frac{\xi^2+\eta^2}{\sqrt{(1-\eta^2)(1+\xi^2)}}\right)\right], \qquad (1.5.53)$$

$$\Delta\psi = \frac{1}{f^2(\xi^2+\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(1+\xi^2)\frac{\partial\psi}{\partial\xi}\right] + \right. \\ \left. + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\psi}{\partial\eta}\right]\right\} + \frac{1}{f^2(\xi^2+1)(1-\eta^2)}\cdot\frac{\partial^2\psi}{\partial\varphi^2}, \qquad (1.5.54)$$

$$\operatorname{rot}\mathbf{A} = \frac{1}{f(\xi^2+\eta^2)}\times \\ \times\begin{vmatrix} \sqrt{\frac{\xi^2+\eta^2}{1+\xi^2}}\,\mathbf{e}_\xi & \sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}}\,\mathbf{e}_\eta\sqrt{(1-\eta^2)(1+\xi^2)}\,\mathbf{e}_\varphi \\ \frac{\partial}{\partial\xi} & \frac{\partial}{\partial\eta} \quad \frac{\partial}{\partial\varphi} \\ \sqrt{\frac{\xi^2+\eta^2}{1+\xi^2}}\,A_\xi & \sqrt{\frac{\xi^2+\eta^2}{1-\eta^2}}\,A_\eta\sqrt{(1-\eta^2)(1+\xi^2)}\,A_\varphi \end{vmatrix} \qquad (1.5.55)$$

At the limit, when the focal distance $f \to 0$, the oblate spheroidal coordinates also convert to spherical coordinates. Here $f\xi \to r$, but $\eta \to \cos\theta$. If though f is finite, but $\xi \to \infty$, then we also have $f\xi \to r$, $\eta \to \cos\theta$, where r and θ are spherical coordinates. /44

Let us note that the substitution

$$\xi \to -i\xi,\ \eta \to \eta,\ f \to if \qquad (1.5.56)$$

formally transforms the system of oblate spheroidal coordinates (1.5.48) into a prolate spheroidal system of coordinates (1.5.40) and, on the contrary, the substitution

$$\xi \to i\xi,\ \eta \to \eta,\ f \to -if \qquad (1.5.57)$$

transforms the prolate spheroidal system of coordinates (1.5.40) into an oblate spheroidal system of coordinates (1.5.48).

On the basis of formulas (1.5.13), (1.5.15), the Maxwell equations (1.1.4) in the system of orthogonal curvilinear coordinates q_1, q_2, q_3 are written in the form

$$\frac{\partial}{\partial q_\alpha}(h_{q_\beta}E_{q_\beta})-\frac{\partial}{\partial q_\beta}(h_{q_\alpha}E_{q_\alpha})= =ik_0\mu\, h_{q_\alpha}h_{q_\beta}H_{q_\gamma}. \tag{1.5.58}$$

For the other equations we hence obtain a circular permutation of the indices $\alpha \neq \beta$, $\alpha \neq \gamma$, $\beta \neq \gamma$, $\alpha, \beta, \gamma = 1, 2, 3$. Analogously we have:

$$\frac{\partial}{\partial q_\alpha}(h_{q_\beta}H_{q_\beta})-\frac{\partial}{\partial q_\beta}(h_{q_\alpha}H_{q_\alpha})=\frac{k^2}{i\mu k_0}h_{q_\alpha}h_{q_\beta}E_{q_\gamma}+ +\frac{4\pi}{c}h_{q_\alpha}h_{q_\beta}J^{(\text{ст})}_{q_\gamma} \tag{1.5.59}$$

and

$$\frac{\partial}{\partial q_1}(h_{q_2}h_{q_3}H_{q_1})+\frac{\partial}{\partial q_2}(h_{q_1}h_{q_3}H_{q_2})+ +\frac{\partial}{\partial q_3}(h_{q_1}h_{q_2}H_{q_3})=0, \tag{1.5.60}$$

$$\frac{\partial}{\partial q_1}(h_{q_2}h_{q_3}E_{q_1})+\frac{\partial}{\partial q_2}(h_{q_1}h_{q_3}E_{q_2})+ +\frac{\partial}{\partial q_3}(h_{q_1}h_{q_2}E_{q_3})=\frac{4\pi}{\rho}h_{q_1}h_{q_2}h_{q_3}\rho. \tag{1.5.61}$$

Formula (1.5.14) permits writing with no effort the Helmholtz scalar equation,

$$\frac{\partial}{\partial q_1}\left(\frac{h_{q_2}h_{q_3}}{h_{q_1}}\cdot\frac{\partial\psi}{\partial q_1}\right)+\frac{\partial}{\partial q_2}\left(\frac{h_{q_1}h_{q_3}}{h_{q_2}}\cdot\frac{\partial\psi}{\partial q_2}\right)+ +\frac{\partial}{\partial q_3}\left(\frac{h_{q_1}h_{q_2}}{hq_3}\cdot\frac{\partial\psi}{\partial q_3}\right)+k^2h_{q_1}h_{q_2}h_{q_3}\psi=0. \tag{1.5.62}$$

In curvilinear coordinates,

$$\delta(\mathbf{r}-\mathbf{r}_0)=h_{q_1}^{-1}h_{q_2}^{-1}h_{q_3}^{-1}\delta(q_1-q_1^0)\delta(q_2-q_2^0)\delta(q_3-q_3^0). \tag{1.5.63}$$

§6. Separation of Maxwell Equations in Orthogonal Curvilinear Coordinates.

In theory, each problem of the diffraction of electromagnetic waves on barriers of any geometrical shape involves a solution of

Maxwell vector equations and is a vector problem. However, in many instances a vector problem can be reduced to solving scalar boundary problems by seeking independent scalar auxiliary functions that satisfy the Helmholtz scalar equation and individual boundary conditions on the surface of the diffracting body. Reduction of the vector problem to solving scalar problems that are independent of one another is usually called separation of the Maxwell equations.

Use of the method of separation of Maxwell equations in problems of the diffraction of an electromagnetic field permits substantially simplifying the process of finding the vectors E, H when the diffracting body possesses a surface that coincides with the complete coordinate surface of one of the orthogonal curvilinear systems of coordinates q_1, q_2, q_3, in which (in turn) the Helmholtz wave equation allows separation of the variables relative to q_1, q_2, q_3 (i.e., when the Helmholtz equation is written in the coordinates q_1, q_2, q_3, after substituting the functions $\psi(q_1, q_2, q_3) = Q_1(q_1)Q_2(q_2)Q_3(q_3)$ into it, i.e., solution to the equation is separated into three ordinary differential equations for $Q_i(q_i)$, $i = 1, 2, 3$).

Let us state the conditions imposed on the orthogonal curvilinear coordinate systems with satisfaction of which it becomes possible to separate the Maxwell equations in these coordinates.

In the region under study, let $J^{(ext)} = 0$, $\rho = 0$. Then, in the orthogonal curvilinear coordinate systems q_1, q_2, q_3, equations (1.5.58) and (1.5.59) take the form

$$\frac{\partial}{\partial q_\alpha}(h_{q_\beta}E_{q_\beta}) - \frac{\partial}{\partial q_\beta}(h_{q_\alpha}E_{q_\alpha}) = ah_{q_\alpha}h_{q_\beta}H_{q_\gamma}, \qquad (1.6.1)$$

$$\frac{\partial}{\partial q_\alpha}(h_{q_\beta}H_{q_\beta}) - \frac{\partial}{\partial q_\beta}(h_{q_\alpha}H_{q_\alpha}) = bh_{q_\alpha}h_{q_\beta}E_{q_\gamma}, \qquad (1.6.2)$$ /46

where, just as before, the indices α, β, γ denote any even permutation of the numbers 1, 2, 3 and where we set $a = ik_0\mu$, $b = k^2/i\mu k_0$. Now let us assume that in a certain system of orthogonal curvilinear coordinates q_1, q_2, q_3, the vector problem posed for equations (1.6.1), (1.6.2) allows a solution of the form,

$$E_{q_l} \neq 0, \; H_{q_l} = 0 \qquad (1.6.3)$$

or of the form,

$$E_{q_l} = 0, \; H_{q_l} \neq 0, \qquad (1.6.4)$$

where the index q_l (l is any of the numbers 1, 2, 3) denotes that the component of the field vector is taken along the coordinate line q_l. For example, let there be the case (1.6.3). Then, after substituting $\gamma = 1$ into (1.6.1), we have

$$\frac{\partial}{\partial q_j}(h_{q_к}E_{q_к}) = \frac{\partial}{\partial q_к}(h_{q_j}E_{q_j}), \tag{1.6.5}$$

where the indices j, K along with l are now assumed to fixed. It is obvious that relationship (1.6.5) will be identically satisfied if we write E_{q_j} and E_{q_K} in the form

$$E_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial u^*}{\partial q_j}, \quad E_{q_к} = \frac{1}{h_{q_к}} \cdot \frac{\partial u^*}{\partial q_к}, \tag{1.6.6}$$

where u^* is a given function. After substituting (1.6.6) into (1.6.2) when $\gamma = j$, K, taking into account that $H_{q_l} = 0$, we obtain the equations

$$\frac{\partial}{\partial q_l}(h_{q_к}H_{q_к}) = -b\frac{h_{q_к}h_{q_l}}{h_{q_j}} \cdot \frac{\partial u^*}{\partial q_j},$$
$$\frac{\partial}{\partial q_i}(h_{q_j}H_{q_j}) = b\frac{h_{q_l}h_{qj}}{h_{q_к}} \cdot \frac{\partial u^*}{\partial q_к}, \tag{1.6.7}$$

which will be identically satisfied if we set

$$u^* = \frac{\partial u}{\partial q_l}, \quad H_{q_к} = -\frac{b}{h_{q_j}} \cdot \frac{\partial u}{\partial q_j},$$
$$H_{q_j} = \frac{b}{h_{q_к}} \cdot \frac{\partial u}{\partial q_к}, \tag{1.6.8}$$

where u is a certain function and if we assume that the Lame coefficients of the orthogonal curvilinear system of coordinates q_1, q_2, q_3 satisfy the conditions /4

$$h_{q_l} = 1, \quad \frac{\partial}{\partial q_l}\left(\frac{h_{q_j}}{h_{q_к}}\right) = 0. \tag{1.6.9}$$

After substituting the expressions in (1.6.8) for the components H_{q_K}, H_{q_j} in those equations of (1.6.2) for which $\gamma = l$, we obtain

$$E_{q_l} = -\frac{1}{h_{q_j} h_{q_\kappa}}\left[\frac{\partial}{\partial q_j}\left(\frac{h_{q_\kappa}}{h_{q_j}} \cdot \frac{\partial u}{\partial q_j}\right) + \frac{\partial}{\partial q_\kappa}\left(\frac{h_{q_j}}{h_{q_\kappa}} \cdot \frac{\partial u}{\partial q_\kappa}\right)\right], \tag{1.6.10}$$

and from (1.6.6), on the basis of (1.6.8), we find

$$E_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial^2 u}{\partial q_j \partial q_l}, \quad E_{q_\kappa} = \frac{1}{h_{q_\kappa}} \cdot \frac{\partial^2 u}{\partial q_\kappa \partial q_l}. \tag{1.6.11}$$

As a result, all the components of the electromagnetic field E, H were found to be expressed through one scalar function u. If we substitute these expressions of the components into the two still unused equations from (1.6.1) for which $\gamma = j$, K, we find that they will be identically satisfied if u is the solution to the differential equation

$$\frac{1}{h_{q_j} h_{q_\kappa}}\left[\frac{\partial}{\partial q_j}\left(\frac{h_{q_\kappa}}{h_{q_j}} \cdot \frac{\partial u}{\partial q_j}\right) + \frac{\partial}{\partial q_\kappa}\left(\frac{h_{q_j}}{h_{q_\kappa}} \cdot \frac{\partial u}{\partial q_k}\right)\right] + \frac{\partial^2 u}{\partial q_l^2} + k^2 u = 0, \tag{1.6.12}$$

where the parameter k is determined as in (1.2.3)

Thus, by proper choice of the auxiliary scalar function u, we can construct the vector E, H through it; solution of the homogeneous system of Maxwell equations (1.1.4) whose components, taking (1.6.12) into account, are associated with the function u by the relationships

$$E_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial^2 u}{\partial q_j \partial q_l}, \quad H_{q_j} = \frac{k^2}{ik_0\mu} \cdot \frac{1}{h_{q_\kappa}} \cdot \frac{\partial u}{\partial q_\kappa},$$

$$E_{q_\kappa} = \frac{1}{h_{q_\kappa}} \cdot \frac{\partial^2 u}{\partial q_\kappa \partial q_l}, \tag{1.6.13}$$

$$H_{q_\kappa} = -\frac{k^2}{ik_0\mu} \cdot \frac{1}{h_{q_j}} \cdot \frac{\partial u}{\partial q_j},$$

$$E_{q_l} = \frac{\partial^2 u}{\partial q_l^2} + k^2 u, \quad H_{q_l} = 0.$$

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In the case of (1.6.4) we can find analogously, for the components of the vectors E, H, the relationships

$$E_{q_j} = \frac{ik_0\mu}{h_{q_k}} \cdot \frac{\partial v}{\partial q_k}, \qquad H_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial^2 v}{\partial q_j \partial q_l},$$

$$E_{q_k} = -\frac{ik_0\mu}{h_{q_j}} \cdot \frac{\partial v}{\partial q_j}, \qquad H_{q_k} = \frac{1}{h_{q_k}} \cdot \frac{\partial^2 v}{\partial q_k \partial q_l}, \qquad (1.6.14)$$

$$E_{q_l} = 0, \qquad H_{q_l} = \frac{\partial^2 v}{\partial q_l^2} + k^2 v,$$

where v is an auxiliary scalar function which satisfies equation (1.6.12).

Consequently, in those cases when the vector problem for the homogeneous system of Maxwell equations (1.1.4) allows posing the problem in the form of a superposition of two partial fields such as (1.6.3) and (1.6.4), for determination of each of them it is sufficient to find one scalar auxiliary function which satisfies equation (1.6.12) and the boundary conditions which guarantee satisfaction of the boundary conditions for the vectors E, H. After this the vectors E, H are found from the relationships in (1.6.13) and (1.6.14). The components of the complete field are found as the sums of the respective components of the partial fields and are determined through u and v by the formulas

$$E_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial^2 u}{\partial q_j \partial q_l} + \frac{ik_0\mu}{h_{q_k}} \cdot \frac{\partial v}{\partial q_k},$$

$$E_{q_k} = \frac{1}{h_{q_k}} \cdot \frac{\partial^2 u}{\partial q_k \partial q_l} - \frac{ik_0\mu}{h_{q_j}} \cdot \frac{\partial v}{\partial q_j},$$

$$E_{q_l} = \frac{\partial^2 u}{\partial q_l^2} + k^2 u,$$

$$H_{q_j} = \frac{1}{h_{q_j}} \cdot \frac{\partial^2 v}{\partial q_j \partial q_l} + \frac{k^2}{ik_0\mu} \cdot \frac{1}{h_{q_k}} \cdot \frac{\partial u}{\partial q_k}, \qquad (1.6.15)$$

$$H_{q_k} = \frac{1}{h_{q_k}} \cdot \frac{\partial^2 v}{\partial q_k \partial q_l} - \frac{k^2}{ik_0\mu} \cdot \frac{1}{h_{q_j}} \cdot \frac{\partial u}{\partial q_j},$$

$$H_{q_l} = \frac{\partial^2 v}{\partial q_l^2} + k^2 v.$$

Usually an electromagnetic field such as (1.6.3) is called a field of electrical type, and a field such as (1.6.4) a magnetic type field. A necessary condition for their existence is orthogonal curvilinear coordinate systems q_1, q_2, q_3 is the feasibility of condition (1.6.9) [48, 187]. Condition (1.6.9) is satisfied by the cylindrical coordinate systems (Cartesian rectangular, polar, elliptical, parabolic, etc., coordinates on the plane z = const in conjunction with the Cartesian coordinate z, figured in the direction

normal to this plane) and by a spherical system of coordinates.

Since $h_\rho = 1$, $h_\phi = \rho$, $h_z = 1$ in the coordinates of a circular cylinder, then condition (1.6.9) is satisfied relative to the coordinate z. Therefore, in the case of an electrical type field, when $E_z \neq 0$, but $H_z = 0$ (in cylindrical coordinates such a field is called a transverse magnetic field or a TM-type field), according to formulas (1.6.13) we have

$$\begin{aligned} E_\rho &= \frac{\partial^2 u}{\partial \rho \partial z}, & H_\rho &= \frac{k^2}{ik_0\mu} \cdot \frac{1}{\rho} \cdot \frac{\partial u}{\partial \varphi}, \\ E_\varphi &= \frac{1}{\rho} \cdot \frac{\partial^2 u}{\partial \varphi \partial z}, & H_\varphi &= -\frac{k^2}{ik_0\mu} \cdot \frac{\partial u}{\partial \rho}, \\ E_z &= \frac{\partial^2 u}{\partial z^2} + k^2 u, & H_z &= 0, \end{aligned} \qquad (1.6.16)$$

where the function u is the solution to the equation,

$$\frac{1}{\rho}\left[\frac{\partial}{\partial \rho}\left(\rho \frac{\partial u}{\partial \rho}\right) + \frac{1}{\rho} \cdot \frac{\partial^2 u}{\partial \varphi^2}\right] + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0, \qquad (1.6.17)$$

which is the Helmholtz equation, written in cylindrical coordinates.

In the case of a magnetic type field, when $E_z = 0$, $H_z \neq 0$ (in cylindrical coordinates such a field is called a transverse electrical field or a TE-type field), we have, on the basis of (1.6.14),

$$\begin{aligned} E_\rho &= \frac{ik_0\mu}{\rho} \cdot \frac{\partial v}{\partial \varphi}, & H_\rho &= \frac{\partial^2 v}{\partial \rho \partial z}, \\ E_\varphi &= -ik_0\mu \frac{\partial v}{\partial \rho}, & H_\varphi &= \frac{1}{} \frac{\partial^2 v}{\partial \varphi \partial z}, \\ E_z &= 0, & H_z &= \frac{\partial^2 v}{\partial z^2} + k^2 v, \end{aligned} \qquad (1.6.18) \qquad /50$$

where v is the solution to equation (1.6.17).

In the coordinates of an elliptical cylinder $h_\xi = h_\eta = f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}$, but $h_z = 1$, and condition (1.6.9) is also satisfied relative to the coordinate z. Therefore, in the case of a TM-type field when $E_z \neq 0$, $H_z = 0$, according to formulas (1.6.13), we obtain

$$E_\xi = \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial^2 u}{\partial \xi \partial z},$$

$$H_\xi = \frac{k^2}{ik_0\mu} \cdot \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial u}{\partial \eta},$$

$$E_\eta = \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial^2 u}{\partial\eta\,\partial z},$$
$$H_\eta = -\frac{k^2}{ik_0\mu} \cdot \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial u}{\partial\xi}, \qquad (1.6.19)$$
$$E_z = \frac{\partial^2 u}{\partial z^2} + k^2 u, \quad H_z = 0,$$

and in the case of a TE-type field, when $E_z = 0$, but $H_z \neq 0$, from (1.6.14) we find

$$E_\xi = \frac{ik_0\mu}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial v}{\partial\eta},$$
$$H_\xi = \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial^2 v}{\partial\xi\,\partial z},$$
$$E_\eta = -\frac{ik_0\mu}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial v}{\partial\xi}, \qquad (1.6.20)$$
$$H_\eta = \frac{1}{f\sqrt{\mathrm{ch}^2\xi - \cos^2\eta}} \cdot \frac{\partial^2 v}{\partial\eta\,\partial z},$$
$$E_z = 0, \quad H_z = \frac{\partial^2 v}{\partial z^2} + k^2 v,$$

where u and v are solutions to the equation, /5

$$\frac{1}{f^2(\mathrm{ch}^2\xi - \cos^2\eta)}\left[\frac{\partial^2 u}{\partial\xi^2} + \frac{\partial^2 u}{\partial\eta^2}\right] + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0 \qquad (1.6.21)$$

i.e., the Helmholtz equations, written in elliptical coordinates.

In spherical coordinates, $h_r = 1$, $h_\theta = r$, $h_\phi = r\sin\theta$ and condition (1.6.9) is satisfied relative to the coordinate r. Therefore, for an electrical type field, when $E_r \neq 0$, $H_r = 0$, according to (1.6.13), we find:

$$E_r = \frac{\partial^2 u}{\partial r^2} + k^2 u, \qquad H_r = 0,$$
$$E_\theta = \frac{1}{r} \cdot \frac{\partial^2 u}{\partial\theta\,\partial r},$$
$$H_\theta = \frac{k^2}{ik_0\mu} \cdot \frac{1}{r\sin\theta} \cdot \frac{\partial u}{\partial\varphi}, \qquad (1.6.22)$$
$$E_\varphi = \frac{1}{r\sin\theta} \cdot \frac{\partial^2 u}{\partial\varphi\,\partial r},$$
$$H_\varphi = -\frac{k^2}{ik_0\mu} \cdot \frac{1}{r} \cdot \frac{\partial u}{\partial\theta},$$

and in the case of a magnetic type field, when $E_r = 0$, $H_r \neq 0$, from (1.6.14), we obtain

$$E_r = 0, \qquad H_r = \frac{\partial^2 v}{\partial r^2} + k^2 v,$$

$$E_\theta = \frac{ik_0\mu}{r\sin\theta} \cdot \frac{\partial v}{\partial \varphi}, \qquad H_\theta = \frac{1}{r} \cdot \frac{\partial^2 v}{\partial \theta \partial r}, \tag{1.6.23}$$

$$E_\varphi = -\frac{ik_0\mu}{r} \cdot \frac{\partial v}{\partial \theta}, \qquad H_\varphi = \frac{1}{r\sin\theta} \cdot \frac{\partial^2 v}{\partial \varphi \partial r},$$

where u and v are solutions to the equation,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2\sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta} \cdot \frac{\partial^2 u}{\partial \varphi^2} + k^2 u = 0, \tag{1.6.24}$$

which, after substitution of /52

$$u = r\bar{u}, \quad v = r\bar{v} \tag{1.6.25}$$

is reduced for $\bar{u}$ and $\bar{v}$ to the Helmholtz equation,

$$\frac{\partial^2 \bar{u}}{\partial r^2} + \frac{2}{r} \cdot \frac{\partial \bar{u}}{\partial r} + \frac{1}{r^2\sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \bar{u}}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta} \cdot \frac{\partial^2 \bar{u}}{\partial \varphi^2} + k^2 \bar{u} = 0, \tag{1.6.26}$$

written in spherical coordinates.

Let us note that the only case when, in the orthogonal curvilinear system of coordinates q_1, q_2, q_3 that differs from the cylindrical and spherical systems, the field vectors E, H may be expressed through two scalar functions, is that case in which the Lame coefficients as well as the vectors E, H are independent of one of the curvilinear coordinates, for example, of q_l. Then

$$E_{q_j} = \frac{ik_0\mu}{k^2 h_{q_k} h_{q_l}} \cdot \frac{\partial u}{\partial q_k},$$

$$H_{q_j} = \frac{1}{ik_0\mu} \cdot \frac{1}{h_{q_k} h_{q_l}} \cdot \frac{\partial v}{\partial q_k},$$

$$E_{q_k} = -\frac{ik_0\mu}{k^2 h_{q_j} h_{q_l}} \cdot \frac{\partial u}{\partial q_j},$$

$$H_{q_\kappa} = -\frac{1}{ik_0\mu} \cdot \frac{1}{h_{q_j}h_{q_l}} \cdot \frac{\partial v}{\partial q_j},$$
$$E_{q_l} = \frac{1}{h_{q_l}} v, \quad H_{q_l} = \frac{1}{h_{q_l}} u, \tag{1.6.27}$$

where u and v are functions which satisfy the equation,

$$\frac{\partial}{\partial q_j}\left(\frac{h_{q_\kappa}}{h_{q_j}h_{q_l}} \cdot \frac{\partial u}{\partial q_j}\right) + \frac{\partial}{\partial q_\kappa}\left(\frac{h_{q_j}}{h_{q_\kappa}h_{q_l}} \cdot \frac{\partial u}{\partial q_\kappa}\right) + \\ + k^2 \frac{h_{q_j}h_{q_\kappa}}{h_{q_l}} u = 0. \tag{1.6.28}$$

If we use formulas (1.2.13) and (1.2.17) we can show that in cylindrical coordinates the components of the electromagnetic field /5
E, H may be expressed by relationships (1.6.13) or (1.6.14) through the only non-zero z-component of the electrical $\vec{\Pi}$ or magnetic $\vec{\Pi}^*$ Hertz vector, respectively, if we assume the Hertz vector to be directed parallel to the z axis of the given cylindrical coordinate system. Here in (1.6.13) we find $u = \Pi_z$, and in (1.6.14), $v = \Pi_z^*$. In this respect, the auxiliary functions u and v are often called Hertz scalar potential functions.

In exactly the same way, if we introduce into analysis (in the spherical system of coordinates) the Hertz vector $\vec{\Pi}$ or the Hertz magnetic vector $\vec{\Pi}^*$ with only the radial component $\Pi_r(\Pi_r^*)$ being non-zero, and if we use relationships (1.2.13) and (1.2.17) we can show that the components of the vectors E, H will be expressed by formulas (1.6.22) or (1.6.23) through Π_r or Π_r^*, respectively, i.e., that $u = \Pi_r$, $v = \Pi_r$ and that Π_r, Π_r^* satisfy equation (1.6.24). The functions $\overline{u}$, $\overline{v}$, related to u and v by the formulas in (1.6.25) are Debye potentials. The functions u, v from (1.6.27) are called Abraham potentials.

§7. Boundary Conditions for Scalar Potential Functions.

Let us establish the type of boundary conditions which must be satisfied on the surface of a diffracting body by the scalar potential functions introduced in §6. To all the values associated with the primary electromagnetic field E^0, H^0 and the secondary fields E^1, H^1 (outside the diffracting body), E^2, H^2 (inside the diffracting body) we can conventionally assign the indices 0, 1, 2, respectively (in the region outside the diffracting body the physical parameters ε, μ, σ are just the same as for the field E^0, H^0 and for the field E^1, H^1, with respect to which, in this region, they will be written without superscripts. Just as before, it is assumed that $k_0 = \omega/c$).

A Circular Infinitely Long Cylinder with Radius $\rho = a$.

1. The case $\sigma_2 = \infty$. The primary field is a TM-type field ($H_z^0 = 0$) and it is assumed that the field E^i, H^i ($i = 0, 1, 2$) is independent of the coordinate z ($\frac{\partial}{\partial z} \equiv 0$, the axis Oz of the cylindrical coordinate system ρ, ϕ, z is superposed with the axis of the cylinder). Since the tangential components of the vectors E and H are their ϕ- and z-components, then from (1.6.16) we can establish that the conditions in (1.1.5) will be satisfied if we require that, /54 on the surface of the cylinder, the function u satisfy the conditions,

$$\left.\begin{aligned} k^2 u &= k_2^2 u^{(2)}, \\ \frac{k^2}{\mu} \cdot \frac{\partial u}{\partial \rho} &= \frac{k_2^2}{\mu_2} \cdot \frac{\partial u^{(2)}}{\partial \rho}, \end{aligned}\right\} \quad \rho = a, \qquad (1.7.1)$$

where $u = u^{(0)} + u^{(1)}$ (the function $u^{(0)}$ is determined by the primary field E^0, H^0).

2. The case $\sigma_2 \neq \infty$. The primary field is a TE-type field ($E_z^0 = 0$) and it is assumed that the field E^i, H^i ($i = 0, 1, 2$) is independent of the coordinate z ($\frac{\partial}{\partial z} \equiv 0$, the coordinate system is the previous one). On the basis of (1.6.18) let us establish that the conditions in (1.1.5) will be satisfied if we require of the function v^i that on the surface of the cylinder it satisfy the conjugate conditions (1.7.1)[6], where $v = v^{(0)} + v^{(1)}$ (the function v^0 is determined by the primary field).

Conditions (1.7.1) for the functions v and u occur, for example, in the problem of the diffraction of a plane electromagnetic wave (1.3.4) on an infinitely long circular cylinder ($\sigma_2 \neq \infty$) under the condition that the vector n, which determines the direction of propagation of the wave front, is normal to the generatrix of the cylinder and that the electromagnetic properties of the cylinder do not vary along its axis.

3. The case $\sigma_2 \neq \infty$. The primary field is a TM-type field ($H_z^0 = 0$). The field E^i, H^i ($i = 0, 1, 2$) depends on the coordinate ($\frac{\partial}{\partial z} \neq 0$). The coordinate system is the previous one. It is easy to see that in such case it is impossible to satisfy the boundary conditions in (1.1.5) on the surface of a cylinder if we assume that the longitudinal component H_z^i of the vector H^i ($i = 1, 2$) of the secondary field is identically equal to zero. Therefore, the

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[6] In this case, in the second of the relationships in (1.7.1), k^2/μ and k_2^2/μ_2 are substituted, respectively, for μ and μ_2.

secondary fields (and consequently the complete field outside the cylinder) must be sought in the form of a superposition both of the TM-type field and of the TE-type, in spite of the fact that the primary field is a field of only one type. On the basis of (1.6.15), written in cylindrical coordinates, we can establish that the conditions in (1.1.5) on the surface of a cylinder will be satisfied if we require of the functions u and v, on the surface of the cylinder when $\rho = a$ that they satisfy the relationships.

$$\frac{1}{\rho}\cdot\frac{\partial^2 u}{\partial\varphi\,\partial z}-ik_0\mu\frac{\partial v}{\partial\rho}=\frac{1}{\rho}\cdot\frac{\partial^2 u^{(2)}}{\partial\varphi\,\partial z}-ik_0\mu_2\frac{\partial v^{(2)}}{\partial\rho},$$

$$\frac{\partial^2 u}{\partial z^2}+k^2u=\frac{\partial^2 u^{(2)}}{\partial z^2}+k_2^2u^{(2)},$$

$$\frac{1}{\rho}\cdot\frac{\partial^2 v}{\partial\varphi\,\partial z}-\frac{k^2}{ik_0\mu}\cdot\frac{\partial u}{\partial\rho}=\frac{1}{\rho}\cdot\frac{\partial^2 v^{(2)}}{\partial\varphi\,\partial z}- \tag{1.7.2}$$

$$-\frac{k_2^2}{ik_0\mu_2}\cdot\frac{\partial u^{(2)}}{\partial\rho},$$

$$\frac{\partial^2 v}{\partial z^2}+k^2v=\frac{\partial^2 v^{(2)}}{\partial z^2}+k_2^2v^{(2)},$$

where $u = u^{(0)} + u^{(1)}$ (the function $u^{(0)}$ is determined by the primary field E^0, H^0), but $v = v^{(1)}$ (for the primary field, $v^{(0)} \equiv 0$).

4. The case $\sigma_2 \neq \infty$, when the primary field is a TE-type field and the field E^i, H^i ($i = 1, 2$) depends on the coordinate z, similar to the above. For satisfaction of conditions (1.1.5) it follows to require of u and v that on the surface of the cylinder they satisfy the conditions in (1.7.2) where we now have $v = v^{(0)} + v^{(1)}$ (the function $v^{(0)}$ is determined by the primary field E^0, H^0), but $u = u^{(1)}$ (for the primary field $u^{(0)} \equiv 0$).

In particular, as an example of the problem leading to the conditions in (1.7.2), for the functions u, v we can use the problem of diffraction of an obliquely incident plane wave (1.3.4) on an infinitely long circular dielectric cylinder.

5. The case $\sigma_2 = \infty$, when the primary field is a type TM field (the coordinate system is the previous one). In this case the boundary conditions (1.1.6) on the surface of a cylinder will be satisfied (with an arbitrary dependence of the fields on the coordinate z), if we seek the secondary field in the same way as a field of only one TM-type, and if we assume (on the basis of (1.1.16) that

$$u = 0 \quad (\rho = a), \tag{1.7.3}$$

where $u = u^{(0)} + u^{(1)}$ (the function $u^{(0)}$ is determined by the primary field E^0, H^0).

6. The case $\sigma_2 = \infty$, when the primary field is a TE-type field, also permits satisfaction of the boundary conditions (1.1.6) (with an arbitrary dependence of fields on the coordinate z), if we seek the secondary field in the form of a TE-type field and if we assume (on the basis of (1.6.18)) that

$$\frac{\partial v}{\partial \rho} = 0 \quad (\rho = a), \tag{1.7.4}$$

where $v = v^{(0)} + v^{(1)}$, but the function $v^{(0)}$ is determined by the primary field.

As an example of problems leading to conditions (1.7.3), (1.7.4), we can use the problems on the diffraction of an electromagnetic field on an ideally conducting infinitely long circular cylinder. /56

An Elliptical Infinitely Long Cylinder. We shall assume that the lateral surface of the cylinder is given by the equation $\xi = \xi^0$ and that the physical properties of the cylinder do not vary along its axis. An elliptical cylindrical system of coordinates is introduced such that its axis Oz coincides with the axis of the cylinder. The tangential components of the vectors E, H are their η- and ϕ-components. Since cases 1 - 6, pertaining to a circular cylinder, are also transferred to an elliptical cylinder, let us cite only the form of the boundary conditions for the functions u, v, assuming that the number of instances is assumed to be a constant of the above situations.

1. Conditions (1.1.5) will be satisfied if we require that on the surface of the cylinder the function u satisfy the conjugate conditions,

$$\left.\begin{aligned} k^2 u &= k_2^2 u^{(2)}, \\ \frac{k^2}{\mu}\cdot\frac{\partial u}{\partial \xi} &= \frac{k_2^2}{\mu_2}\cdot\frac{\partial u^{(2)}}{\partial \xi}, \end{aligned}\right\} \quad \xi = \xi^0, \tag{1.7.5}$$

where $u = u^{(0)} + u^{(1)}$ ($u^{(0)}$ is determined by the primary field).

2. Conditions (1.1.5) are satisfied if on the surface of the cylinder the function v satisfies relationship (1.7.5), where $v = v^{(0)} + v^{(1)}$ ($v^{(0)}$ is determined by the primary field and where in the second equation it follows to substitute k^2/μ and k_2^2/μ_2, respectively, for μ and μ_2.

3. Conditions (1.1.5) are satisfied if on the surface of the cylinder the functions u and v satisfy the relationships,

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial \eta \partial z} - ik_0 \mu \frac{\partial v}{\partial \xi} = \frac{\partial^2 u^{(2)}}{\partial \eta \partial z} - ik_0 \mu_2 \frac{\partial v^{(2)}}{\partial \xi},\\
&\frac{\partial^2 u}{\partial z^2} + k^2 u = \frac{\partial^2 u^{(2)}}{\partial z^2} + k^2 u^{(2)},\\
&\frac{\partial^2 v}{\partial \eta \partial z} - \frac{k^2}{ik_0 \mu} \cdot \frac{\partial u}{\partial \xi} = \frac{\partial^2 v^{(2)}}{\partial \eta \partial z} - \\
&\qquad - \frac{k_2^2}{ik_0 \mu_2} \cdot \frac{\partial u^{(2)}}{\partial \xi},\\
&\frac{\partial^2 v}{\partial z^2} + k^2 v = \frac{\partial^2 v^{(2)}}{\partial z^2} + k^2 v^{(2)},
\end{aligned}\right\} \xi = \xi^0, \qquad (1.7.6)$$

where $u = u^{(0)} + u^{(1)}$ (the function $u^{(0)}$ is determined by the primary field E^0, H^0), but $v = v^{(1)}$ (for the primary field, $v^{(0)} \equiv 0$).

4. This is analogous to case 3. In (1.7.6) we now have $v = v^{(0)} + v^{(1)}$, where the function $v^{(0)}$ is determined by the primary field E^0, H^0, but $u = u^{(1)}$ (the function $u^{(0)}$ for the primary field is identically equal to zero). /

5. Condition (1.1.6) is satisfied if on the surface of the cylinder, we set

$$u = 0 \quad (\xi = \xi^0), \qquad (1.7.7)$$

where $u = u^{(0)} + u^{(1)}$ ($u^{(0)}$ is determined by the primary field E^0, H^0).

6. Condition (1.1.6) is satisfied if on the surface of the cylinder, we set

$$\frac{\partial v}{\partial \xi} = 0 \quad (\xi = \xi^0), \qquad (1.7.8)$$

where $v = v^{(0)} + v^{(1)}$ ($v^{(0)}$ is determined by the primary field E^0, H^0).

A Spherically Shaped Body. We shall assume that the surface of a sphere is given by the equation $r = a$ (a system of spherical coordinates r, θ, ϕ is introduced such that its origin coincides with the center of the sphere). The tangential components of the vectors E, H are their θ- and ϕ-components.

1. The case $\sigma_2 \neq \infty$. By using formulas (1.6.15), (1.6.22), (1.6.23) we find that conditions (1.1.5) will be satisfied on the surface of a sphere if we require of the potential functions u and v

that when $r = a$ they satisfy the relationships

$$\begin{aligned}
&\frac{1}{r}\cdot\frac{\partial^2 u}{\partial\theta\,\partial r}+\frac{ik_0\mu}{r\sin\theta}\cdot\frac{\partial v}{\partial\varphi}=\\
&=\frac{1}{r}\cdot\frac{\partial^2 u^{(2)}}{\partial\theta\,\partial r}+\frac{ik_0\mu_2}{r\sin\theta}\cdot\frac{\partial v^{(2)}}{\partial\varphi},\\
&\frac{1}{r\sin\theta}\cdot\frac{\partial^2 u}{\partial\varphi\,\partial r}-\frac{ik_0\mu}{r}\cdot\frac{\partial v}{\partial\theta}=\\
&=\frac{1}{r\sin\theta}\cdot\frac{\partial^2 u^{(2)}}{\partial\varphi\,\partial r}-\frac{ik_0\mu_2}{r}\cdot\frac{\partial v^{(2)}}{\partial\theta},\\
&\frac{1}{r}\cdot\frac{\partial^2 v}{\partial\theta\,\partial r}+\frac{k^2}{ik_0\mu r\sin\theta}\cdot\frac{\partial u}{\partial\varphi}=\\
&=\frac{1}{r}\cdot\frac{\partial^2 v^{(2)}}{\partial\theta\,\partial r}+\frac{k_2^2}{ik_0\mu_2 r\sin\theta}\cdot\frac{\partial u^{(2)}}{\partial\varphi},\\
&\frac{1}{r\sin\theta}\cdot\frac{\partial^2 v}{\partial\varphi\,\partial r}-\frac{k^2}{ik_0\mu r}\cdot\frac{\partial u}{\partial\theta}=\\
&=\frac{1}{r\sin\theta}\cdot\frac{\partial^2 v^{(2)}}{\partial\varphi\,\partial r}-\frac{k_2^2}{ik_0\mu_2 r}\cdot\frac{\partial u^{(2)}}{\partial\theta}.
\end{aligned}\tag{1.7.9}$$

For this, it is sufficient to require that the Debye potentials connected with u and v by formulas (1.6.25), when $r = a$, satisfy the conditions /58

$$\left.\begin{aligned}
&\frac{k^2}{\mu}\,\bar{u}=\frac{k_2^2}{\mu_2}\,\bar{u}^{(2)},\\
&\frac{\partial(r\bar{u})}{\partial r}=\frac{\partial(r\bar{u}^{(2)})}{\partial r},
\end{aligned}\right\}\tag{1.7.10}$$

$$\left.\begin{aligned}
&\mu\bar{v}=\mu_2\bar{v}^{(2)},\\
&\frac{\partial(r\bar{v})}{\partial r}=\frac{\partial(rv^{(2)})}{\partial r},
\end{aligned}\right\}\tag{1.7.11}$$

where, just as previously, $\bar{u} = \bar{u}^{(0)} + \bar{u}^{(1)}$, $\bar{v} = \bar{v}^{(0)} + \bar{v}^{(1)}$ ($\bar{u}^{(0)}$, $\bar{v}^{(0)}$ are determined by the primary field E^0, H^0).

We can arrive at conditions (1.7.10) and (1.7.11) for the functions $\bar{u}$, $\bar{v}$, for example, in problems of the diffraction of electromagnetic waves on spheres, possessing finite conductivity.

2. The case $\sigma_2 = \infty$. By using formular (1.6.15), (1.6.22) and (1.6.23), we establish that condition (1.1.6) will be satisfied on the surface of a sphere if we require of $u = u^{(0)} + u^{(1)}$ and $v = v^{(0)} + v^{(1)}$ that, when $r = a$, they satisfy the equations,

$$\left.\begin{aligned} \frac{\partial u}{\partial r} &= 0, \\ v &= 0, \end{aligned}\right\} \qquad (1.7.12)$$

which, for the Debye potentials, can be rewritten in the form,

$$\left.\begin{aligned} &\frac{\partial(r\bar{u})}{\partial r} = 0, \quad \bar{u} = \bar{u}^{(0)} + \bar{u}^{(1)}, \\ &\bar{v} = 0, \quad \bar{v} = \bar{v}^{(0)} + \bar{v}^{(1)}. \end{aligned}\right\} \qquad (1.7.13)$$

The boundary conditions (1.7.12), (1.7.13) occur, in particular, in problems of the diffraction of electromagnetic waves on an ideally conducting sphere.

A Spheroidally Shaped Body. As the diffracting body let there be a prolate spheroid, whose surface is given by the equation $\xi = \xi^0$ (a system of prolate spheroidal coordinates ξ, η, ϕ is introduced such that its origin coincides with the center of the spheroid, and the surface of spheroid coincides with one of the coordinate surfaces of the system). The tangential components of the vectors E, H of the field are their η- and ϕ- components. As noted already, the spheroidal coordinates do not satisfy condition (1.6.9) and the /59
separation of Maxwell equations, on the strength of which in the general case in these coordinates it is impossible to introduce potential functions which would satisfy the Helmholtz equation and which would also satisfy the simple boundary conditions. However, in the particular case of a symmetrical excitation of the spheroid, the vectors E, H of the primary and secondary fields are independent of the asimuth coordinate ϕ and the components of the field may be expressed through the Abraham potentials u, v by formulas (1.6.27). Thus, if the spheroid is excited symmetrically by the field of the electrical unit dipole (1.3.9) located on the axis of rotation of the spheroid and possessing the moment p, oriented along the axis, then it is easy to see that if we use formulas (1.2.13) and (1.3.9), we find that $v \equiv 0$, and from (1.6.27),

$$\begin{aligned} E_\xi &= \frac{ik_0\mu(\xi^2-1)^{-1/2}}{k^2f^2(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial u}{\partial \eta}, \quad H_\xi = 0, \\ E_\eta &= \frac{-ik_0\mu(1-\eta^2)^{-1/2}}{k^2f^2(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial u}{\partial \xi}, \quad H_\eta = 0, \\ E_\varphi &= 0, \qquad H_\varphi = \frac{1}{f\sqrt{(\xi^2-1)(1-\eta^2)}}\, u. \end{aligned} \qquad (1.7.14)$$

Consequently, when $\sigma_2 \neq \infty$, the conditions in (1.1.5) will be satisfied on the surface of a spheroid, if we require that when $\xi = \xi^0$, the function u will satisfy the conjugate conditions ($u = u^{(0)} + u^{(1)}$)

$$\left.\begin{aligned} u &= u^{(2)}, \\ \frac{\mu}{k^2}\cdot\frac{\partial u}{\partial \xi} &= \frac{\mu_2}{k_2^2}\cdot\frac{\partial u^{(2)}}{\partial \xi}. \end{aligned}\right\} \qquad (1.7.15)$$

Let us note that in this case it is more convenient as the potential function to take rather than the Abraham potential u, the only non-zero ϕ- component of the vector H associated with u by the relationship $u = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}H_\phi$. After substituting this expression u into (1.7.15) when $\xi = \xi^0$, we

$$\left.\begin{aligned} H_\varphi &= H_\varphi^{(2)}, \\ \frac{\mu}{k^2}\cdot\frac{\partial}{\partial \xi}\left(\sqrt{\xi^2-1}\,H_\varphi\right) &= \frac{\mu_2}{k_2^2}\cdot\frac{\partial}{\partial \xi}\left(\sqrt{\xi^2-1}\,H_\varphi^{(2)}\right), \end{aligned}\right\} \qquad (1.7.16)$$

where, as follows from (1.6.28), the function H_ϕ is the solution to the equation

$$\begin{aligned} &\sqrt{\xi^2-1}\,\frac{\partial^2}{\partial \xi^2}\left(\sqrt{\xi^2-1}\,H_\varphi\right) + \\ &+\sqrt{1-\eta^2}\,\frac{\partial^2}{\partial \eta^2}\left(\sqrt{1-\eta^2}\,H_\varphi\right) + \\ &+k^2f^2(\xi^2-\eta^2)u = 0, \end{aligned} \qquad (1.7.17)$$

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which respresents the Helmholtz equation with an azimuth number m = 1.

In the case $\sigma_2 = \infty$ and with the previous position of the dipole, the relationships in (1.7.14) are retained. Now condition (1.1.6) will be satisfied if we require of the Abraham potential u that on the surface of the spheroid it satisfy the equation

$$\frac{\partial u}{\partial \xi} = 0 \quad (\xi = \xi^0), \qquad (1.7.18)$$

which for the function H_ϕ can be rewritten in the form

$$\frac{\partial}{\partial \xi}\left(\sqrt{\xi^2-1}\,H_\varphi\right) = 0 \quad (\xi = \xi^0). \qquad (1.7.19)$$

In both cases it was assumed that $u = u^{(0)} + u^{(1)}$, where the potential $u^{(0)}$ is determined by the primary field, and that $H_\phi = H_\phi^0 + H_\phi^1$. If the spheroid is symmetrically excited by the magnetic dipole, also located on the axis of the spheroid with the moment m oriented along the axis, then by using (1.2.17) and (1.3.9) we can show that $u \equiv 0$, and from (1.6.27) we find

$$E_\xi = 0, \quad H_\xi = \frac{1}{ik_0\mu} \cdot \frac{(\xi^2-1)^{-1/2}}{f^2(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial v}{\partial \eta},$$
$$E_\eta = 0, \; H_\eta = -\frac{1}{ik_0\mu} \cdot \frac{(1-\eta^2)^{-1/2}}{f^2(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial v}{\partial \xi}, \qquad (1.7.20)$$
$$E_\varphi = \frac{1}{f\sqrt{(\xi^2-1)(1-\eta^2)}}\, v, \; H_\varphi = 0.$$

Consequently, when $\sigma_2 \neq \infty$, conditions (1.1.5) will be satisfied if the function v satisfies on the surface of the spheroid the conjugate conditions,

$$\left.\begin{aligned} \frac{1}{\mu} \cdot \frac{\partial v}{\partial \xi} &= \frac{1}{\mu_2} \cdot \frac{\partial v^{(2)}}{\partial \xi}, \\ v &= v^{(2)}, \end{aligned}\right\} \quad \xi = \xi^0. \qquad (1.7.21)$$

Just as in the case of an electric dipole, as the potential function here it is more convenient to take the ϕ-component of the vector E, which is connected with the Abraham potential by the relationship $v = f\sqrt{(\xi^2 - 1)(1 - \eta^2)}\, E_\phi$, on the basis of which we have, from (1.7.21) when $\xi = \xi^0$,

$$\left.\begin{aligned} \frac{1}{\mu} \cdot \frac{\partial}{\partial \xi}(\sqrt{\xi^2-1}\, E_\varphi) &= \frac{1}{\mu_2} \cdot \frac{\partial}{\partial \xi}(\sqrt{\xi^2-1}\, E_\varphi^{(2)}), \\ E_\varphi &= E_\varphi^{(2)}, \end{aligned}\right\} \qquad (1.7.22)$$

where on the basis of (1.6.28) the function E_ϕ satisfies the Helmholtz equation (1.7.17).

In the case $\sigma_2 = \infty$ and with the previous position of the dipole we find that for satisfaction of condition (1.1.6) it is necessary that the function v satisfy, on the surface of the spheroid, the equation,

$$v = 0 \quad (\xi = \xi^0), \qquad (1.7.23)$$

which for the function E_ϕ can be rewritten in the form

$$E_\varphi = 0 \quad (\xi = \xi^0). \qquad (1.7.24)$$

Just as before, here it is assumed that $v = v^{(0)} + v^{(1)}$, where the potential $v^{(0)}$ is determined by the primary field, and that $E_\phi = E_\phi^{(0)} E_\phi^{(1)}$.

If the diffracting body is an oblate spheroid (with the surface $\xi = \xi^0$ in the system of oblate spheroidal coordinates ξ, η, ϕ,

associated with the spheroid), then in the same situations, as for the prolate spheroid, the boundary conditions for the Abraham potentials u, v or for the components H_ϕ, E_ϕ of the vectors E, H may be obtained from (1.7.14) - (1.7.24) by substitution there of ξ by $i\xi$ and f by if.

If we summarize the above we find that, in the cases of circular and elliptical cylinders, complete separation of the Maxwell equations (taking into account the boundary conditions also) in the cylindrical coordinates is possible only if the diffracting cylinder is an ideally conducting one ($\sigma_2 = \infty$) or when $\sigma_2 \neq \infty$, when the electromagnetic field is independent of the space coordinate z. In the case of a sphere, such separation of the Maxwell equations in the spherical coordinates is always possible.

In the past if we spoke of boundary conditions for the vectors E, H of an electromagnetic field it was assumed everywhere that the surfaces of the diffracting bodies are smooth without edges. The presence of edges on the surface of a diffracting body may lead to additional difficulties in solving the boundary problems. For example, at the corner point (such points are the points of the edge), the normal to the surface is determined singularly on the strength of which at this point, the normal derivative will be also uniquely /62
determined. Therefore, without giving additional conditions at the corner point, the ordinary homogeneous boundary condition containing the normal derivative will not uniquely determine the sought solution to the problem, i.e., the boundary problem for the system of Maxwell equations in this case may have, rather than one, a set of solutions which differ from one another by the character of the singularity on the edge. To guarantee uniqueness of the solution of boundary problems in the case of bodies bounded by surfaces with corner points (edges), the boundary conditions for the desired solutions must be formulated for all points of the surface, including the corner points.

Additional requirements have various mathematical terms at the corner points, between which usually there is no distinction. Any condition which pertains to the corner points (edges) guaranteeing uniqueness of the solution to the diffraction problem, is called the condition on the edge. From the physical point of view the condition on the edge consists either of the fact that the edge must not be radiated, i.e., the current of energy through the surface of the thin surrounding edge of the ring-shaped cylinder ρ at the limit when $\rho \to 0$ must tend toward zero, or of the fact that the density of energy $\overline{W}$ (1.4.4) is spatially integrable in the vicinity of the edge (the Meixner condition). Both conditions are equally strong. In the case of an edge of a thin ideally conducting screen (disk), it follows from them that the components of the vectors E, H, parallel to the edge (tangents to the lines of the edge) are always bounded (or equal to zero) and the components normal to the edge have on the edge a singularity on the order of $\rho^{-1/2}$. The distribution of surface currents on the screen (disk) is characterized by

the fact that, on the edge of the screen (disk), it can not have any non-zero normal component, whereas the component of the density of the surface currents parallel to the edge becomes infinite at the edge.

In mathematical terms the condition on the edge, in the case of an ideally conducting infinitely thin circular disk, can be written in the form

$$\frac{\partial \Pi_{\parallel}}{\partial \xi} = \frac{\partial \Pi_{\parallel}}{\partial \eta} = 0 \quad (\xi = \eta = 0), \tag{1.7.25}$$

where ξ, η are the coordinates of the oblate spheroid with origin at the center of the disk, by which degeneration a disk appears and where through $\Pi_{\parallel}$ the components of the Hertz vector $\vec{\Pi}$ are denoted which are parallel to the plane of the disk or in the form (the Meixner condition)

$$j_{\rho} = 0, \quad \rho = a, \quad z = 0, \tag{1.7.26}$$

where a is the radius of the disk lying in the plane $z = 0$; j_{ρ} is the radial component of the vector j (the density of the complete surface current, induced on the disk).

Let us note that at the remaining points of the disk the boundary conditions, for example, for the components of the Hertz vector are derived from the equations in (1.1.6) and formula (1.2.13) or (1.2.17). /6

§8. Boundary Problems in Diffraction Theory. Uniqueness of the Solution.

In an infinite space T_0 with electromagnetic constants ε_0, μ_0, σ_0, let there be propagated an electromagnetic wave E^0, H^0 (the primary field of the assigned currents and charges or the incident field). If there is a heterogeneous inclusion in the space in the form of a certain body T with electromagnetic constants ε_1, μ_1, σ_1, bounded by a surface S_1, then the resulting electromagnetic field E, H in the part of space $T_0 - T$, which is extraneous with respect to T, will differ from the field which would exist there with propagation of the wave in a homogeneous isotropic space, i.e., in free space. This can be explained by the fact that, with incidence of the electromagnetic field on a body, in this latter there occur induced oscillations in the free or bound charges, as a result of which inside the body in the space surrounding it a secondary electromagnetic field E, H occurs which varies in time at the frequency of the field of the incident wave (the secondary wave outside the region T is usually called a scattered, reflected or re-emitted wave). The vector sum of the incident and secondary fields also forms a resulting (complete field in the region $T_0 = T$. The

escribed phenomenon is called the diffraction of electromagnetic aves. Usually by diffraction, in the broader sense of the word, e mean a set of all phenomena which take place with propagation f a wave in a nonhomogeneous medium: reflection, scattering, refraction of waves, the bending of barriers by the waves, etc. Therefore, many wave phenomena can be studied as specific cases of diffraction. The problems associated with establishing the secondary resulting) field are called diffraction problems. These are the sual boundary problems of mathematical physics. In the most common ormulation, the diffraction problem consists of seeking complex ectors of the electrical and magnetic intensities, which satisfy he system of Maxwell equations (homogeneous, if in the region being tudied, there are no sources of the field, and nonhomogeneous when hey exist), given by the boundary conditions on the surface S_1:

$$E_t^0 + E_t^1 = E_t^2, \quad H_t^0 + H_t^1 = H_t^2, \tag{1.8.1}$$

hen $\sigma_1 \neq \infty$, or

$$E_t^0 + E_t^1 = 0, \tag{1.8.2}$$

hen $\sigma_1 = \infty$, and the condition of radiation at infinity: /64

$$\lim_{r\to\infty} r\left(\frac{\partial \mathbf{A}}{\partial r} - ik\,\mathbf{A}\right) = 0, \quad \lim_{r\to\infty} \mathbf{A} = 0. \tag{1.8.3}$$

ere A denotes the vectors E^1 and H^1; E_t, H_t are the tangential components of the vectors E, H; the index 1 is ascribed to the scattered ield, and the index 2 to the field inside the region T. In the ystem of Maxwell equations the quantities ε, μ, σ have values ε_0, $_0$, σ_0 in the region $T_0 - T$ and values ε_1, μ_1, σ_1 inside T. When a iffracting body T of sharp boundaries (edges) exists on the surface $_1$, formulation of the problem must be supplemented still by the ondition on the edge.

A diffracting body T may have any shape and may consist of any aterial characterized by the electromagnetic constants ε_1, μ_1, σ_1.

The uniqueness of the solution to the diffraction problem is stablished by the following theorem: the solution to the homogeneous ystem of Maxwell equations,

$$\begin{aligned} \operatorname{rot}\mathbf{H} &= \frac{k^2}{ik_0\mu}\,\mathbf{E}, \\ \operatorname{rot}\mathbf{E} &= ik_0\mu\,\mathbf{H}, \\ \operatorname{div}\mathbf{H} &= \operatorname{div}\mathbf{E} = 0, \end{aligned} \tag{1.8.4}$$

which satisfies the homogeneous boundary conditions (1.8.1) or (1.8.2) on the surface S_1 and the condition of radiation (1.8.3) at infinity is identically zero (in the presence of edges on S_1 to guarantee uniqueness of the solution, the problem is supplemented by the condition on the edge).

In the space T_0 if several, for example N, separately distributed diffracting bodies T_j exist with electromagnetic constants ε_j, μ_j, σ_j, bound by closed surfaces S_j, $j = 1, 2, \ldots, N$, the diffraction problem (diffraction problem in a multiply connected region) also consists in seeking the vectors E^1, H^1 of the secondary field, which satisfies, in each of the regions T_j ($j = 0, 1, \ldots, N$), the homogeneous system of Maxwell equations, the boundary conditions such as (1.8.1), (1.8.2) on the surfaces S_j and the condition of radiation at infinity (in the region T_0). In the presence of edges on the surfaces, the problem is supplemented by the condition on the edge. The formulated theorem of uniqueness of the solution is extended also to this case [57 - 59].

The diffraction problems are vector ones. However in the case when the surface of the diffracting body coincides with one of the complete coordinate surfaces of a certain orthogonal curvilinear system of coordinates, associated with a given body and permitting a complete separation of the Maxwell equations (including the boundary conditions as well), the vector boundary problem relative to E, H may be reduced, as noted above, to scalar boundary problems for the Helmholtz equation relative to the auxiliary potential functions u, v or the rectangular components of the Hertz vectors $\vec{\Pi}$, $\vec{\Pi}^*$, through which the vectors E, H are expressed by the formulas in §6 and (1.2.13), (1.2.17), respectively.

In the simplest case of one body the scalar boundary problem consists of seeking an ordinary solution to the equation,

$$\Delta\psi^j + k_j^2\psi^j = 0, \quad j = 1, 2 \tag{1.8.5}$$

(nonhomogeneous in the presence of field sources in the region under study), which satisfies the given boundary conditions on the surface S_1 and the supplementary condition at infinity, superposed on the function ψ^1, which determines the secondary field in the region $T_0 = T$. Here the index 1 is ascribed to the scattered field, and 2 to the field inside the region T. In the presence of edges on the surface S_1, the problem is supplemented by the condition on the edge.

The type of boundary conditions for those bodies, the diffraction on which is studied below, is given in §7.

In diffraction theory, the most important particular boundary problems are:

The Dirichlet, or first boundary problem, when

$$\Delta\psi + k^2\psi = 0,\ \psi \in T;\ \psi|_{S_1} = f(P),\ P \in S_1; \tag{1.8.6}$$

The Neumann, or second boundary problem, when

$$\Delta\psi + k^2\psi = 0,\ \psi \in T;\ \left.\frac{\partial\psi}{\partial n}\right|_{S_1} = f(P), \quad P \in S_1, \tag{1.8.7}$$

ıere n denotes differentiation over the direction of the outer nor-
ıl to S_1;

The third boundary problem, when

$$\begin{gathered}\Delta\psi + k^2\psi = 0,\ \psi \in T;\\ \left(\frac{\partial\psi}{\partial n} + \alpha\psi\right)\Big|_{S_1} = f(P),\ P \in S_1,\end{gathered} \tag{1.8.8}$$

ıere α is a continous function determined on the surface S_1. The
udy of diffraction on ideally conducting bodies leads to such
ypes of boundary problems. If a body T possesses a finite conduc-
vity, then on its surface several conjugate conditions, for ex-
ıple, (1.7.1), (1.7.10), etc., must be satisfied.

The uniqueness of solving the scalar boundary problems for the
ılmholtz equation may be established on the basis of the following
ıeorems:

The function ψ which, in the region $T_0 = T$, satisfies the Helm- /66
ıltz equation (1.8.5), and on the boundary S_1, satisfies one of
ıe homogeneous boundary conditions

$$\psi = 0 \quad \text{or} \quad \partial\psi/\partial n = 0$$

ı is the vector outside the normal to S_1), at infinity also satis-
ies the condition of radiation and is identically equal to zero
ı $T_0 - T$;

The function ψ, which in the region $T_0 - T$, satisfies the equa-
ion,

$$\Delta\psi^1 + k^2\psi^1 = 0,$$

in the region T satisfies the equation,

$$\Delta\psi^2 + k_1^2\psi^2 = 0,$$

on the boundary S_1 it satisfies the conjugate conditions,

$$\left.\begin{aligned} \psi^1 &= \psi^2, \\ \beta_1 \frac{\partial\psi^1}{\partial n} &= \beta_2 \frac{\partial\psi^2}{\partial n} \end{aligned}\right\}$$

and at infinity satisfies the condition of radiation and is identically equal to zero in all the space (here β_j are parameters which depend on the physical properties of the medium, β_j = const, $j = 1,2$).

Here it is assumed that the boundary S_1 is sufficiently smooth, without corner points. In the presence of edges on its surface, to guarantee uniqueness of the solution to the respective boundary problem, its formulation must be supplemented by the condition on the edge.

In the case of a multiply connected region, the basic boundary problems for the Helmholtz equation are formulated in a similar manner and the uniqueness theorems for the solution given above are generalized [2, 57 - 59] for them.

§9. Comments on a Method of Solving Boundary Problems. Infinite Systems of Linear Equations.

Various approaches are used, within the framework of the strict diffraction theory, for a strict solution to boundary problems, formulated for a Helmholtz wave equation in the presence of one or several scattering bodies.

In the case of bodies having a conventional shape with a sufficiently smooth surface (S), the solution to the boundary problems may be written in closed form, for example, in the form of integrals of closed functions, using the Green functions. The Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ depends on the position of two points: the observation point determined in a fixed coordinate system $Oxyz$ with the radius-vector $\mathbf{r}$ with coordinates (x, y, z) and the source point, in which is placed a unit field source determined in the same coordinates with the radius-vector $\mathbf{r}_0$ with coordinates (x_0, y_0, z_0). The Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ of the Helmholtz scalar wave equation,

$$\Delta u + k^2 u = 0, \qquad (1.9.1)$$

given in a certain region (V), bounded by a closed sufficiently smooth surface (S) is the solution (that is symmetrical relative to $\mathbf{r}$ and $\mathbf{r}_0$ and singular when $\mathbf{r} = \mathbf{r}_0$) to the Helmholtz equation,

$$\Delta G_k(\mathbf{r}|\mathbf{r}_0) + k^2 G_k(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r}-\mathbf{r}_0) \qquad (1.9.2)$$

(homogeneous everywhere, with the exception only of the point $\mathbf{r} = \mathbf{r}_0$), which satisfies the several given homogeneous boundary conditions on the surface (S).

Using the Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ we can construct a solution to the nonhomogeneous Helmholtz equation,

$$\Delta u + k^2 u = -\rho \qquad (1.9.3)$$

with the given homogeneous boundary conditions on (S) (in this case, the point $\mathbf{r} = \mathbf{r}_0$ is located in space outside (S)) or a solution to the homogeneous equation (1.9.1) with the given nonhomogeneous boundary conditions on (S) (in this case the point $\mathbf{r} = \mathbf{r}_0$ is located on the surface (S)). On the strength of the linearity of the Helmholtz equation, we can also construct a solution to the nonhomogeneous equation (1.9.3) with the given nonhomogeneous boundary conditions on (S) by superposing particular solutions.

To construct the particular solutions we use the formula,

$$\iint_{(S)} \left[G_k(\mathbf{r}|\mathbf{r}_0') \frac{\partial u(\mathbf{r}_0')}{\partial n} - u(\mathbf{r}_0') \frac{\partial G_k(\mathbf{r}|\mathbf{r}_0')}{\partial n} \right] d\mathbf{r}_0' + \iiint_{(V)} \rho(\mathbf{r}_0) G_k(\mathbf{r}|\mathbf{r}_0)\, d\mathbf{r}_0 = \begin{cases} u(\mathbf{r}), & \text{if } \mathbf{r} \text{ is inside and on } (S), \\ 0, & \text{if } \mathbf{r} \text{ is outside } (S), \end{cases} \qquad (1.9.4)$$

which is obtained from formulas (1.9.2) and (1.9.3) using the Green theorem and determining the Dirac δ-function (first (1.9.2) is multiplied by u, and (1.9.3) by G_k and the second result is subtracted from the first, then substitution by $\mathbf{r}$ and $\mathbf{r}_0$ and integration over all the coordinates of the source $\mathbf{r}_0$ is made, after which the Green theorem is used). Here $\mathbf{n}$ is the normal outside (S) and $\mathbf{r}_0'$ determines the points located on the surface (S). /68

If $u|_S = 0$, the Green function is then constructed in such a manner that $G_k|_S = 0$. As a result, in (1.9.4) the surface integral vanishes and the solution to the Dirichlet problem for the nonhomogeneous equation (1.9.3) is written in the form,

$$u(\mathbf{r}) = \iiint_{(V)} \rho(\mathbf{r}_0) G_k(\mathbf{r}|\mathbf{r}_0)\, d\mathbf{r}_0. \qquad (1.9.5)$$

Solution to the Neumann problem $\left(\frac{\partial u}{\partial n}\Big|_S = 0\right)$ for equation (1.9.3) is described by the same type of formula as (1.9.5), if the Green function is constructed in such a manner that it satisfies the condition $\frac{\partial G_k}{\partial n}\Big|_S = 0.$ on the surface (S).

The Dirichlet problem for the homogeneous equation (1.9.1), under the nonhomogeneous condition $u|_S = f$, has the solution,

$$u(\mathbf{r}) = -\iint\limits_{(S)} f(\mathbf{r}_0') \frac{\partial}{\partial n} G_k(\mathbf{r}|\mathbf{r}_0')\, dr_0' \qquad (1.9.6)$$

($\mathbf{r}$ is inside and on (S)). In this case the Green function is so constructed that $G_k|_S = 0$ on the surface (S).

The Neumann problem for equation (1.9.1) with the nonhomogeneous boundary condition $\frac{\partial u}{\partial n}\Big|_S = f$ has the solution,

$$u(\mathbf{r}) = \iint\limits_{(S)} f(\mathbf{r}_0')\, G_k(\mathbf{r}|\mathbf{r}_0')\, d\mathbf{r}_0' \qquad (1.9.7)$$

($\mathbf{r}$ is inside and on (S)), if the Green function is so constructed that $\frac{\partial}{\partial n} G_k|_S = 0.$ on the surface (S).

Solution to the third boundary problem for equation (1.9.1), with the nonhomogeneous boundary condition,

$$\left(\frac{\partial u}{\partial n} + \alpha u\right)\Big|_S = f,\ \alpha = \alpha(\mathbf{r}),\ \mathbf{r} = \mathbf{r}_0',$$

will have the form,

$$u(\mathbf{r}) = \begin{cases} \iint\limits_{(S)} \alpha(\mathbf{r}_0')\, G_k(\mathbf{r}|\mathbf{r}_0')\, d\mathbf{r}_0', \\ -\iint\limits_{(S)} \frac{f(\mathbf{r}_0)}{\alpha(\mathbf{r}_0')} \cdot \frac{\partial}{\partial n} G_k(\mathbf{r}|\mathbf{r}_0')\, d\mathbf{r}_0' \end{cases} \qquad (1.9.8)$$

(both expressions are equivalent), if the Green function is so constructed that it satisfies the condition /6

$$\left(\frac{\partial G_k}{\partial n} + \alpha G_k\right)\Big|_S = 0.$$

on the surface (S).

The surface (S) in (1.9.4) may consist of two closed surfaces: S' and S'', one of which (the outer) S' encompasses the other (inner) S''. The surface S' may also be infinite (in extent). In this case the region of integration (V) in (1.9.4) has an infinite volume; on S' the normal $\mathbf{n}$ is directed outside, and on S'' it is directed inside the region included inside S''

In the three-dimensional case, when the boundary of the region is located at infinity and a solution is sought which represents waves that travel from the source point, the Green function is used,

$$G_k(\mathbf{r}\,|\,\mathbf{r}_0) = \frac{e^{ikR}}{4\pi R}, \tag{1.9.9}$$

where

$$R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2},$$

and in the two-dimensional case,

$$G_k(\mathbf{r}\,|\,\mathbf{r}_0) = \frac{i}{4} H_0^{(1)}(kR), \tag{1.9.10}$$

where

$$R = |\mathbf{r} - \mathbf{r}_0| = \sqrt{(x-x_0)^2 + (y-y_0)^2},$$

if the time dependence is given by the time factor exp ($-i\omega t$). The functions (1.9.9) and (1.9.10) are usually called Green functions of free space. They correspond to spherical and cylindrical waves, traveling from the point $\mathbf{r} = \mathbf{r}_0$. Using the Green function of free space the solution to the Helmholtz homogeneous equation in the unbounded region outside the surface (S) of the scattering barrier, is given by the formula,

$$u(\mathbf{r}) = u^0(\mathbf{r}) + \oint_{(S)} \left\{ \psi(\mathbf{r}_0') G_k(\mathbf{r}\,|\,\mathbf{r}_0') - \varphi(\mathbf{r}_0') \frac{\partial}{\partial n} G_k(\mathbf{r}\,|\,\mathbf{r}_0') \right\} d\mathbf{r}_0', \tag{1.9.11}$$

where the differentiation is carried out over the direction of the outer normal to the surface (S). Here $u^0(\mathbf{r})$ is the potential of the primary field but $\phi(\mathbf{r}_0') = u(\mathbf{r}_0')$ and $\psi(\mathbf{r}_0' = \frac{\partial u(\mathbf{r}_0')}{\partial n}$ are the values $u(\mathbf{r})$ and its normal derivative on the surface (S).

Formula (1.9.11) allows graphic interpretation. It represents /7
a mathematical formulation of the Huygens theory of unit waves: the field $u(\mathbf{r})$ at each point of the region under study is represented in the form of a superposition of spherical waves (in the three-dimensional case) e^{ikR}/R with a distribution density of the sources ψ and the dipole waves $\partial(e^{ikR}/R)/\partial n$ with the density distribution ϕ, originating from the points of the surface (S) (from the elementary point of view, according to Huygens theory, the field in the given point through a certain moment of time is equal to the sum of the fields of each of the point sources, located on (S), and the envelope of these fields for all points is the wave front at a given time).

On the surface (S) if one of the conditions is satisfied,

$$u|_S = 0 \quad \text{or} \quad \left.\frac{\partial u}{\partial n}\right|_S = 0,$$

then the solution to the respective external boundary problem for equation (1.9.1) will be given by the formula,

$$u(\mathbf{r}) = u^0(\mathbf{r}) + \oint_{(S)} \psi(\mathbf{r}_0') G_k(\mathbf{r}\,|\,\mathbf{r}_0')\, d\mathbf{r}_0' \qquad (1.9.12)$$

in the first case or by

$$u(\mathbf{r}) = u^0(\mathbf{r}) - \oint_{(S)} \varphi(\mathbf{r}_0') \frac{\partial G_k(\mathbf{r}\,|\,\mathbf{r}_0')}{\partial n}\, d\mathbf{r}_0' \qquad (1.9.13)$$

in the second case.

In the electrodynamic sense the quantities ψ and ϕ are usually interpreted as functions of the density distribution of surface currents, induced on the surface (S). For example, for an infinitely long ideally conducting cylinder we can set $u(\mathbf{r}) = E_z(\mathbf{r})$, if the vector of the electrical intensity of the incident field is parallel to the axis of the cylinder ($\mathbf{E} = \{0, 0, E_z\}$). In this case it follows, from the second equation of the Maxwell system (1.1.4) and from the boundary condition (1.1.7) on the surface of the cylinder, that

$$\psi(\mathbf{r}_0') = \frac{\partial u(\mathbf{r}_0')}{\partial n} = \left.\frac{\partial E_z}{\partial \rho}\right|_S = -\frac{i\omega\mu}{4\pi} j_z,$$

and the respective equation for the function E_z can be written in the form,

$$E_z(\mathbf{r}) = E_z^0(\mathbf{r}) - \frac{i\omega\mu}{4\pi} \oint_{(S)} j_z(\mathbf{r}_0') G_k(\mathbf{r} \mid \mathbf{r}_0')\, d\mathbf{r}_0'.$$

In the case of a TE-type field ($\mathbf{H} = \{0, 0, H_z\}$) we can set $u(\mathbf{r}) = H_z(\mathbf{r})$, where on the surface of the cylinder we will have /71

$$\varphi(\mathbf{r}_0') = H_z(\mathbf{r}_0') = -\frac{c}{4\pi} j_\varphi(\mathbf{r}_0').$$

As a result the respective equation for H_z can be written in the form,

$$H_z(\mathbf{r}) = H_z^0(\mathbf{r}) + \frac{c}{4\pi} \oint_{(S)} j_\varphi(\mathbf{r}_0') \frac{\partial G_k(\mathbf{r} \mid \mathbf{r}_0')}{\partial n} d\mathbf{r}_0'.$$

From the expressions obtained for E_z and H_z, it is clear that the scattered field is completely determined by giving the density distribution of surface currents on (S).

In analogy with theory of the potential, we can say that (1.9.12 is a description of the solution using "simple sources" with surface density ψ, and (1.9.13) is a description of the solution using "dipole sources" with density ϕ.

If the functions ψ, ϕ in (1.9.12) and (1.9.13) are unknown, then determination of the solutions to the respective boundary problems may be reduced to solving the integral equations relative to the functions ψ and ϕ, which are obtained directly from the boundary conditions on the surface (S) for $\partial u/\partial n$ and u. Usually this is a Fredholm equation, either of the first or second sort with nonsingular kernels (see, for example, [10, 48, 116]).

In the past we have been actually concerned only with the simplest case of one body. However the formulas given above have been expanded also to the case of an arbitrary number of scattering bodies. In the presence of, for example, N scattering bodies, bounded by the surfaces (S_i), $i = 1, 2, \ldots, N$, the diffraction field may be represented through the Green function of free space by the integral relationship

$$u(\mathbf{r}) = u^0(\mathbf{r}) + \sum_{i=1}^{N} \oint_{(S_i)} \psi_i(\mathbf{r}_i') G_k(\mathbf{r} \mid \mathbf{r}_i')\, d\mathbf{r}_i' \qquad (1.9.14)$$

or

$$u(\mathbf{r}) = u^0(\mathbf{r}) - \sum_{i=1}^{N} \oint_{(S_i)} \varphi_i(\mathbf{r}_i') \frac{\partial}{\partial n_i} G_k(\mathbf{r} \mid \mathbf{r}_i') d\mathbf{r}_i' \qquad (1.9.15)$$

respectively, where the functions ψ_i and ϕ_i give the law of density distribution of the surface currents induced on the surfaces (S_i), $i = 1, 2, \ldots, N$. In this case, the solution to the diffraction problem may be reduced to solving a system of N integral equations of Fredholm type of first or of second sort relative to N unknown functions or ϕ_i or ψ_i which are obtained directly from the boundary conditions on the surfaces (S_i), $i = 1, 2, \ldots, N$, for the function /7
u or its normal derivative $\partial u/\partial n$.

The integral equations (systems), obtained by solving the diffraction problems, may often be solved using computers for sufficiently general boundary conditions having a high degree of accuracy using straight numerical methods. They can also be solved by the iteration method, for example, by the method of perturbation theory when the diffraction process is represented as a sequence of actions, alternately influenced by the effect of the bodies on one another. However, this method is suitable only in the case when the dimensions of the bodies are small and the distance between them is great in comparison with the wave length of the excitation.

Let us note that the solution to the diffraction problem, obtained by direct application of the Huygens theory which satisfies the condition of radiation at infinity, is always unique.

Another method of solving the problem of the interaction of two scattering bodies, based on use of the Green functions, is the Schwarzschild method [48], which apparently first studied such a problem in connection with diffraction on slits, where the interaction of both sides of the slit was taken into account [142]. The Schwarzschild method assumes the Green function of the individual body to be known (whereas direct use of the Huygens theory requires knowledge only of the Green function of free space) and leads to a system of two Fredholm integral equations of the second sort, solvable by the method of successive approximations. However, in spite of the proven convergence of the sequence of iterations, this method has not received wide-spread use because of the slow convergence of the process. It may be suitable only in the case when the interacting bodies are so far from one another that the amplitude of the wave, scattered by one body at the site of the other, is small in comparison with the amplitude of the incident wave.

The problem of the interaction of several scattering bodies may also be solved without reverting to the Green function if the perturbation method is used directly to seek its solution.

For example, let the superficial Dirichlet problem be solved,

$$\Delta u + k^2 u = 0,$$
$$(u + u^0)|_{S_i} = 0, \quad i = \pm 1, \tag{1.9.16}$$

for two scattering bodies with surfaces (S_{-1}) and (S_{+1}). Let us denote through ${}^{i}u^{1}$ the potential of the secondary field, scattered by the i-th body (i = -1 or i = +1) acted on by the incident field u^0 without taking into account the secondary field produced by the presence in space of the other, i.e., the $-i$-th body (i = -1 or i = +1). Let us call this particular secondary field a field of the first order of scattering. To determine the potential ${}^{i}u^{1}$ let us require that the function /73

$$u_1 = u_0 + {}^{i}u^{1}$$

be the solution to equation (1.9.16), which satisfies the homogeneous Dirichlet condition on the surface of the i-th body (i = -1 or i = +1). The potential

$$u_{(1)} = u^0 + \sum_{i=\pm 1} {}^{i}u^{1},$$

which determines the sum of the secondary field and the fields reflected from each of the bodies separately (in the absence of the other), we can call the potential of the field in first approximation. Acted on by the field of the first order of scattering, excited by the $-i$-th body (i = -1 or i = +1, such a field is represented by the potential ${}^{-i}u^{1}$), the i-th body (i = -1 or i = +1) excites secondary field of the second order of scattering which is represented by the potential ${}^{i}u^{2}$ (i = -1 or i = +1). For its determination, let us require that the function

$$u_2 = {}^{-i}u^{1} + {}^{i}u^{2}$$

be the solution to equation (1.9.16), which satisfies the homogeneous Dirichlet condition on the surface (S_i) (i = -1 or i = +1). The potential

$$u_{(2)} = u^0 + \sum_{i=\pm 1} {}^{i}u^{1} + \sum_{i=\pm 1} {}^{i}u^{2},$$

which determines the sum of the primary field and the secondary fields of first and second orders of scattering, can be called the potential of the field in second approximation, etc. The process

of constructing approximations for the scattered secondary fields can be expanded to an arbitrary (infinite) number of similar stages. Here the potential of the field in m-th approximation will be given by the formula

$$u_{(m)} = u^0 + \sum_{i=\pm 1} \sum_{j=1}^{m} {}^{i}u^{j}, \quad m = 1, 2, \ldots$$

The sum

$$\tilde{u}_i = \sum_{j=1}^{\infty} {}^{i}u^{j}$$

is the potential of the secondary field, scattered by the i-th body as a result of the repeated effect on it of the secondary waves of all successive orders of scattering, excited by the $-i$-th body as a result of its interaction with the i-th body. The total diffraction field is determined by the potential /74

$$u = u^0 + \sum_{i=\pm 1} \tilde{u}_i = u^0 + \sum_{i=\pm 1} \sum_{j=1}^{\infty} {}^{i}u^{j},$$

which in its construction will satisfy the conditions of the problem (1.9.16) (of ${}^{i}u^{j}$ it is only required that it satisfy the condition of radiation at infinity).

In the case when the surfaces (S_i) allow separation of the Helmholtz equation in the respective curvilinear orthogonal coordinates associated with each body, representation of the potentials ${}^{i}u^{j}$ in the form of infinite series of wave eigenfunctions of each body reduces solution of the problem to seeking the coefficients of the series. These latter are found from the boundary condition given on the surface of each body (for example, the expansion coefficients of the potential ${}^{i}u^{j}$ of the secondary field of the j-th order of scattering are found from the condition $({}^{i}u^{j} + {}^{-i}u^{j-1})|_{S_i} = 0$, $i = \pm 1$, $j = 1, 2, \ldots$). Here it is suitable to use special formulas, i.e., addition theorems for the wave functions, with the aid of which the wave eigenfunctions of one body are described through the wave eigenfunctions of the other body. As a result, the expansion coefficients of the potential ${}^{i}u^{j}$ are expressed by a certain infinite series, whose terms are products of the expansion coefficients of the potential ${}^{-i}u^{j-1}$ and the wave functions from the fixed values of the arguments. The expansion coefficient of the field potential of an arbitrary order of scattering is expressed ultimately through the expansion coefficient of the potential ${}^{i}u^{1}$ of the field of the first order of scattering. In the general case,

omputation of the expansion coefficients is a quite time-consuming ffort, with respect to which this method is feasible to use only n the case when the dimensions of the bodies are small, and the istance between them is sufficiently high in comparison with the ave length (see, for example, [119, 120, 135, 136]).

The method of strict solution to the problems of wave scattering n several bodies, based on the use of wave eigenfunctions, and the ddition theorem for them is the classical method of separation of ariables. In finding the solution by this method of diffraction roblems the field outside, the diffracting body is usually repre- ented in the form of a sum of a known primary field and an unknown econdary field, scattered on barriers, superposed in the form of n infinite series of primary wave functions, each of which satis- ies the condition of radiation at infinity. The field inside the cattering body is taken in the form of an infinite series of pri- ary wave functions, not having singularities in the volume of the ody. The coefficients of the series are found from the boundary /75 onditions which must be satisfied on the surface of the body. In he case of one diffracting body, the coefficients are obtained in losed form.

As it applies to two scattering bodies, for example, with ideally onducting surfaces (S_1) and (S_2), the method of separation consists f the following.

With each body (S_1) and (S_2) there are associated local coordin- te systems in which the surface (S_1) or, respectively, (S_2) coin- ides with one of the complete coordinate surfaces. The field cattered on (S_1) and (S_2) is sought in the form of a sum of two iffraction series of primary wave functions, in which one of the eries is composed of wave functions of one body, and the second eries is composed of wave functions of a second body. Using the ddition theorem, from the boundary conditions on (S_1) and on (S_2), nfinite systems of linear equations are obtained with an infinite umber of unknowns, i.e., the unknown expansion coefficients. None f the known methods of solving infinite systems can be justifiably pplied to such systems directly. However, after substitution of ie initial unknown coefficients by new ones, based on a certain aw for the new unknowns, infinite systems of linear equations are btained, which are found to be quasi-regular in the general case id uniquely solvable by the method of truncation (reduction) under ie condition that the bodies are not tangent to one another.

The method of separation of variables, in those cases when it s applicable, is an effective means of finding the numerical (ap- roximate) values of the diffraction coefficients and all functions hich characterize the unknown fields, using computers and tabulated ave functions for a sufficiently wide range of variation in the arameters of the problem. However, just as in the case of a single arrier, the computational feasibility here is limited first of all

because of the problems arising in tabulating the wave functions. In fact, numerical solution of the problem is always feasible, within allowable computational accuracy, for all values of the parameters for which numerical solution of the respective diffraction problem on a single barrier is possible (the range of wave lengths and waves comparable with the dimensions of the barriers).

The infinite systems of linear equations which arise in problems of the interaction of two or several scattering bodies involve systems of normal type

$$x_n + \sum_{m=1}^{\infty} c_{nm} x_m = b_n, \quad n = 1, 2, \ldots, \tag{1.9.17}$$

with complex coefficients c_{nm} and with complex right-hand parts (free terms) b_n.

The theories of infinite systems of linear equations with an infinite number of unknowns have been studied in a number of papers, from which we wish to cite [20] where the bases of this theory are discussed, as well as [21, 60 - 66] where the properties of systems similar to those obtained in the problems solved below are studied.

A system such as (1.9.17) is called regular if the sum of the modulae of its coefficients in each series is less than unity,

$$\sum_{m=1}^{\infty} |c_{nm}| < 1, \quad n = 1, 2, \ldots,$$

and is called a completely regular system if its coefficients c_{nm} satisfy the condition,

$$\sum_{m=1}^{\infty} |c_{nm}| \leqslant 1 - \alpha < 1, \quad n = 1, 2, \ldots,$$

where α is a certain constant ($\alpha > 0$). The regular system of equations (1.9.17) whose right-hand parts satisfy the condition,

$$|b_n| \leqslant C \rho_n,$$

where C = const > 0, but $\rho_n = 1 - \sum_{m=1}^{\infty} |c_{nm}|$, $n = 1, 2, \ldots$, always has a bound solution $|x_n| < C$, which may be found by the method of successive approximations. Solution to a completely regular system exists with any restricted right-hand parts b_n and it is always unique. A regular system of equations may also have several restricted solutions.

Solution to the system (1.9.17), found by the method of successive approximations with a zero initial value, is called the principal solution to the system. The principal solution to the regular system (1.9.17), with free terms satisfying the condition $|b_n| \leq C\rho_n$, may be found also by the method of truncation (reduction), i.e., if x_n^N is the solution to the finite system

$$x_n + \sum_{m=1}^{N} c_{nm} x_m = b_n, \quad n = 1, 2, \ldots, N,$$

then

$$\tilde{x}_n = \lim_{N \to \infty} x_n^N,$$

where $\tilde{x}_n$ is the principal solution to the system (1.9.17).

A system for which the condition of regularity is not satisfied in all series, but only beginning from a certain one, i.e.,

$$\sum_{m=1}^{\infty} |c_{nm}| < 1, \quad n = N+1, N+2, \ldots, \qquad /77$$

and for which, furthermore,

$$\sum_{m=1}^{\infty} |c_{nm}| < +\infty, \quad n = 1, 2, \ldots, N,$$

$$|b_n| \leqslant C\rho_n = C\left(1 - \sum_{m=N+1}^{\infty} |c_{nm}|\right), \quad n = N+1, \ldots,$$

is called quasi-regular. For such systems the question of the existence of a solution is reduced to the question of the existence of a solution to a finite system of N equations with N first unknowns $x_1, x_2, \ldots, x_N$.

The system of equations

$$X_n + \sum_{m=1}^{\infty} C_{nm} X_m = B_n, \quad n = 1, 2, \ldots, \qquad (1.9.18)$$

is called a majorant for the system (1.9.17) if the inequality

$$|c_{nm}| \leqslant C_{nm}, \quad n = 1, 2, \ldots, \quad m = 1, 2, \ldots,$$
$$|b_n| \leqslant B_n, \quad n = 1, 2, \ldots$$

is satisfied.

If the coefficients of the system (1.9.17) are such that the series $\sum_{n,m=1}^{\infty} |c_{nm}|^2$, comprised of them, converges:

$$\sum_{n,m=1}^{\infty} |c_{nm}|^2 < +\infty, \tag{1.9.19}$$

then we can say that system (1.9.17) possesses a completely continuous form (completely continuous linear operator in the Gilbert coordinate space l^2). For the systems in (1.9.17) with a completely continuous form, the following argument is valid (Gilbert alternative): with convergence of the series,

$$\sum_{n=1}^{\infty} |b_n|^2 < \infty \tag{1.9.20}$$

or this system (1.9.17) has one definitive solution $\{x_n\}$ which satisfies the condition

$$\sum_{n=1}^{\infty} |x_n|^2 < \infty, \tag{1.9.21}$$

or the homogeneous system corresponding to the nonhomogeneous system (1.9.17) has a solution that is non-zero.

Let us note that if

$$\sum_{n,m=1}^{\infty} |c_{nm}|^2 < 1 \tag{1.9.22}$$

and condition (1.9.20) is satisfied then the system (1.9.17) has a unique solution $\{x_n\}$ which satisfies the condition (1.9.21).

One of the widely-used methods of solving the nonhomogeneous system (1.9.17), with completely continuous form, is the method of truncation [20, 21, 60 - 66], consisting, as mentioned above, of a substitution of the infinite system (1.9.17) by the finite system

$$x_n + \sum_{m=1}^{N} c_{nm} x_m = b_n, \quad n = 1, 2, \ldots, N, \tag{1.9.23}$$

from N equations with N unknowns, the solution to which (let us denote it through x_n^N) is studied as an approximate solution to the original infinite system (1.9.17). If system (1.9.17), with a

npletely continuous form, is uniquely solvable, then with a suf-
ciently large value of N system (1.9.23) will also be solvable and
e approximate solutions will converge to precise ones: $\lim x_n^N = x_n$,
$\to \infty$. By comparing the successive approximations of the solution
the system for the various increasing values of N-magnitude of
ıncation, approximations may be obtained with *a priori* given de-
ee of accuracy.

Let us note that in a number of cases, a system which is not
gular (quasi-regular) or which does not possess a completely con-
nuous form, may be reduced to a regular (quasi-regular) system or
a system with completely continuous form relative to the unknowns
(for example, by substitution of the unknowns x_n with other un-
owns z_n using the formula $x_n = \alpha_n z_n$).

All the above discussions refer to infinite systems, forming a
t of two systems,

$$x_n + \sum_{m=1}^{\infty} \alpha_{nm} x_m = b_n,$$

$$y_n + \sum_{m=1}^{\infty} \beta_{nm} y_m = c_n, \quad n = 1, 2, \ldots,$$

th two series of unknowns $x_1, x_2, x_3, \ldots, y_1, y_2, \ldots$, which after
troduction of new designations may be reduced to one system of
rmal type and to infinite systems of equations of the form,

$$x_n + \sum_{m=-\infty}^{\infty} c_{nm} x_m = b_n, \quad n = 0, \pm 1, \pm 2, \ldots$$

CHAPTER 2

CYLINDRICAL WAVE FUNCTIONS AND CERTAIN OF THEIR PROPERTIES

§1. Helmholtz Equation in Cylindrical Coordinates. Cylindrical Wave Functions.

The Helmholtz scalar wave equation in the system of cylindrical coordinates ρ, ϕ, z has the form of (1.6.17). If we seek the particular solution to equation (1.6.17) in the form of the product of three functions,

$$u(\rho, \varphi, z) = R(\rho)\Phi(\varphi)Z(z) \not\equiv 0, \tag{2.1.1}$$

each of which depends only on one coordinate, then after substituting (2.1.1) into (1.6.17) and subsequent separation of variables in the equations thus obtained, we come to the conclusion that, for construction of the particular solution to equation (1.6.17) in the form (2.1.1), as the functions R, Φ and Z it follows to take any solutions to the respective equations,

$$\frac{d^2\Phi}{d\varphi^2} + \lambda^2\Phi = 0, \tag{2.1.2}$$

$$\frac{d^2Z}{dz^2} + h^2Z = 0, \tag{2.1.3}$$

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\cdot\frac{dR}{d\rho} + \left(v^2 - \frac{\lambda^2}{\rho^2}\right)R = 0, \tag{2.1.4}$$

in which λ and h are separation constants, but $v^2 = k^2 - h^2$.

The independent particular solutions to equations (2.1.2) and (2.1.3) are, respectively, the functions $\Phi(\phi) = \begin{matrix}\cos\\ \sin\end{matrix}(\lambda\phi)$, $Z(z) = \begin{matrix}\cos\\ \sin\end{matrix}(hz)$ or, in complex form, the functions $\Phi(\phi) = e^{\pm i\lambda\phi}$, $Z(z) = e^{\pm ihz}$. Equation (2.1.4) is the Bessel equation. The particular solutions to it will be the functions $R_\lambda(v\rho)$, where through $R_\lambda(x)$ we denote any Bessel function of λ-th order. Consequently, the particular solutions to equation (1.6.17), represented in the form of (2.1.1), will be the functions,

$$u_{\lambda,h}(\rho, \varphi, z) = R_\lambda(v\rho)\begin{matrix}\cos\\ \sin\end{matrix}(\lambda\varphi)e^{ihz}. \tag{2.1.5}$$

are interested only in the unique solutions to equation (1.6.17), connection with which it follows to assume the separation constant λ to be a whole number: $\lambda^2 = n^2$, $n = 0, \pm 1, \ldots$ (in this case the solutions to (2.1.5) are periodic functions with the period 2π relative to the variable ϕ). With the whole number $\lambda = n$, it follows to take as $R_\lambda(v\rho)$, depending on the region being studied and the supplementary conditions, either the Bessel function of first sort $J_n(v\rho)$ of n-th order, or the first Hankel function $H_n^{(1)}(v\rho)$ n-th order. Then the particular solutions to equation (1.6.17) ill be the functions,

$$J_n(v\rho)e^{in\varphi}e^{ihz}, \tag{2.1.6}$$

$$H_n^{(1)}(v\rho)e^{in\varphi}e^{ihz}. \tag{2.1.7}$$

he several properties of the Bessel functions from those which we hall require further on will be given below. Here we note only hat the function $J_n(x)$ is regular at the point $x = 0$, whereas the nction $H_n^{(1)}(x)$ has at this point a logarithmic singularity. rom the asymptotic concept of the Hankel function

$$H_n^{(1)}(k\rho) \approx e^{i\left(k\rho - \frac{\pi n}{2} - \frac{\pi}{4}\right)} \sqrt{\frac{2}{\pi k\rho}}$$

ith large absolute values of the argument $k\rho$, it follows that the unction $H_n^{(1)}(k\rho)e^{-i\omega t}$ represents at infinity a propagating wave hich satisfies the condition of radiation there. Therefore, the unctions in (2.1.6) are ordinarily used for constructing a general olution to equation (1.6.17) inside the cylindrical region conaining within itself $\rho = 0$, but the functions in (2.1.7) are used or constructing the general solution to equation (1.6.17) which atisfies, at infinity in the unbounded region outside the cylinder, he condition of radiation. Since the separation constant h in 2.1.5) is an arbitrary value, and equation (1.6.17) is linear and omogeneous relative to u, then the general solution to it is regular at the points of the axis $\rho = 0$, and may be represented in he form of a series of Fourier-type integrals,

$$\sum_{n=-\infty}^{\infty} e^{in\varphi} \int_{-\infty}^{\infty} a_n(h) J_n(v\rho) e^{ihz}\, dh, \tag{2.1.8}$$

ut the solution which satisfies the condition of radiation at in- /81
inity, is given in the form of series such as

$$\sum_{n=-\infty}^{\infty} e^{in\varphi} \int_{-\infty}^{\infty} b_n(h) H_n^{(1)}(v\rho) e^{ihz} dh, \qquad (2.1.9)$$

in which $a_n(h)$, $b_n(h)$ are arbitrary values that depend on n and h and are subject to determination from the boundary conditions.

Functions (2.1.6) and (2.1.7) are ordinarily called cylindrical wave functions. In two-dimensional problems, when equation (1.6.17) is independent of the space coordinate z, the wave functions will be

$$J_n(k\rho) e^{in\varphi}, \qquad (2.1.10)$$

$$H_n^{(1)}(k\rho) e^{in\varphi} \qquad (2.1.11)$$

respectively, and the general solution to the equation inside the circle and outside it will be represented by a series such as

$$\sum_{n=-\infty}^{\infty} a_n J_n(k\rho) e^{in\varphi} \qquad (2.1.12)$$

and

$$\sum_{n=-\infty}^{\infty} b_n H_n^{(1)}(k\rho) e^{in\varphi}. \qquad (2.1.13)$$

§2. Bessel Functions and Certain of Their Properties

If we make the substitution of variables in equation (2.1.4) according to the formula $x = \sqrt{k^2 - h^2}\,\rho$, it is then reduced to the normal form of the Bessel equation,

$$\frac{d^2y}{dx^2} + \frac{1}{x} \cdot \frac{dy}{dx} + \left(1 - \frac{\lambda^2}{x^2}\right) y = 0, \qquad (2.2.1)$$

which is characterized by two singular points: the regular singular point $x = 0$ and the irregular singular point $x = \infty$. In the case of non-integral values of λ, the Bessel function has two solutions, which can be represented by the generalized power series ($\lambda \geq 0$),

$$J_\lambda(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k + \lambda + 1)} \left(\frac{x}{2}\right)^{\lambda+2k}, \qquad (2.2.2)$$

$$J_{-\lambda}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k-\lambda+1)} \left(\frac{x}{2}\right)^{-\lambda+2k}, \qquad (2.2.3)$$ /82

which converge with any finite value of x. Solution to (2.2.2), which has no singularity at the point $x = 0$, is called a Bessel function of the first sort of λ-th order. The second solution has at the point $x = 0$ a pole of λ-th order, by virtue of which it is linear independently of $J_\lambda(x)$. Therefore, the general solution to equation (2.2.1) in the region, not containing the point $x = 0$, may be represented by the linear combination

$$y = C_1 J_\lambda(x) + C_2 J_{-\lambda}(x),$$

where C_1, C_2 are arbitrary constants.

The function $J_{-\lambda}(x)$ is determined by formula (2.2.3) only for non-integral values of λ. However, if we continue it uninterruptedly to integral values of $\lambda = n$, we find

$$J_{-n}(x) = (-1)^n J_n(x). \qquad (2.2.4)$$

This relationship indicates that with the integral values $\lambda = n$, the functions $J_\lambda(x)$ and $J_{-\lambda}(x)$ are linearly dependent and a general solution to equation (2.2.1) can never be constructed from them. In this case as the particular solution to (2.2.1), which is linearly independent of $J_\lambda(x)$, we take the function,

$$N_n(x) = \lim_{\lambda \to n} N_\lambda(x) = \frac{1}{\pi}\left\{\frac{\partial}{\partial \lambda} J_\lambda(x) - (-1)^n \frac{\partial}{\partial \lambda} J_{-\lambda}(x)\right\}_{\lambda=n} \qquad (2.2.5)$$

where

$$N_\lambda(x) = \frac{J_\lambda(x)\cos\lambda\pi - J_{-\lambda}(x)}{\sin\lambda\pi}. \qquad (2.2.6)$$

It is obvious that both (2.2.6) and (2.2.5) are solutions to the Bessel equation. On the basis of (2.2.5) and (2.2.2), (2.2.3), the function $N_n(x)$ can be represented by the expansion ($\gamma = 0.577215\ldots$)

$$N_n(x) = \frac{2}{\pi} J_n(x) \ln\frac{\gamma x}{2} - \frac{1}{\pi}\left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(n+k)!}\left(\frac{x}{2}\right)^{2k}\left(\sum_{m=1}^{k}\frac{1}{m} + \sum_{m=1}^{k+n}\frac{1}{m}\right) -$$

$$-\frac{1}{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{x}{2}\right)^{2k}, \quad n=0, 1, \ldots \qquad (2.2.7)$$

If in the future by $N_\lambda(x)$ we mean the function (2.2.6) with non-integral values of λ and the function (2.2.5) with integral values $\lambda = n$, we find that the general solution to equation (2.2.1) for integral and non-integral values of λ, in the region not containing the point $x = 0$, may be written in the form of the linear combination, /8

$$y = C_1 J_\lambda(x) + C_2 N_\lambda(x),$$

where C_1, C_2 are arbitrary constants.

The function $N_\lambda(x)$, determined by formulas (2.2.5), (2.2.6), is called a Bessel function of the second sort of λ-th order or the Neumann function. The particular solutions to equation (2.2.1) constructed from $J_\lambda(x)$ and $N_\lambda(x)$, in the form of their linear combinations,

$$H_\lambda^{(1)}(x) = J_\lambda(x) + iN_\lambda(x), \quad H_\lambda^{(2)}(x) = J_\lambda(x) - iN_\lambda(x), \qquad (2.2.8)$$

are called Bessel functions of the third sort or the first and second Hankel functions, respectively. From (2.2.5) and (2.2.4) it follows that with the integral values $\lambda = n$,

$$H_{-n}^{(1),(2)}(x) = (-1)^n H_n^{(1),(2)}(x). \qquad (2.2.9)$$

For non-integral values of λ, we have the relationship,

$$H_{-\lambda}^{(1)}(x) = e^{i\lambda\pi} H_\lambda^{(1)}(x), \quad H_{-\lambda}^{(2)}(x) = e^{-i\lambda\pi} H_\lambda^{(2)}(x).$$

The theory of Bessel functions has been rather thoroughly discussed in [24, 28-30, 33-36, 178, 179, 181]. Let us look at several of the properties of Bessel functions.

1. Bessel functions satisfy the recurrent relationships,

$$\pm\frac{dR_\lambda(x)}{dx} = R_{\lambda\mp1}(x) - \frac{\lambda}{x}R_\lambda(x), \qquad (2.2.10)$$

$$R_{\lambda+1}(x) = \frac{2\lambda}{x} R_\lambda(x) - R_{\lambda-1}(x), \qquad (2.2.11)$$

where $R_\lambda(x)$ denotes any of the Bessel functions. Formulas (2.2.10) and 2.2.11) are suitable for successive computation of the values of the Bessel functions and their derivative through two functions of the first two orders: $J_0(x)$, $J_1(x)$ and $N_0(x)$, $N_1(x)$, for which detailed tables exist [178, 179, 181].

2. From (2.2.2) and (2.2.3) we can establish that

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x, \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x. \qquad (2.2.12)$$

These relationships and the successive use of recurrent formula (2.2.11) permit expressing the Bessel function $J_{n+\frac{1}{2}}(x)$, and from formulas (2.2.6), (2.2.8) the functions $N_{n+\frac{1}{2}}(x)$, $H^{(1),(2)}_{n+\frac{1}{2}}(x)$ through the unit functions. For Bessel functions of half-integral order we have the relationships

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$$N_{n+\frac{1}{2}}(x) = (-1)^{n+1} J_{-n-\frac{1}{2}}(x),$$
$$N_{-n-\frac{1}{2}}(x) = (-1)^n J_{n+\frac{1}{2}}(x). \qquad (2.2.13)$$

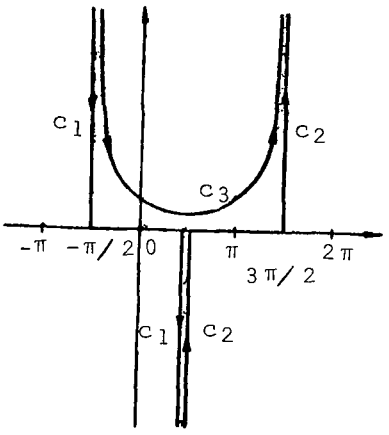

Fig. 12. Integration Boundaries c_1, c_2 and c_3.

3. The Bessel functions can be represented by Sommerfeld integrals,

$$H^{(1)}_\lambda(x) = \frac{e^{-i\lambda\pi/2}}{\pi} \int_{c_1} e^{ix\cos\varphi + i\lambda\varphi} d\varphi, \qquad (2.2.14)$$

$$H^{(2)}_\lambda(x) = \frac{e^{-i\lambda\pi/2}}{\pi} \int_{c_2} e^{ix\cos\varphi + i\lambda\varphi} d\varphi, \qquad (2.2.15)$$

$$J_\lambda(x) = \frac{e^{-i\lambda\pi/2}}{2\pi} \int_{c_3} e^{ix\cos\varphi + i\lambda\varphi} d\varphi \qquad (2.2.16)$$

(the integration boundaries are shown on Figure 12), which are generalized to the case of complex values of x and λ (Re x > 0 in the integrals for the Hankel functions). The Bessel functions are analytical functions of the argument x. From the integral representations (2.2.14) - (2.2.16) it is clear that, with a constant value of x [Re x > 0 for the Hankel functions], their subintegral expressions are analytical functions of the parameter λ, but the integrals themselves converge uniformly to any finite region of variation in the parameter λ. Consequently, for each

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x (Re $x > 0$), the Bessel functions are also analytical functions of the parameter λ (integral transcendental functions of λ).

In the case of the integral values $\lambda = n$,

$$H_n^{(1)}(x) = \frac{(-i)^n}{\pi} \int_{-\pi/2+i\infty}^{\pi/2-i\infty} e^{ix\cos\varphi + in\varphi}\, d\varphi, \quad \operatorname{Re} x > 0, \tag{2.2.17}$$

$$H_n^{(2)}(x) = \frac{(-i)^n}{\pi} \int_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{ix\cos\varphi + in\varphi}\, d\varphi, \quad \operatorname{Re} x > 0, \tag{2.2.18}$$

$$J_n(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ix\sin\varphi + in\varphi}\, d\varphi. \tag{2.2.19}$$

4. After applying the saddle-point method to the Sommerfeld integrals (2.2.14), (2.2.15), we can obtain a general concept as to the behavior of the Bessel functions with a large (real and positive) argument x and arbitrary (complex) index λ. The saddle points of the integrals are found from the equation (relative to z) $\sin z = \lambda/x$, which for any values of x and λ always has two roots, symmetrically arranged relative to the point $z = \pi/2$ in the band $0 < \operatorname{Re} z < \pi$. Here the type of asymptotic expression of each Bessel function involves the choice of saddle point: the contribution given by one of the saddle points is found to be exponentially small in comparison with the contribution made by the other saddle point, and only in individual cases do both saddle points make contributions of the same order and then both become determinant: in obtaining the asymptotic expression. The first asymptotic expressions of the Bessel functions were obtained on the basis of Debye's saddle-point method [114]. Debye theory was then supplemented and developed by others. A discussion of it can be found, for example, in [28, 29, 38, 48]. In particular, from Debye theory it follows that, for real positive values of the index λ, the asymptotic expressions of the Bessel functions have the form:

when $\lambda < x$,

$$H_\lambda^{(1),(2)}(x) \simeq \sqrt{\frac{2}{\pi}}\,(x^2-\lambda^2)^{-1/4} \exp\left\{\pm i\left(\sqrt{x^2-\lambda^2} - \lambda \arccos\frac{\lambda}{x} - \frac{\pi}{4}\right)\right\}, \tag{2.2.20}$$

$$J_\lambda(x) \simeq \sqrt{\frac{2}{\pi}}\,(x^2-\lambda^2)^{-1/4} \cos\left(\sqrt{x^2-\lambda^2} - \lambda \arccos\frac{\lambda}{x} - \frac{\pi}{4}\right); \tag{2.2.21}$$

when $\lambda > x$

$$H_\lambda^{(1),(2)}(x) \simeq \mp i \sqrt{\frac{2}{\pi}} (\lambda^2 - x^2)^{-1/4} \times$$

$$\times \exp\left\{ \lambda \ln \frac{\lambda + \sqrt{\lambda^2 - x^2}}{x} - \sqrt{\lambda^2 - x^2} \right\}, \qquad (2.2.22)$$

$$J_\lambda(x) \simeq \frac{1}{\sqrt{2\pi}} (\lambda^2 - x^2)^{-1/4} \times$$

$$\times \exp\left\{ -\left(\lambda \ln \frac{\lambda + \sqrt{\lambda^2 - x^2}}{x} - \sqrt{\lambda^2 - x^2} \right) \right\}. \qquad (2.2.23)$$

From these expressions it is clear that when $\lambda < x$ the Bessel functions vary in absolute value very slowly with change in λ, whereas when $\lambda > x$, with increase in λ the function $J\lambda(x)$ drops more rapidly than the exponent, but the function $H_\lambda^{(1),(2)}(x)$ increases more rapidly than the exponent. For values of $\lambda << x$, from (2.2.20) and (2.2.21), we find that

$$H_\lambda^{(1),(2)}(x) \simeq \sqrt{\frac{2}{\pi x}} e^{\pm i(x - \pi\lambda/2 \mp \pi/4)},$$

$$J_\lambda(x) \simeq \sqrt{\frac{2}{\pi x}} \cos(x - \pi\lambda/2 - \pi/4), \qquad (2.2.24)$$

$$N_\lambda(x) \simeq \sqrt{\frac{2}{\pi x}} \sin(x - \pi\lambda/2 - \pi/4),$$

and in the case if $\lambda >> x$,

$$H_\lambda^{(1),(2)}(x) \simeq \pm i \sqrt{\frac{2}{\pi\lambda}} \left(\frac{2\lambda}{ex} \right)^\lambda,$$

$$J_\lambda(x) \simeq \frac{1}{\sqrt{2\pi\lambda}} \left(\frac{2\lambda}{ex} \right)^{-\lambda} \qquad (2.2.25)$$

or with the integral values $\lambda = n >> x$,

$$H_n^{(1)}(x) \simeq -\frac{i}{\pi} (n-1)! \left(\frac{2}{x} \right)^n, \qquad (2.2.26)$$

$$H_n^{(2)}(x) \simeq \frac{i}{\pi} (n-1)! \left(\frac{2}{x} \right)^n, \qquad (2.2.27)$$

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$$J_n(x) \simeq \frac{1}{n!} \left(\frac{x}{2} \right)^n. \qquad (2.2.28)$$

In the vicinity of the point $\lambda = x$, the Debye asymptotic expressions (2.2.20) - (2.2.23) are supplemented by the representations,

$$H_{\lambda}^{(1),(2)}(x) \simeq \frac{2}{\pi} e^{\mp i\pi/3} \left(\frac{6}{x}\right)^{1/3} A(q), \tag{2.2.29}$$

$$J_{\lambda}(x) \simeq \frac{1}{\pi} \left(\frac{6}{x}\right)^{1/3} A(q) \tag{2.2.30}$$

(the so-called Hankel approximation), which change to Debye representations in the case if λ does not lie too near to x. Here through $A(q)$ we denote the Airy integral [176, 177],

$$A(q) = \frac{1}{2} \int_{-\infty}^{\infty} e^{i(q\tau - \tau^3)} d\tau = \int_{0}^{\infty} \cos(\tau^3 - q\tau) d\tau, \tag{2.2.31}$$

where

$$q = (\lambda - x) \left(\frac{6}{x}\right)^{1/3} \begin{cases} e^{\mp i\pi/3} \\ -1 \end{cases} \tag{2.2.32}$$

respectively, for the functions $H_{\lambda}^{(1),(2)}(x)$ and $J\lambda(x)$.

In [49], Fok gave other asymptotic representations for the Bessel functions, applicable in the region for which $|x^2 - \lambda^2| < A^2 |\lambda|^{4/3}$, where $|\lambda| >> 1$, but A is not a very large number, when the Debye approximations can never be used. The first terms of the Fok approximations have the form,

$$H_{\lambda}^{(1),(2)}(x) \simeq \frac{\mp i}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{-1/3} w_{1,2}(t), \tag{2.2.33}$$

$$J_{\lambda}(x) \simeq \frac{1}{\sqrt{\pi}} \left(\frac{x}{2}\right)^{-1/3} v(t), \tag{2.2.34}$$

where through $w_{1,2}(t)$ we denote the Fok - Airy functions that are suitable for application and which are determined for the real values of the argument t by the integrals

$$w_{1,2}(t) = \frac{1}{\sqrt{\pi}} \int_{\infty \exp(\mp 2\pi i/3)}^{\infty} e^{t\tau - \tau^3/3} d\tau = u(t) \pm iv(t) \tag{2.2.35}$$

(The Fok - Airy functions may also be extended to the case of complex values of t). The functions $u(t)$, $v(t)$ for real values of t were tabulated by Fok. The tables of their values are given, for example, in [49]. For large positive values of t, the functions /80

$u(t)$, $v(t)$ and their derivatives have asymptotic expressions

$$u(t) \simeq t^{-1/4} e^{\frac{2}{3} t^{3/2}}, \quad v(t) \simeq \frac{1}{2} t^{-1/4} e^{-\frac{2}{3} t^{3/2}},$$
$$u'(t) \simeq t^{1/4} e^{\frac{2}{3} t^{3/2}}, \quad v'(t) \simeq -\frac{1}{2} t^{1/4} e^{-\frac{2}{3} t^{3/2}}, \qquad (2.2.36)$$

from which it is clear that with increase in t, the function $u(t)$ grows monotonically, but $v(t)$ drops monotonically. For large negative values of t the asymptotic expressions of the functions $u(t)$, $v(t)$ will be

$$u(t) \simeq (-t)^{-1/4} \cos\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right],$$
$$v(t) \simeq (-t)^{-1/4} \sin\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right],$$
$$u'(t) \simeq (-t)^{1/4} \sin\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right], \qquad (2.2.37$$
$$v'(t) \simeq -(-t)^{1/4} \cos\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right],$$

from which, in turn, it is clear that with negative values of t the functions $u(t)$ and $v(t)$ have a fluctuating character.

5. In solving problems of diffraction involving the use of Bessel functions, there often arises the necessity of computing origins of the Bessel functions and of having knowledge of their distribution on the respective planes. Since each Bessel function is analytical both from the argument x on the entire plane of the complex variable x (with the exception possibly only of the point $x = 0$) and from the index λ on the plane of the variable λ, then it is obvious that each origin will appear within known limits of the analytical functions both of the argument x and the index λ (the general questions of the theory of origins of Bessel functions are discussed in [28, 29]). In problems of diffraction, it is usually necessary to know the distribution and the value of the origins only of the Hankel functions and their derivatives as a function of the variation in their complex index with fixed large /89 and positive values of the argument x (x is real and $x > 0$).

It is known that the Hankel function $H_\lambda^{(1)}(x)$ for any real index λ has no origins with the arguments 0, $\pi/2$, π, but the function $H_\lambda^{(2)}(x)$ has no origins with the arguments 0, $-\pi/2$, $-\pi$.

Therefore, in the case when $x > 0$ ($\arg x = 0$), both Hankel functions $H_\lambda^{(1),(2)}(x)$ and their derivatives $H_\lambda^{(1),(2)'}(x)$ for a given x are integral functions along the entire real axis on the plane of the complex variable λ, by virtue of which all possible origins λ_n, λ_n' of the functions $H_\lambda^{(1),(2)}(x)$ and $H_\lambda^{(1),(2)'}(x)$ for the fixed positive argument x are found inside the real axis of the plane λ and are complex numbers (the Hankel functions $H_\lambda^{(1),(2)}(x)$ of the purely imaginary index λ have no origins). The approximate values of the origins λ_n and λ_n' of the function $H_\lambda^{(1)}(x)$ and its derivative $H_\lambda^{(1)'}(x)$, found on the basis of the Debye asymptote, are found from the formulas

$$\lambda_n = x + \frac{x^{1/3}}{2} e^{i\pi/3} \left(\frac{3\pi}{4}\right)^{2/3} (4n-1)^{2/3}, \quad n = 1, 2, \ldots, \tag{2.2.38}$$

$$\lambda_n' = x + \frac{x^{1/3}}{2} e^{i\pi/3} \left(\frac{3\pi}{4}\right)^{2/3} (4n+1)^{2/3}, \quad n = 0, 1, \ldots \tag{2.2.39}$$

From the relationships,

$$H_\lambda^{(1)}(x) = e^{-i\lambda\pi} H_{-\lambda}^{(1)}(x), \quad H_\lambda^{(1)'}(x) = e^{-i\lambda\pi} H_{-\lambda}^{(1)'}(x) \tag{2.2.40}$$

it is clear that the origins λ_n, as well as λ_n', are arranged symmetrically in the first and fourth quarters of the plane λ, since if λ_n, λ_n' are the zero functions $H_\lambda^{(1)}(x)$, $H_\lambda^{(1)'}(x)$, then their origins will also be $-\lambda_n$, $-\lambda_n'$. The origins of the functions $H_\lambda^{(2)}(x)$ and $H_\lambda^{(2)'}(x)$ are arranged symmetrically in the second and the third quarters and are symmetrical with respect to λ_n, λ_n'. The values of the origins obtained from the Hankel approximation of the Bessel functions are more precise than those in (2.2.38) and (2.2.39). They are found from the formulas,

$$\lambda_n = x + \left(\frac{x}{6}\right)^{1/3} e^{i\pi/3} q_n, \tag{2.2.41}$$

$$\lambda_n' = x + \left(\frac{x}{6}\right)^{1/3} e^{i\pi/3} q_n', \tag{2.2.42}$$

where q_n, q_n' are the origins of the Airy function (2.2.31) and its derivative

$$A(q_n) = 0, \quad A'(q_n') = 0. \tag{2.2.43}$$

Let us give tables of the first five origins of the equations /90 in (2.2.43) and the origins of the functions $H_\lambda^{(1)}(x)$, and $H_\lambda^{(1)'}(x)$, borrowed from [48].

TABLE 1. ORIGINS OF THE AIRY FUNCTIONS AND ITS DERIVATIVES

n	q_n	$A'(q_n)$	q_n'	$A(q_n')$
1	3,372134	—1,05905	1,469354	1,16630
2	5,895843	1,21295	4,684712	—0,91272
3	7,962025	—1,30673	6,951786	0,82862
4	9,788127	1,37568	8,889027	—0,77962
5	11,457423	—1,43078	10,632519	0,74562

TABLE 2. ORIGINS OF THE FUNCTIONS $H_\lambda^{(1)}(x)$ AND $H_\lambda^{(1)'}(x)$

$$\left(\lambda_n = x + x^{1/3} e^{i\pi/3} a_n,\quad \lambda_n' = x + x^{1/3} e^{i\pi/3} a_n'\right)$$

n	a_n	a_n'
1	1,855757	0,808617
2	3,244608	2,578096
3	4,381671	3,825715
4	5,386614	4,891820
5	6,305263	5,851301

The method of computing the origins of the Hankel functions, based on the Hankel-Langer approximation [150],

$$H_\lambda^{(2)}(x) \simeq \left(\frac{\operatorname{tg}\alpha - \alpha}{\operatorname{tg}\alpha}\right)^{1/2} e^{-i\frac{\pi}{6}} H_{1/3}^{(2)}[\lambda \operatorname{tg}\alpha - \alpha], \qquad (2.2.44)$$

is suitable for large values of the argument x with the complex index λ, shown in [52] ($\lambda = x \cos \alpha$).

From the recurrent relationships (2.2.10) we obtain the formula,

$$2\frac{\partial R_\lambda(x)}{\partial x} = R_{\lambda-1}(x) - R_{\lambda+1}(x), \qquad (2.2.45)$$

which leads to the asymptotic relationship [125],

* Translators Note: tg = tan.

$$\frac{\partial R_\lambda(x)}{\partial x} \sim -\frac{\partial R_\lambda(x)}{\partial \lambda}. \qquad (2.2.46)$$

6. The two functions $I_\lambda(x)$ and $K_\lambda(x)$ which are often encountered in the appendix are closely associated with the Bessel functions and appear as particular solutions to the equation, /91

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\cdot\frac{\partial y}{\partial x} - \left(1 + \frac{\lambda^2}{x^2}\right) y = 0, \qquad (2.2.47)$$

which is obtained from the Bessel function (2.2.1) by substitution of x by ix. The function $I_\lambda(x)$ is associated with the Bessel function $J_\lambda(x)$ by the relationship,

$$I_\lambda(x) = e^{-\lambda\pi\frac{i}{2}} J_\lambda(ix) \qquad (2.2.48)$$

and is called a modified or Bessel function of the first sort of λ order. The function $K_\lambda(x)$ is the second particular solution to equation (2.2.47) and is linearly independent of (2.2.48) for all values of λ. It is related to the Bessel function $H_\lambda^{(1)}(x)$ by the relationship,

$$K_\lambda(x) = \frac{\pi i}{2} e^{\lambda\pi\frac{i}{2}} H_\lambda^{(1)}(ix) \qquad (2.2.49)$$

and is called a modified or Bessel function of the second sort of λ order. It is also called a MacDonald function of λ order.

If λ is a real number and x is positive, the modified functions assume real values.

In the case of non-integral values of λ,

$$K_{-\lambda}(x) = K_\lambda(x).$$

In the case of the integral values $\lambda = n$,

$$I_{-n}(x) = I_n(x),$$

$$K_n(x) = \lim_{\lambda \to n} K_\lambda(x) = \frac{(-1)^n}{2}\left[\frac{\partial I_{-\lambda}(x)}{\partial \lambda} - \frac{\partial I_\lambda(x)}{\partial \lambda}\right]_{\lambda=n}$$

7. If λ is not equal to an integral number, then the behavior of the independent solutions $J_\lambda(x)$ and $J_{-\lambda}(x)$ of the Bessel equation (2.2.1), when $x \to 0$, is determined by the formulas,

$$J_\lambda(x) \approx \frac{\left(\frac{x}{2}\right)^\lambda}{\Gamma(1+\lambda)}, \quad J_{-\lambda}(x) \approx \frac{\left(\frac{x}{2}\right)^{-\lambda}}{\Gamma(1-\lambda)}. \tag{2.2.50}$$

In the case of the integer $\lambda = n$ when $x \to 0$, we find the following relationship to be valid,

$$J_0(x) \approx 1, \quad J_n(x) \approx \frac{1}{n!}\left(\frac{x}{2}\right)^n,$$

$$N_0(x) \approx \frac{2}{\pi}\ln\frac{x}{2}, \quad N_n(x) \approx -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n},$$

$$H_0^{(1)}(x) \approx 1 + \frac{2i}{\pi}\ln\frac{x}{2}, \quad H_n^{(1)}(x) \approx -\frac{i}{\pi}(n-1)!\left(\frac{2}{x}\right)^n, \tag{2.2.51}$$

$$n = 1, 2, \ldots,$$

$$H_0^{(2)}(x) \approx 1 - \frac{2i}{\pi}\ln\frac{x}{2}, \quad H_n^{(2)}(x) \approx \frac{i}{\pi}(n-1)!\left(\frac{2}{x}\right)^n,$$

$$K_0(z) \approx -\ln\frac{x}{2}, \quad K_n(x) \approx \frac{1}{2}(n-1)!\left(\frac{2}{x}\right)^n.$$

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8. The Wronskian determinant (Wronskian) of the system of linearly independent solutions to the Bessel equation (2.2.1) is computed from the formulas,

$$W\{J_\lambda(x), J_{-\lambda}(x)\} = -\frac{2\sin\lambda\pi}{\pi x},$$

$$W\{J_\lambda(x), N_\lambda(x)\} = \frac{2}{\pi x},$$

$$W\{J_\lambda(x), H_\lambda^{(1),(2)}(x)\} = \frac{\pm 2i}{\pi x}, \tag{2.2.52}$$

$$W\{H_\lambda^{(1)}(x), H_\lambda^{(2)}(x)\} = -\frac{4i}{\pi x},$$

$$W\{I_\lambda(x), K_\lambda(x)\} = -\frac{1}{x},$$

where, by definition,

$$W\{y_1(x),\ y_2(x)\} = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}.$$

§3. Expansion of a Plane Wave for Cylindrical Wave Functions. Integral Representations of Wave Functions.

All components of the vectors E,H of the plane electromagnetic wave (1.3.4) have as their coefficient the function,

$$\psi = e^{ik(nR)}, \tag{2.3.1}$$

where (nR) is determined, just as in (1.3.6). The function (2.3.1), which obviously satisfies the scalar homogeneous equation of Helmholtz ($n_x^2 + n_y^2 + n_z^2 = 1$), is called a plane scalar wave. In the system of cylindrical coordinates it can be written on the basis of (1.3.7) in the form[1]

$$\psi = e^{ik\rho \sin\theta_0 \cos(\varphi-\varphi_0)} e^{ikz\cos\theta_0}, \tag{2.3.2}$$

where the function,

$$f(\rho,\ \varphi) = e^{ik\rho\sin\theta_0\cos(\varphi-\varphi_0)} \tag{2.3.3}$$

is restricted and is denoted along with its derivatives relative to the arguments ρ, ϕ and is periodic for ϕ with a period of 2π. Therefore, it can be expanded into a complex Fourier series,

$$f(\rho,\ \varphi) = \sum_{n=-\infty}^{\infty} a_n(\rho)\, e^{in\varphi}, \tag{2.3.4}$$

whose coefficients, depending on ρ, are equal to

$$a_n(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{ik\rho\sin\theta_0\cos(\varphi-\varphi_0)-in\varphi}\, d\varphi. \tag{2.3.5}$$

From comparison of (2.3.5) and (2.2.19) we find, if we take into account (2.2.4), that

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[1] In (2.3.1), it is assumed that the direction of the vector n relative to the coordinate axes of the system *Oxyz* is determined by the angles θ_0, ϕ_0.

$$a_n(\rho) = e^{-in(\varphi_0 - \pi/2)} J_n(k\rho \sin\theta_0).$$

As a result the expansion (2.3.4) assumes the form,

$$f(\rho, \varphi) = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho \sin\theta_0) e^{in(\varphi - \varphi_0)}, \qquad (2.3.6)$$

on the basis of which, from (2.3.2) we find the unknown expansion of the plane scalar wave (2.3.1) for the cylindrical wave functions,

$$\psi = e^{ik\mathbf{nR}} = e^{ikz\cos\theta_0} \sum_{n=-\infty}^{\infty} i^n J_n(k\rho \sin\theta_0) e^{in(\varphi - \varphi_0)}. \qquad (2.3.7)$$

If we set here $\theta_0 = \pi/2$, $\phi_0 = \pm \pi/2$, then

$$e^{\pm ik\rho \sin\varphi} = \sum_{n=-\infty}^{\infty} (\pm 1)^n J_n(k\rho) e^{in\varphi}. \qquad (2.3.8)$$

If we use (2.2.28) it is easy to see the absolute and uniform convergence of the obtained expansions for all finite values of the argument $k\rho$ from the interval $[0, \infty)$. Consequently, they allow term-by-term integration. In (2.3.7), let $\theta_0 = \pi/2$. Then, by multiplying both parts of (2.3.7) by $e^{im\phi_0} d\phi_0$ and integrating them over ϕ_0 from 0 to 2π, we obtain /94

$$J_m(k\rho) e^{im\varphi} = \frac{(-i)^m}{2\pi} \int_0^{2\pi} e^{ik(\mathbf{nR}) + im\varphi_0} d\varphi_0, \qquad (2.3.9)$$

where $(\mathbf{nR}) = \rho\cos(\phi - \phi_0)$. Let us find the integral representations of the Hankel wave functions $H_m^{(1),(2)}(x) e^{im\phi}$. For this we shall proceed from the known fact that the function,

$$u(\rho, \varphi) = \int_c e^{ik(\mathbf{nR})} v(\varphi_0) d\varphi_0, \qquad (2.3.10)$$

[where $(\mathbf{nR}) = \rho\cos(\phi - \phi_0)$, but $v(\phi_0)$ is one of the solutions to the equation $v'' + m^2 v = 0$] is the solution to the two-dimensional Helmholtz equation if the integration boundary c is taken such that at its ends, the expression,

$$\left\{ v(\varphi_0) \frac{\partial}{\partial \varphi_0} e^{ik(\mathbf{nR})} - e^{ik(\mathbf{nR})} \frac{\partial}{\partial \varphi_0} v(\varphi_0) \right\} \Bigg|_c$$

vanishes. In particular, if as $v(\phi_0)$ we take the function $e^{im\phi_0}$, then such condition will be satisfied by the boundaries c_1, c_2, c_3 determined just as in the Sommerfeld integrals (2.2.14) - (2.2.16). Since the Helmholtz equation in polar coordinates allows separation of the variables, it is then obvious that the function $u(\rho,\phi)$ may be represented in the form of the product $u(\rho,\phi) = H(\rho)e^{im\phi}$, and then (2.3.10) is written in the form,

$$H(\rho)e^{im\varphi} = \int_{c_{1,2}} e^{ik(nR)+im\varphi_0}\,d\varphi_0. \qquad (2.3.11)$$

Let us set here $\phi = 0$, such that $(nR) = \rho\cos\phi_0$, but

$$H(\rho) = \int_{c_{1,2}} e^{ik\rho\cos\varphi_0+im\varphi_0}\,d\varphi_0. \qquad (2.3.12)$$

If we compare (2.3.12) with (2.2.17) and (2.2.18), we find

$$H(\rho) = \pi i^m H_m^{(1),(2)}(k\rho),$$

as a result of which, from (2.3.11), we find the unknown integral representations of the other wave functions that are valid when Re $k > 0$ for $\rho \neq 0$ and $|\phi| < \frac{\pi}{2}$ (in the case $\frac{\pi}{2} < \phi < \frac{3\pi}{2}$, in the representations $\exp[ik(nR)]$ must be substituted by $\exp[-ik(nR)]$): /9

$$H_m^{(1)}(k\rho)e^{im\varphi} = \frac{(-i)^m}{\pi}\int_{-\pi/2+i\infty}^{\pi/2-i\infty} e^{ik(nR)+im\varphi_0}\,d\varphi_0, \qquad (2.3.13)$$

$$H_m^{(2)}(k\rho)e^{im\varphi} = \frac{(-i)^m}{\pi}\int_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{ik(nR)+im\varphi_0}\,d\varphi_0, \qquad (2.3.14)$$

where $(nR) = \rho\cos(\phi - \phi_0)$. Similarly we can show that

$$J_m(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{(-i)^m}{2\pi}\int_0^{2\pi} e^{ik(nR)}\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\,d\varphi_0,$$

$$H_m^{(1)}(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{(-i)^m}{\pi}\int_{-\pi/2+i\infty}^{\pi/2-i\infty} e^{ik(nR)}\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\,d\varphi_0,$$

$$H_m^{(2)}(k\rho)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{(-i)^m}{\pi}\int_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{ik(nR)}\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\,d\varphi_0. \qquad (2.3.15)$$

§4. Addition Theorems for Cylindrical Wave Functions.

On the surface Oxy let there be two systems of Cartesian rectangular coordinates O, x, y, and O_2, x_2, y_2 with identically oriented and parallel respective axes Ox, Oy and the polar coordinate systems ρ, ϕ related to them by the conversion formulas,

$$x_1 = \rho_1 \cos\varphi_1, \quad y_1 = \rho_1 \sin\varphi_1, \quad x_2 = \rho_2 \cos\varphi_2, \quad y_2 = \rho_2 \sin\varphi_2$$

Furthermore, in each of the coordinate systems let there be determined the wave functions (2.1.10) and (2.1.11) (for generality of the results we will also look at the function $H_m^{(2)}(\rho)e^{im\phi}$),

$$J_m(k\rho_s)e^{im\varphi_s}, \quad s = 1, 2; \quad m = 0, \pm 1, \ldots, \qquad (2.4.1)$$

$$H_m^{(1),(2)}(k\rho_s)e^{im\varphi_s}, \quad s = 1, 2; \quad m = 0, \pm 1, \ldots, \qquad (2.4.2)$$

where ρ_s, ϕ_s are the polar coordinates of the arbitrary point P of the plane Oxy in a system with the origin at O_1 and O_2, respectively (Fig. 13). Let us find formulas which permit each wave function (2.4.1) and (2.4.2), written for example in an s-th system of coordinates (s = 1 or s = 2), be expressed through the same wave functions, but written in another, the j-th system of coordinates ($j \neq s$; j = 2 or j = 1). For this, let us introduce into study the plane scalar wave (2.3.1) with $\phi_0 = \pi/2$ and $\phi_0 = \alpha$, /96

$$\psi = e^{ik(nR_s)} \qquad (2.4.3)$$

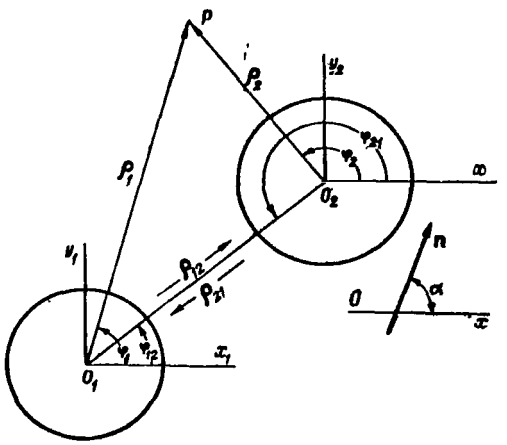

Fig. 13. Addition Theorem for Cylindrical Wave Functions.

($(nR_s) = \rho_s \cos(\phi_s - \alpha)$), written in the s-th system of coordinates. Since $R_s = R_j + R_{sj}$ ($s \neq j$; s, j = 1, 2), then $e^{ik(nR_s)} = e^{ik(nR_j)} e^{ij(nR_{sj})}$, and then on the basis of (2.3.7) ($\theta_0 = \pi/2$) we obtain

$$e^{ik(nR_s)} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho_j) e^{in(\varphi_j - \alpha)} e^{ik(nR_{sj})}. \qquad (2.4.4)$$

Now if we multiply both parts of (2.4.4) by the factor $e^{im\alpha}\,d\alpha$ and integrate them over α along the boundaries c_1, c_2, c_3, we find on the basis of formulas (2.2.17) - (2.2.19) the unknown addition theorems:

$$H_m^{(1),(2)}(k\rho_s)e^{im\varphi_s} =$$

$$= \sum_{n=-\infty}^{\infty} H_{m-n}^{(1),(2)}(kl)J_n(k\rho_j)e^{i(m-n)\varphi_{sj}+in\varphi_j}, \quad l > \rho_j, \tag{2.4.5}$$

$$J_m(k\rho_s)e^{im\varphi_s} =$$

$$= \sum_{n=-\infty}^{\infty} J_{m-n}(kl)J_n(k\rho_j)e^{i(m-n)\varphi_{sj}+in\varphi_j}, \quad l \gtrless \rho_j, \tag{2.4.6}$$

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on the assumption that $l = \rho_{sj}$, $s \neq j$, s, $j = 1,2$. When $l < \rho_j$, we can then similarly find

$$H_m^{(1),(2)}(k\rho_s)e^{im\varphi_s} =$$

$$= \sum_{n=-\infty}^{\infty} H_{m-n}^{(1),(2)}(k\rho_j)J_n(kl)e^{i(m-n)\varphi_j+in\varphi_{sj}}, \tag{2.4.7}$$

if first instead of (2.4.4) we take the expansion

$$e^{ik(\mathbf{nR}_s)} = \sum_{n=-\infty}^{\infty} i^n J_n(k\rho_{sj})e^{in(\varphi_{sj}-\alpha)}\ e^{ik(\mathbf{nR}_j)}.$$

In the obtained formulas, through ρ_{sj}, ϕ_{sj} we denote the polar coordinates of the origin O_j in a system with the origin at O_s. With the above relationships between l and ρ_j, all the cited expansions converge absolutely and uniformly.

Formulas (2.4.5) - (2.4.7) play an important role in the problems of diffraction of waves on several circular cylinders, solved by the method of separation of variables in cylindrical coordinates.

The substitution of the order of summation and integration for the integral with the boundary c_3, carried out in deriving formulas (2.4.5) - (2.4.7), is justified by the uniform convergence of the series (2.4.4). For the integrals with boundaries c_1, c_2, which diverge to infinity, the rule of substitution may be justified by using theorems of the integration of infinite series at an infinite interval (see [28], Supplement II [29]).

The addition theorems (2.4.5) - (2.4.7) may be written in the form,

$$R_m(k\rho_s)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_s) =$$

$$= \sum_{n=-\infty}^{\infty} R_{m-n}(kl)J_n(k\rho_j)\begin{matrix}\cos\\ \sin\end{matrix}[(m-n)\varphi_{sj}+n\varphi_j], \tag{2.4.8}$$

if $l > \rho_j$, or in the form,

$$R_m(k\rho_s)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_s) = \\ = \sum_{n=-\infty}^{\infty} J_n(kl)\, R_{m-n}(k\rho_j)\begin{matrix}\cos\\ \sin\end{matrix}[(m-n)\varphi_{sj} + n\varphi_j], \qquad (2.4.9)$$

if $l < \rho_j$, where $R_m(x)$ is one of the Bessel functions.

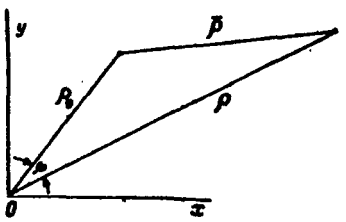

Fig. 14. Addition theorem for the function $H_0^{(1),(2)}(k\bar{\rho})$.

As the particular case from (2.4.5), (2.4.7) we obtain an addition theorem of the simplest form, /98

$$H_0^{(1),(2)}(k\bar{\rho}) = \\ = \sum_{n=-\infty}^{\infty} e^{in\varphi}\begin{cases} J_n(k\rho)\, H_n^{(1),(2)}(k\rho_0), & \rho_0 > \rho, \\ J_n(k\rho_0)\, H_n^{(1),(2)}(k\rho), & \rho > \rho_0, \end{cases} \qquad (2.4.10)$$

for the Hankel function of zero order that is often encountered in different types of applications ($\rho^{-2} = \rho^2 + \rho_0^2 - 2\rho\rho_0\cos\phi$, see Fig. 14).

§5. Expansion of a Spherical Wave for Cylindrical Wave Functions.

The function

$$\psi = e^{ikR}/R, \qquad (2.5.1)$$

which enters into the expressions of the Hertz vectors (1.3.9), (1.3.10) and satisfies the homogeneous Helmholtz equation ($R \neq 0$) is called a spherical wave. Let us find its expansion according to the cylindrical wave functions. For this it is convenient to use the Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ of the equation,

$$\Delta G_k(\mathbf{r}|\mathbf{r}_0) + k^2 G_k(\mathbf{r}|\mathbf{r}_0) = -\delta(\mathbf{r} - \mathbf{r}_0), \qquad (2.5.2)$$

given in the entire space. By definition the Green function $G_k(\mathbf{r}|\mathbf{r}_0)$, being the solution to equation (2.5.2) and being symmetrical relative to the coordinates (x, y, z) and (x_0, y_0, z_0) of the vectors $\mathbf{r}$, $\mathbf{r}_0$ ($\mathbf{r}$, $\mathbf{r}_0$ are vectors which connect the origin with the observation point and with the point sources, respectively; δ is a three-dimensional Dirac δ-function) and also satisfying at infinity the condition of radiation, can be represented in the form, /99

$$G_k(\mathbf{r}|\mathbf{r}_0) = \frac{e^{ik|\mathbf{r}-\mathbf{r}_0|}}{4\pi|\mathbf{r}-\mathbf{r}_0|}. \qquad (2.5.3)$$

When $\mathbf{r} \neq \mathbf{r}_0$ since the Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ is the solution to the homogeneous Helmholtz equation, in a cylindrical coordinate system then it can obviously be represented by an expansion of Fourier integrals such as

$$G_k(\mathbf{r}|\mathbf{r}_0) = \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi_0)} \times$$
$$\times \int_{-\infty}^{\infty} a_n(h) \begin{cases} J_n(v\rho_0) H_n^{(1)}(v\rho), & \rho > \rho_0 \\ J_n(v\rho) H_n^{(1)}(v\rho_0), & \rho < \rho_0 \end{cases} e^{ih(z-z_0)}\, dh. \qquad (2.5.4)$$

In cylindrical coordinates, equation (2.5.2) can be written in the form,

$$\frac{1}{\rho} \cdot \frac{\partial}{\partial \rho}\left(\rho \frac{\partial G_k}{\partial \rho}\right) + \frac{1}{\rho^2} \cdot \frac{\partial^2 G_k}{\partial \varphi^2} + \frac{\partial^2 G_k}{\partial z^2} + k^2 G_k =$$
$$= -\frac{1}{\rho}\delta(\rho-\rho_0)\,\delta(\varphi-\varphi_0)\,\delta(z-z_0). \qquad (2.5.5)$$

Consequently, after multiplying both parts of (2.5.5) by ρ and integrating them over ρ for the small vicinity $\rho = \rho_0$ from $\rho_0 - \varepsilon$ to $\rho_0 + \varepsilon$ after passage to the limit when $\varepsilon \to 0$ we find

$$\rho \frac{\partial G_k}{\partial \rho}\Big|_{\rho=\rho_0-\varepsilon}^{\rho=\rho_0+\varepsilon} = -\delta(\varphi-\varphi_0)\,\delta(z-z_0), \qquad (2.5.6)$$

since the Green function $G_k(\mathbf{r}|\mathbf{r}_0)$ remains continuous when $\rho = \rho_0$. After multiplying both parts of (2.5.6) [where, as $G_k(\mathbf{r}|\mathbf{r}_0)$, we take the expansion (2.5.4)] by $e^{-in\phi}\, d\phi$ and integrating them over ϕ from 0 to 2π, we find

$$4i \int_{-\infty}^{\infty} a_n(h)\, e^{ih(z-z_0)}\, dh = -\delta(z-z_0). \qquad (2.5.7)$$

Since

$$\delta(z \mp z_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ih(z-z_0)} dh \qquad (2.5.8)$$

from comparison of (2.5.7) and (2.5.8) we find that $a_n(h) = \frac{i}{8\pi}$. /100

If we substitute this value $a_n(h)$ into (2.5.4) and assume $R = |\mathbf{r} - \mathbf{r}_0|$, we find the unknown expansion formula,

$$\frac{e^{ikR}}{R} = \frac{i}{2}\sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi_0)} \times$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} J_n(v\rho_0)\, H_n^{(1)}(v\rho), & \rho > \rho_0 \\ J_n(v\rho)\, H_n^{(1)}(v\rho_0), & \rho < \rho_0 \end{array} \right\} e^{ih(z-z_0)}dh \qquad (2.5.9)$$

or

$$\frac{e^{ikR}}{R} = \frac{i}{2}\sum_{n=0}^{\infty} \varepsilon_n \cos n(\varphi - \varphi_0) \times$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} J_n(v\rho_0)\, H_n^{(1)}(v\rho), & \rho > \rho_0 \\ J_n(v\rho)\, H_n^{(1)}(v\rho_0), & \rho < \rho_0 \end{array} \right\} e^{ih(z-z_0)}dh, \qquad (2.5.10)$$

where $\varepsilon_0 = 1$, but $\varepsilon_n = 2$, when $n \geq 1$.

We can arrive at a similar expansion by another means as well, for example, using the representation,

$$\frac{e^{ikR}}{R} = \frac{i}{2}\int_{-\infty}^{\infty} H_0^{(1)}(v\bar{\rho})\, e^{ih(z-z_0)}\, dh \qquad (2.5.11)$$

and the addition theorems (2.4.10) applied to the subintegral Hankel function in (2.5.11): $\rho^{-2} = \rho^2 + \rho_0^2 - 2\rho\rho_0 \cos(\phi - \phi_0)$.

CHAPTER 3

SPHERICAL WAVE FUNCTIONS AND CERTAIN OF THEIR PROPERTIES

§1. The Helmholtz Equation in Spherical Coordinates. Spherical Wave Functions.

The Helmholtz scalar wave equation in the system of spherical coordinates r, θ, ϕ has the form of (1.6.26). If we seek the particular solution in the form of the product of three functions, /101

$$\bar{u}(r, \theta, \varphi) = R(r)\Theta(\theta)\Phi(\varphi) \not\equiv 0, \tag{3.1.1}$$

each of which depends only on a single coordinate, then after substitution of (3.1.1) into (1.6.26) and subsequent separation of variables in the equation, we find that the particular solution to equation (1.6.26) may be represented in the form of (3.1.1) if, as the functions R, and Φ, we take any solutions of the respective equations:

$$\frac{d^2R}{dr^2} + \frac{2}{r} \cdot \frac{dR}{dr} + \left(k^2 - \frac{\lambda}{r^2}\right) R = 0, \tag{3.1.2}$$

$$\frac{1}{\sin\theta} \cdot \frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + \left(\lambda - \frac{\mu}{\sin^2\theta}\right)\Theta = 0, \tag{3.1.3}$$

$$\frac{d^2\Phi}{\partial\varphi^2} + \mu\Phi = 0, \tag{3.1.4}$$

in which the parameters λ, μ are separation constants. The values of the separation constants are determined from the requirement of uniqueness and continuity of the solution to (3.1.1) at any given point of space. In particular, the function $\Phi(\phi)$ must be periodic with a period 2π. Since the particular solutions to equation (3.1.4) are the functions $\begin{matrix}\cos \\ \sin\end{matrix}(\sqrt{\mu\phi})$, then it is obvious that the region of variation in the parameter μ must be bounded by the squares of the integers: $\mu = m^2$, where $m = 0, \pm 1, \pm 2, \ldots$ Then the periodic solutions to equation (3.1.4) will be the functions /102

$$\Phi(\varphi) = \begin{matrix}\cos \\ \sin\end{matrix}(m\varphi), \tag{3.1.5}$$

or in complex form,

$$\Phi(\varphi) = e^{im\varphi}. \tag{3.1.6}$$

If we substitute $\mu = m^2$ into equation (3.1.3), then this latter equation will have solutions that are restricted at the singular points $\theta = 0$, $\theta = \pi$ only under the condition that the parameter $\lambda = n(n + 1)$, where the values of n are integers and $n \geq 0$. Such solutions are the so-called Legendre adjoint functions,

$$\Theta(\theta) = P_n^m(\cos\theta). \tag{3.1.7}$$

The properties of the Legendre adjoint functions are described in [24, 29, 31, 34 - 36, 41]. Several of them will be given below.

When $\lambda = n(n + 1)$ equation (3.1.2), by substitution of $R = \frac{1}{\sqrt{kr}} v(r)$, will be reduced to the normal form of the Bessel equation. Its particular solutions will be the functions

$$R(r) = Z_n(kr), \tag{3.1.8}$$

where through $Z_n(x)$ we denote one of the so-called Bessel spherical functions,

$$j_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x),\ n_n(x) =$$

$$= \sqrt{\frac{\pi}{2x}}\, N_{n+1/2}(x),\ h_n^{(1),(2)}(x) = \sqrt{\frac{\pi}{2x}}\, H_{n+1/2}^{(1),(2)}(x), \tag{3.1.9}$$

where $h_n^{(1),(2)}(x) = j_n(x) \pm i n_n(x)$, but $J_{n+1/2}(x)$, $N_{n+1/2}(x)$, $H_{n+1/2}^{(1),(2)}(x)$ are Bessel functions of half-integral order (the factor $\sqrt{\pi/2}$ is added for convenience of computation). The function $j_n(x)$ has no singularity at the point $x = 0$, whereas the function $h_n^{(1)}(x)$ has a band of n-th order there; at infinity it satisfies the condition of radiation. Consequently, the partial solutions to the Helmholtz equation (1.6.26), represented in the form of (3.1.1) are the functions,

$$j_n(kr)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} (m\varphi) \text{ and } j_n(kr)\, P_n^m(\cos\theta)\, e^{im\varphi}, \tag{3.1.10}$$

$$h_n^{(1)}(kr)\, P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix} (m\varphi) \text{ and } h_n^{(1)}(kr)\, P_n^m(\cos\theta)\, e^{im\varphi}, \tag{3.1.11}$$

the first of which are periodic for ϕ with the period 2π; they are restricted when $\theta = 0$ and $\theta = \pi$ and regular in the region containing the origin ($r = 0$), and the second of which are periodic for ϕ with the same period; they are restricted when $\theta = 0$ and $\theta = \pi$ and regular in the region not containing the origin and furthermore /103

satisfy the condition of radiation at infinity. Functions (3.1.10), (3.1.11) are called spherical wave functions. By virtue of the linearity and homogeneity of equation (1.6.26), its general solution may be represented in the form of a series,

$$\sum_{n=0}^{\infty}\sum_{m=0}^{n} j_n(kr)(a_{mn}\cos m\varphi + b_{mn}\sin m\varphi) P_n^m(\cos\theta) \qquad (3.1.12)$$

or in the form,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} A_{mn} j_n(kr) P_n^m(\cos\theta) e^{im\varphi} \qquad (3.1.13)$$

and respectively,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} h_n^{(1)}(kr)(c_{mn}\cos m\varphi + d_{mn}\sin m\varphi) P_n^m(\cos\theta) \qquad (3.1.14)$$

or in the form,

$$\sum_{n=0}^{\infty}\sum_{m=-n}^{n} B_{mn} h_n^{(1)}(kr) P_n^m(\cos\theta) e^{im\varphi}. \qquad (3.1.15)$$

The expansion coefficients A_{mn}, B_{mn} and others depend on m and n and are subject to determination from the boundary conditions.

§2. The Legendre Equation. The Legendre Functions.

The differential equation,

$$\frac{d}{dz}\left[(1-z^2)\frac{dy}{dz}\right] + \lambda y = 0 \qquad (3.2.1)$$

is called the Legendre equation Equation (3.1.3) is reduced to the Legendre equation by substituting $z = \cos\theta$ when $\mu = 0$. It is charactized by two regular singular points $z = +1$ and $z = -1$. In the general case the argument z and the parameter λ of the Legendre equation may assume any real and complex values. The particular solutions to equation (3.2.1) with arbitrary λ are called Legendre /10
functions. In the particular case of real values $\lambda = n(n+1)$ with negative integral values of n, the Legendre equation allows two linearly independent solutions[1] $y_1 = P_n(z)$ and $y_2 = Q_n(z)$, which

[1] We make no distinctions in this book between the designations of the Legendre functions and the adjoint Legendre functions, which refer to the area $-1 \leq z \leq 1$ and to all other cases.

can be represented in the form,

$$P_n(z) = \frac{2n!}{2^n (n!)^2} z^n F\left(-\frac{n}{2}, \frac{1-n}{2}; \frac{1}{2} - n; \frac{1}{z^2}\right), \qquad (3.2.2)$$

$$Q_n(z) = \frac{n!}{(2n+1)!!} \cdot \frac{1}{z^{n+1}} \times$$

$$\times F\left(\frac{n+1}{2}, \frac{n+2}{2}; n + \frac{3}{2}; \frac{1}{z^2}\right), \qquad (3.2.3)$$

if we seek them in the form of a generalized power series with increasing and decreasing exponents of the argument z, respectively. Here through F we denote the hypergeometric series,

$$F(\alpha, \beta; \gamma; z) = \sum_{m=0}^{\infty} \frac{(\alpha)_m (\beta)_m}{m! (\gamma)_m} z^m, \qquad (3.2.4)$$

where $(a)_m = a(a+1)\ldots(a+m-1)$, $m \geq 1$, but $(a)_0 = 1$. It is easy to see than the series (3.2.2) contains only a finite number of terms that are non-zero, regardless of the choice of values of the argument z, which may also be complex (if n is an even number, then the series (3.2.2) is cut off at the term of n-th power, if n is an odd number, then the series (3.2.2) is cjt off at the $n-1$ - th power). Its sum when $z = 1$ is equal to unity. Consequently, the particular solution to the Legendre equation $P_n(z)$, determined by the series (3.2.2), is an n-th power polynomial relative to the argument z. The solution to equation (3.2.1) is called a Legendre polynomial. The series (3.2.3) is infinite, converging for all values $|z| > 1$ and diverging inside the interval $(-1, +1)$ and at its ends $z = \pm 1$ (at the singular points of the equation). The function $Q_n(z)$ determined by it, the second particular solution to the Legendre equation, is called a Legendre function of the second sort. Thus, the general solution to the Legendre equation in the region where $|z| > 1$, can be represented by the linear combination,

$$y(z) = C_1 P_n(z) + C_2 Q_n(z), \qquad (3.2.5)$$

where C_1, C_2 are arbitrary constants.

If we know the first solution to the Legendre equation in the /105
interval $(-1, +1)$, i.e., the Legendre polynomial $P_n(z)$, the second solution inside this interval which is linearly independent of $P_n(z)$ can be sought in the form of an integral,

$$P_n(z) \int \frac{dz}{(1-z^2) P_n^2(z)},$$

by computing which and denoting the result of the computation through $Q_n(z)$, we obtain

$$Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{1+z}{1-z} - W_{n-1}(z) \qquad (3.2.6)$$

(z is real and z (- (-1, +1)), where

$$W_{n-1}(z) = \sum_{s=1}^{n} \frac{1}{s} P_{s-1}(z) P_{n-s}(z), \qquad (3.2.7)$$

but $P_n(z)$ is the Legendre polynomial. As is obvious from (3.2.6), the Legendre function of the second sort $Q_n(z)$, determined in (-1, +1) by this expression, has at the ends an interval of logarithmic singularity. The general solution to the Legendre equation inside the interval (-1, +1) can also be represented in the form of the linear combination (3.2.5), where just as earlier $P_n(z)$ is the Legendre polynomial, but $Q_n(z)$ is given by expression (3.2.6).

Let us note that if we seek the Legendre function of the second sort outside the interval (-1, +1) in a similar manner, through the first solution $P_n(z)$, we then find that both for real and for complex values of the argument z, which satisfy the condition $|z| > 1$, it is represented by the expression,

$$Q_n(z) = \frac{1}{2} P_n(z) \ln \frac{z+1}{z-1} - W_{n-1}(z), \qquad (3.2.8)$$

which differs from (3.2.6) only by the substitution of $\ln \frac{1+z}{1-z}$ by $\ln \frac{z+1}{z-1}$. It is easy to see that the function $Q_n(z)$, given by the series (3.2.3), and the function $Q_n(z)$, given by expression (3.2.8), agree in accuracy for positive integral values of n.

From (3.2.6) and (3.2.8) it is clear that on the plane of the complex variable z, the Legendre function of the second sort has two branching points: $z = +1$ and $z = -1$. Therefore, to find the unique branch of this function, we must join the points $z = +1$, $z = -1$ with the branch cut.

The Legendre polynomials $P_n(z)$ and the functions $Q_n(z)$ of the integral positive index n are particular forms of the Legendre equation with arbitrary real or complex values of the parameter λ. In the general case, the solutions to the Legendre equation are denoted through $P_\lambda(z)$ and $Q_\lambda(z)$ and are called Legendre functions of first and second sort, respectively. Their explicit form is found from equation (3.2.1) by the usual manner using generalized power series and is written in the form, /1

$$P_\lambda(z) = F\left(-\lambda,\ \lambda+1;\ 1;\ \frac{1-z}{2}\right), \tag{3.2.9}$$

$$Q_\lambda(z) = -\frac{\sqrt{\pi}\,\Gamma(\lambda+1)}{\Gamma\left(\lambda+\frac{3}{2}\right)(2z)^{\lambda+1}} \times$$

$$\times F\left(\frac{\lambda}{2}+1,\ \frac{\lambda}{2}+\frac{1}{2};\ \lambda+\frac{3}{2};\ \frac{1}{z^2}\right) \tag{3.2.10}$$

on the assumption that in (3.2.9) the values of z satisfy the inequality $|z - 1| < 2$, but in (3.2.10) they satisfy the condition $|z| > 1$, $|\arg z| < \pi$, $\lambda \neq -1, -2, \ldots$ Using these formulas, the functions $P_\lambda(z)$, $Q_\lambda(z)$ are determined only in the bound part of the complex plane z, however, they allow analytical continuation also to a wider region of variation in the argument z. For the function $P_\lambda(z)$ such a continuation is given by the formula,

$$P_\lambda(z) = \frac{2}{\pi}\int_0^{\pi/2} F_\lambda\left(\frac{z-1}{2}\sin^2\varphi\right)d\varphi, \tag{3.2.11}$$

where $|\arg(z+1)| < \pi$, but

$$F_\lambda(z) = \frac{\left[(1+z)^{1/2}+z^{1/2}\right]^{2\lambda+1}+\left[(1+z)^{1/2}+z^{1/2}\right]^{-2\lambda-1}}{2(1+z)^{1/2}}.$$

The analytical continuation of the function $Q_\lambda(z)$ is given by the formula,

$$Q_\lambda(z) = \int_0^\infty G_\lambda\left(\frac{z-1}{2}\operatorname{ch}^2\varphi\right)d\varphi, \tag{3.2.12}$$

where $|\arg(z-1)| < \pi$, $\operatorname{Re}\lambda > -1$, but

$$G_\lambda(z) = \frac{\left[(1+z)^{1/2}+z^{1/2}\right]^{-2\lambda-1}}{(1+z)^{1/2}}.$$

The function $P_\lambda(z)$, determined by the integral (3.2.11) satisfies the Legendre equation; when $z = 1$, it is equal to unity and is a regular function of the complex variable z on the entire plane z with the branch cut along the segment of the real axis $(-\infty, -1)$. In the region $|z - 1| < 2$, it is identical to the function $P_\lambda(z)$, determined by formula (3.2.9). From (3.2.9) and (3.2.11) it is clear that $P_\lambda(z)$ is an integral function of the index λ. /107

The function $Q_\lambda(z)$ determined by the integral (3.2.12) satis-

fies the Legendre equation and is a regular function of the complex variable z on the entire plane with the branch cut along the real axis from $-\infty$ to $+1$. When $z \to 1$ from the right the function $Q_\lambda(z) \to +\infty$. Consequently it is independent of $P_\lambda(z)$. The values of $Q_\lambda(z)$, given by formulas (3.2.10) and (3.2.12), coincide in the common region of determination, and they are identical there. From these same formulas it is clear that in the half-plane Re $\lambda > -1$, the function $Q_\lambda(z)$ is an analytical function of the index λ (on the entire λ-plane, the function $Q_\lambda(z)$ is meromorphic with simple poles at the points $\lambda = -1, -2, \ldots$).

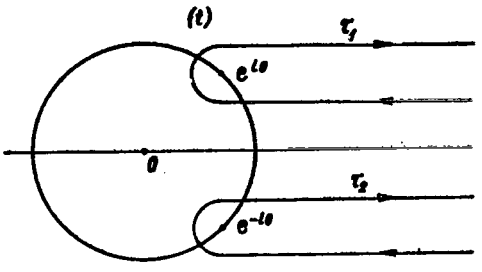

Fig. 15. Integration Boundaries τ_1, τ_2.

Since equation (3.2.1) does not vary its form when the parameter λ is substituted in it by the parameter $-\lambda$ or when the argument z is substituted by the argument $-z$, then along with the functions $P_\lambda(z)$ and $Q_\lambda(z)$, its solution will also be the functions $P_{-\lambda-1}(z)$, $Q_{-\lambda-1}(z)$, $P_\lambda(-z)$ $Q_\lambda(-z)$, and

$$\begin{aligned} P_{-\lambda-1}(z) &= P_\lambda(z), \\ Q_\lambda(-z) &= -e^{\pm i\lambda\pi} Q_\lambda(z), \\ \frac{2}{\pi} \sin(\lambda\pi) Q_\lambda(z) &= P_\lambda(z) e^{\mp i\lambda\pi} - P_\lambda(-z) \end{aligned} \qquad (3.2.13)$$

(in the two latter formulas it is assumed that $\lambda \neq -1, -2, \ldots$; the superscript is taken for Im $z > 0$, and the subscript for Im $z < 0$ when a branch cut exists along $(-\infty, 1)$). In the case of integral positive values of $\lambda = n$,

$$P_n(-z) = (-1)^n P_n(z), \quad Q_n(-z) = (-1)^{n+1} Q_n(z). \qquad (3.2.14)$$

In applications to problems of diffraction it appears to be useful /1 if we represent the function $P_\lambda(z)$ in the form of a sum of two functions,

$$P_\lambda(\cos\theta) = Q_\lambda^{(1)}(\theta) + Q_\lambda^{(2)}(\theta), \quad z = \cos\theta, \qquad (3.2.15)$$

where the functions $Q_\lambda^{(j)}(\theta)$ for Re $\lambda > -1$ are determined by the integrals [125],

$$Q_\lambda^{(j)}(\theta) = \frac{1}{2\pi i}\int\limits_{(\tau_j)} \frac{dt}{t^{\lambda+1}\sqrt{1+t^2-2t\cos\theta}}, \quad j=1,\ 2,$$

with the integration boundaries shown on Figure 15.

The functions $Q_\lambda^{(1)}(\theta)$ and $Q_\lambda^{(2)}(\theta)$ are mutually related by the relationships,

$$Q_\lambda^{(1)}(\theta) = e^{-i\pi\lambda} Q_\lambda^{(2)}(\pi-\theta),\ Q_\lambda^{(2)}(\theta) = e^{i\pi\lambda} Q_\lambda^{(1)}(\pi-\theta). \qquad (3.2.16)$$

From (3.2.15) and (3.2.16) it follows that

$$P_\lambda(-\cos\theta) = e^{i\pi\lambda} P_\lambda(\cos\theta) - 2i\sin(\lambda\pi)\, Q_\lambda^{(2)}(\theta), \qquad (3.2.17)$$

$$P_\lambda(\cos\theta) = e^{i\pi\lambda} P_\lambda(-\cos\theta) - 2i\sin(\lambda\pi)\, Q_\lambda^{(2)}(\pi-\theta). \qquad (3.2.18)$$

When $|\lambda \sin\theta| >> 1$ and Re $\lambda > -1$, for the functions $P_\lambda(\cos\theta)$, $Q_\lambda^{(1),(2)}(\theta)$, we find the asymptotic representations:

$$Q_\lambda^{(1)}(\theta) \simeq e^{-i\left(\lambda+\frac{1}{2}\right)\theta+i\pi/4} \Big/ \sqrt{2\pi\lambda\sin\theta}, \qquad (3.2.19)$$

$$Q_\lambda^{(2)}(\theta) \simeq e^{\left(\lambda+\frac{1}{2}\right)\theta-i\pi/4} \Big/ \sqrt{2\pi\lambda\sin\theta}, \qquad (3.2.20)$$

$$P_\lambda(\cos\theta) = \sqrt{\frac{2}{\pi\lambda\sin\theta}}\cos\left[\left(\lambda+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right]. \qquad (3.2.21)$$

If $|\lambda| >> 1$, $\theta \approx 0$, then

$$P_\lambda(\cos\theta) \simeq J_0\left[\left(\lambda+\frac{1}{2}\right)\theta\right]. \qquad (3.2.22)$$

If $|\lambda| >> 1$, but the quantity $|\lambda(\pi-\theta)|$ is not too high in comparison with unity, then asymptotically,

$$P_\lambda(-\cos\theta) \simeq J_0\left[\left(\lambda+\frac{1}{2}\right)(\pi-\theta)\right]. \qquad (3.2.23)$$

In the interval (-1, +1), the Legendre functions satisfy the inequalities, /109

$$\left|P_\lambda(\cos\theta)\right| \leqslant \frac{2}{\sqrt{\lambda\pi\sin\theta}}, \quad |Q_\lambda(\cos\theta)| \leqslant \frac{\sqrt{\pi}}{\sqrt{\lambda\sin\theta}}. \qquad (3.2.24)$$

Now let us look at several properties of the Legendre polynomials. Their important properties, which play a large role in the theory of expansion of arbitrary functions into series of Legendre polynomials, are the orthogonality and completeness of the system of polynomials $P_n(z)$ in the interval [-1, +1], and

$$\int_{-1}^{+1} P_n(z)\,P_m(z)\,dz = \begin{cases} 0, & n \neq m, \\ \dfrac{2}{2n+1}, & n = m. \end{cases} \qquad (3.2.25)$$

We can show that any function $f(z)$ that is sectionally continuous, along with the first order derivative at each point of continuity $f(z)$, is expanded into a Fourier series for the Legendre polynomials, that converge to the function $f(z)$. When $\lambda = n(n+1)$ (n is an integer, $n \geq 0$), the Legendre equation (3.2.1) has no solutions that are bounded at the singular points $z = \pm 1$ that differ from the Legendre polynomials.

If we assume in (3.2.2) consecutively that $n = 0, 1, 2, \ldots$, we find

$$P_0(z) = 1, \quad P_1(z) = z, \quad P_2(z) = \frac{1}{2}(3z^2 - 1) \qquad (3.2.26)$$

etc. The particular solutions to the Legendre polynomials are:

$$P_{2k}(0) = (-1)^k \frac{(2k-1)!!}{2k!!}, \quad P_{2k+1}(0) = 0, \qquad (3.2.27)$$

$$P'_{2k}(0) = 0, \quad P_{2k+1}(0) = (-1)^k \frac{(2k+1)!!}{2^k k!} \qquad (3.2.28)$$

For all values of n and z (- [-1, +1] we find the inequality,

$$|P_n(z)| \leqslant 1.$$

If in (3.2.26) we assume consecutively that $n = 0, 1, \ldots$, we find

$$Q_0(z) = \frac{1}{2}\ln\frac{1+z}{1-z},$$

$$Q_1(z) = \frac{z}{2} \ln \frac{1+z}{1-z} - 1, \qquad (3.2.29)$$
$$Q_2(z) = \left(\frac{3z^2}{4} - \frac{1}{4}\right) \ln \frac{1+z}{1-z} - \frac{3}{2} z$$

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etc., where $z \in (-1, +1)$. After substituting here $\ln \frac{1+z}{1-z}$ with the expression $\ln \frac{z+1}{z-1}$, we find the formula for computing the particular values of the functions $Q_n(z)$ in the region $|z| > 1$.

When $z = 0$ we have

$$Q_{2k}(0) = 0, \quad Q_{2k+1}(0) = (-1)^{k+1} \frac{(2k)!!}{(2k+1)!!}, \qquad (3.2.30)$$

and when $z = \pm 1$

$$Q_n(+1) = +\infty, \quad Q_{2k}(-1) = -\infty, \quad Q_{2k+1}(-1) = +\infty. \qquad (3.2.31)$$

For the function $Q_n(z)$ determined in the region $|z| > 1$,

$$\lim_{|z| \to \infty} Q_n(z) = 0.$$

§3. Equation of the Adjoint Legendre Functions. Adjoint Legendre Functions.

The equation,

$$\frac{d}{dz}\left[(1-z^2)\frac{dy}{dz}\right] + \left[\lambda(\lambda+1) - \frac{\mu^2}{1-z^2}\right] y = 0 \qquad (3.3.1)$$

with arbitrary values of the argument z and the parameters λ, μ is called the equation of the adjoint Legendre functions [equation (3.1.3) is reduced to (3.3.1) by substituting $z = \cos\theta$). It is characterized by two regular singular points: $z = +1$ and $z = -1$. The particular solutions to equation (3.3.1) are called the adjoint Legendre functions. Since, when $\mu = 0$, equation (3.3.1) converts to the Legendre equation (3.2.1), then it is obvious that they can be studied as a generalization of the Legendre functions $P_\lambda(z)$ and $Q_\lambda(z)$. The linearly independent solutions to equation (3.3.1) are denoted through $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ and are called adjoint Legendre functions of first and second sort, with the respective power of λ and of μ order.

In application we are interested in the solutions to the equation in (3.3.1) which represent on the one hand the real values of the argument z belonging to the segment [-1, +1] and on the other hand solutions representing any complex value of z on the z-plane. /1

When z is real and $z \in (-1, +1)$, the adjoint Legendre functions of first sort are determined for all values of λ and μ by the expression,

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{1+z}{1-z}\right)^{\mu/2} \times \times F\left(-\lambda, \lambda+1; 1-\mu; \frac{1-z}{2}\right), \qquad (3.3.2)$$

where F is a hypergeometric series (3.2.4). Equation (3.3.1) does not change sign with substitution in it of λ by $-\lambda -1$. Therefore, along with $P_\lambda^\mu(z)$ its solution will also be the function $P^\mu_{-\lambda-1}(z)$. From the symmetry of the series (3.2.4) relative to the first two parameters, it follows that

$$P^\mu_{-\lambda-1}(z) = P_\lambda^\mu(z). \qquad (3.3.3)$$

Equation (3.3.1) also does not change with substitution of z by $-z$, by virtue of which it will be satisfied also by the functions $P_\lambda^\mu(z)$ which is linearly independent of (3.3.2) with non-integral values of $\lambda + \mu$. In this case the general solution to equation (3.3.1) in the interval (-1, +1) will be represented by the linear combination,

$$y = c_1 P_\lambda^\mu(z) + c_2 P_\lambda^\mu(-z), \qquad (3.3.4)$$

where c_1, c_2 are arbitrary constants. With integral values of $\lambda + \mu$ the functions $P_\lambda^\mu(z)$ and $P_\lambda^\mu(-z)$ are connected by the relationship,

$$P_\lambda^\mu(z) = (-1)^{\lambda+\mu} P_\lambda^\mu(-z), \qquad (3.3.5)$$

from which their linear dependence follows. In this case the second solution to equation (3.3.1) $Q_\lambda^\mu(z)$, which is independent of $P_\lambda^\mu(z)$ in the interval (-1, +1) may be determined by the relationship,

$$Q_\lambda^\mu(z) = \lim_{(\lambda+\mu)\to n} \frac{\pi}{2\sin(\lambda+\mu)\pi} \times \times \{P_\lambda^\mu(z)\cos(\lambda+\mu)\pi - P_\lambda^\mu(-z)\}, \qquad (3.3.6)$$

where n is an integral number.

The adjoint Legendre functions for real values of $z \in (-1, +1)$ may be determined also by the expressions, /112

$$P_\lambda^\mu(z) = \frac{\sqrt{\pi}\,2^\mu}{(1-z^2)^{\mu/2}} \left\{ \frac{F\left(-\frac{\lambda+\mu}{2}, \frac{1-\mu+\lambda}{2}; \frac{1}{2}; z^2\right)}{\Gamma\left(\frac{1-\lambda-\mu}{2}\right)\Gamma\left(1+\frac{\lambda-\mu}{2}\right)} - \right.$$

$$\left. - \frac{2zF\left(\frac{1-\lambda-\mu}{2}, 1+\frac{\lambda-\mu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)\Gamma\left(-\frac{\lambda+\mu}{2}\right)} \right\}, \qquad (3.3.7)$$

$$Q_\lambda^\mu(z) = \frac{\pi^{3/2}2^\mu}{(1-z^2)^{\mu/2}} \times$$

$$\times \left\{ \frac{\operatorname{ctg}\left[\frac{\pi}{2}(\lambda+\mu)\right] zF\left(\frac{1-\lambda-\mu}{2}, 1+\frac{\lambda-\mu}{2}; \frac{3}{2}; z^2\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)\Gamma\left(-\frac{\lambda+\mu}{2}\right)} - \right.$$

$$\left. - \frac{\operatorname{tg}\left[\frac{\pi}{2}(\lambda+\mu)\right] F\left(-\frac{\lambda+\mu}{2}, \frac{1+\lambda-\mu}{2}; \frac{1}{2}; z^2\right)}{2\Gamma\left(1+\frac{\lambda-\mu}{2}\right)\Gamma\left(\frac{1-\lambda-\mu}{2}\right)} \right\}, \qquad (3.3.8)$$

which are convenient for computation of these functions in the vicinity of $z = 0$. The functions $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ for any values of λ and μ satisfy, in the interval $(-1, +1)$, the relationships:

$$2Q_\lambda^\mu(z)\sin(\mu\pi) =$$

$$= \pi\left[P_\lambda^\mu(z)\cos(\mu\pi) - \frac{\Gamma(\lambda+\mu+1)}{\Gamma(\lambda-\mu+1)} P_\lambda^{-\mu}(z)\right], \qquad (3.3.9)$$

$$P_\lambda^\mu(-z) = P_\lambda^\mu(z)\cos[\pi(\lambda+\mu)] -$$

$$- \frac{2}{\pi} Q_\lambda^\mu(z)\sin[\pi(\lambda+\mu)], \quad 0 < z < 1, \qquad (3.3.10)$$

$$Q_\lambda^\mu(-z) = -Q_\lambda^\mu(z)\cos[\pi(\lambda+\mu)] -$$

$$- \frac{\pi}{2} P_\lambda^\mu(z)\sin[\pi(\lambda+\mu)], \quad 0 < z < 1, \qquad (3.3.11)$$

$$Q_{-\lambda-1}^\mu(z) = \frac{\sin[\pi(\lambda+\mu)]}{\sin[\pi(\lambda-\mu)]} Q_\lambda^\mu(z) - \qquad (3.3.12) \quad /11$$

$$- \frac{\pi\cos(\lambda\pi)\cos(\mu\pi)}{\sin[\pi(\lambda+\mu)]} P_\lambda^\mu(z),$$

$$P_\lambda^{-\mu}(z) = \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)} \times$$

$$\times \left[P_\lambda^\mu(z)\cos(\mu\pi) - \frac{2}{\pi}\sin(\mu\pi)\, Q_\lambda^\mu(z) \right], \qquad (3.3.13)$$

$$Q_\lambda^{-\mu}(z) = \frac{\Gamma(\lambda-\mu+1)}{\Gamma(\lambda+\mu+1)} \times$$

$$\times \left[Q_\lambda^\mu(z)\cos(\mu\pi) + \frac{\pi}{2}\sin(\mu\pi)\, P_\lambda^\mu(z) \right]. \qquad (3.3.14)$$

From (3.3.11) it follows that for any integral values of $\lambda = n$ and $\mu = m$,

$$Q_n^m(-z) = (-1)^{m+n+1} Q_n^m(z) \qquad (3.3.15)$$

When $z = 0$, from (3.3.7) and (3.3.8) it follows that:

$$P_\lambda^\mu(0) = \frac{2^\mu}{\sqrt{\pi}} \cos\left[\frac{\pi}{2}(\lambda+\mu)\right] \frac{\Gamma\left(\frac{1+\lambda+\mu}{2}\right)}{\Gamma\left(1+\frac{\lambda-\mu}{2}\right)}, \qquad (3.3.16)$$

$$Q_\lambda^\mu(0) = -2^{\mu-1}\sqrt{\pi}\sin\left[\frac{\pi}{2}(\lambda+\mu)\right] \frac{\Gamma\left(\frac{1+\lambda+\mu}{2}\right)}{\Gamma\left(1+\frac{\lambda-\mu}{2}\right)}, \qquad (3.3.17)$$

$$P_\lambda^\mu(0) = \frac{2^{\mu+1}}{\sqrt{\pi}}\sin\left[\frac{\pi}{2}(\lambda+\mu)\right] \frac{\Gamma\left(1+\frac{\lambda+\mu}{2}\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)}, \qquad (3.3.18)$$

$$Q_\lambda^{\mu\prime}(0) = 2^\mu\sqrt{\pi}\cos\left[\frac{\pi}{2}(\lambda+\mu)\right] \frac{\Gamma\left(1+\frac{\lambda+\mu}{2}\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)}, \qquad (3.3.19)$$

From these expression for the adjoint Legendre functions it is obvious that for arbitrary values of the parameters λ and μ, equation (3.3.1) has no solutions that are bounded in the entire interval $[-1, +1]$. Equation (3.3.1) will possess such solutions only for integral values of the parameters $\lambda = n$ and $\mu = m$, when $n \geq 0$. /11

In the case of integral positive values of $\lambda = n$ and $\mu = m$ the adjoint Legendre functions of first and second sort are given by the expressions,

$$P_n^m(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \frac{d^m P_n(z)}{dz^m}, \qquad (3.3.20)$$

$$z \in (-1, +1)$$

$$Q_n^m(z) = (-1)^m (1-z^2)^{\frac{m}{2}} \frac{d^m Q_n(z)}{dz^m}, \qquad (3.3.21)$$

in which $P_n(z)$ and $Q_n(z)$ for integral values of $n \geq 0$ were determined in the preceding section. From the properties of the functions $P_n(z)$ and $Q_n(z)$ it follows that functions (3.3.20), (3.3.21) are regular functions of the argument z inside the interval (-1, +1) and the unique finite solutions to equation (3.3.1) in the interval [-1, +1] give only the Legendre functions of first sort $P_n^m(z)$. From (3.3.20) it is obvious that when $m > n$, $P_n^m(z) \equiv 0$. For all other values of $0 \leq m \leq n$, the function $P_n^m(z)$ is a polynomial relative to the z power $n - m$.

For integral values of $\lambda = n$, $\mu = m$,

$$P_n^{-m}(z) = (-1)^m \frac{(n-m)!}{(n+m)!} P_n^m(z), \qquad (3.3.22)$$

$$z \in (-1, +1)$$

$$Q_n^{-m}(z) = (-1)^m \frac{(n-m)!}{(n+m)!} Q_n^m(z). \qquad (3.3.23)$$

In the interval (-1, +1), the functions $P_n^m(z)$ satisfy the recurrent relationships,

$$P_n^{m+2}(z) + \frac{2z(m+1)}{\sqrt{1-z^2}} \times$$

$$\times P_n^{m+1}(z) + (n-m)(n+m+1) P_n^m(z) = 0,$$

$$P_{n+1}^m(z) - P_{n-1}^m(z) - (2n+1)\sqrt{1-z^2}\, P_n^{m-1}(z) = 0. \qquad (3.3.24)$$

$$(2n+1) z P_n^m(z) - (n+m) P_{n-1}^m(z) - (n-m+1) P_{n+1}^m(z) = 0$$

and others which remain valid for the function $Q_n^m(z)$ as well. The /115
particular values of the adjoint Legendre functions with integral and positive indices $\lambda = n$, $\mu = m$ are:

$$P_n^m(0) = \begin{cases} 0, & (n-m) \text{ odd} \\ \dfrac{(-1)^{\frac{n-m}{2}} (n+m)!}{2^n \left(\frac{n-m}{2}\right)! \left(\frac{n+m}{2}\right)!}, & (n-m) \text{ even} \end{cases} \qquad (3.3.25)$$

$$P_n^{m'}(0) = \begin{cases} 0, & (n-m) \text{ even} \\ \dfrac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}, & (n-m) \text{ odd} \end{cases} \tag{3.3.26}$$

$$Q_n^m(0) = \begin{cases} 0, & (n-m) \text{ even} \\ \dfrac{(-1)^{\frac{n-m+1}{2}} 2^{n-1}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m-1}{2}\right)!}{(n-m)!}, & (n-m) \text{ odd} \end{cases} \tag{3.3.27}$$

$$Q_n^{m'}(0) = \begin{cases} 0, & (n-m) \text{ odd} \\ (-1)^{\frac{n-m}{2}} 2^{n-1} \dfrac{\left(\frac{n-m-2}{2}\right)!\left(\frac{n+m}{2}\right)!}{(n-m-1)!}, & (n-m) \text{ even.} \end{cases} \tag{3.3.28}$$

When $z = \pm$ ($m \neq 0$)

$$P_n^m(\pm 1) = 0, \; Q_n^m(\pm 1) = \infty. \tag{3.3.29}$$

From (3.3.20) and (3.3.21) it follows that

$$\begin{aligned} P_1^1(z) &= (1-z^2)^{1/2}, \\ P_2^1(z) &= 3z(1-z^2)^{1/2}, \\ P_2^2(z) &= 3(1-z^2) \end{aligned} \tag{3.3.30}$$

/1

etc.,

$$\begin{aligned} Q_1^1(z) &= (1-z^2)^{1/2}\left[\frac{1}{2}\ln\frac{1+z}{1-z} + \frac{z}{1-z^2}\right], \\ Q_2^1(z) &= (1-z^2)^{1/2}\left[\frac{3z}{2}\ln\frac{1+z}{1-z} + \frac{3z^2-2}{1-z^2}\right], \\ Q_2^2(z) &= (1-z^2)\left[\frac{3}{2}\ln\frac{1+z}{1-z} - \frac{3z^3-5z}{1-z^2}\right] \end{aligned} \tag{3.3.31}$$

etc. The functions $P_n^m(z)$ and $Q_n^m(z)$ are connected for any integral values of $\lambda = n$ and $\mu = m$ by the functional relationships,

$$P_n^m(z) = P_{-n-1}^m(z),$$

$$Q_{-n-1}^m(z) = Q_n^m(z) + \tag{3.3.32}$$

$$+ (-1)^m \cos(\pi n)(n-m)!\,(m-n-1)!\,P_n^m(z).$$

For them we have the asympotitic representations,

$$P_n^m(\cos\theta) \simeq \frac{2^{n+\frac{1}{2}}}{\pi}\,\frac{(n+m)!}{(2n+1)!!}\times$$

$$\times\,\frac{\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{\pi m}{2}\right]}{\sqrt{2\sin\theta}}, \tag{3.3.33}$$

if m is real and $n >> 1$, $n >> m$, $z = \cos\theta$,

$$P_n^m(\cos\theta) \simeq n^m \sqrt{\frac{2}{\pi n \sin\theta}}\times$$

$$\times \cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4} + \frac{\pi m}{2}\right],$$

$$Q_n^m(\cos\theta) \approx n^m \sqrt{\frac{2}{\pi n \sin\theta}}\times \tag{3.3.34}$$

$$\times \cos\left[\left(n+\frac{1}{2}\right)\theta + \frac{\pi}{4} + \frac{\pi m}{2}\right],$$

if n, m are real and positive numbers, $\varepsilon \leq \theta \leq \pi - \varepsilon$ and $\varepsilon > 0$, /117
$i >> 1/\varepsilon$.

For integral values of $\lambda = n$, $\mu = m$ the adjoint functions $P_n^m(z)$ on the interval $(-1, +1)$ possess the property of orthogonality, and

$$\int_{-1}^{+1} P_n^m(z)\,P_k^l(z)\,dz =$$

$$= \begin{cases} 0, & \text{if } n \neq k,\ m \neq l, \\ \dfrac{2}{2n+1}\cdot\dfrac{(n+m)!}{(n-m)!}, & \text{if } n = k,\ m = l. \end{cases} \tag{3.3.35}$$

Another important property of the adjoint Legendre funtions $P_n^m(z)$, which are regular in $[-1, +1]$, is the property of completeness of the system $\{P_n^m(z)\}_{n,m}$, which exhausts all bound solutions to equation (3.3.1) in $[-1, +1]$. From these properties there follows the possibility of expanding the arbitrary function in $(-1, +1)$ into a series of adjoint Legendre functions $P_n^m(z)$.

The Wronskian of the functions $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ for any values of λ and μ is equal to

$$W\{P_\lambda^\mu(z),\ Q_\lambda^\mu(z)\} = \frac{2^{2\mu}}{1-z^2} \times$$

$$\times \frac{\Gamma\left(1+\frac{\lambda+\mu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu}{2}\right)}{\Gamma\left(1+\frac{\lambda-\mu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right)},\ z\in(-1,\ +1). \tag{3.3.36}$$

In the case of complex values of the argument z and arbitrary values of λ and μ the adjoint Legendre functions will be single-valued and regular functions on the z-plane only in the presence of a branch cut along the real axis from $-\infty$ to 1. For complex values of z, which satisfy the condition $|z| < 1$, the linearly independent solutions to equation (3.3.1) are given by the expressions,

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} \times$$

$$\times F\left(-\lambda,\ 1+\lambda;\ 1-\mu;\ \frac{1-z}{2}\right),\ |\,|1-z|<2|, \tag{3.3.37}$$

or

$$P_\lambda^\mu(z) = \Gamma\left(\frac{1}{2}\right)2^\mu(z^2-1)^{-\mu/2} \times$$

$$\times\left\{\frac{F\left(\frac{1+\lambda-\mu}{2},\ -\frac{\lambda+\mu}{2};\ \frac{1}{2};\ z^2\right)}{\Gamma\left(\frac{1-\lambda-\mu}{2}\right)\Gamma\left(1-\frac{\mu-\lambda}{2}\right)} - \right.$$

$$\left. -2z\frac{F\left(\frac{1-\lambda-\mu}{2},\ 1-\frac{\mu-\lambda}{2};\ \frac{3}{2};\ z^2\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)\Gamma\left(-\frac{\lambda+\mu}{2}\right)}\right\} \tag{3.3.38}$$

(in the case of an integral positive $\lambda + \mu$, one of the equations in (3.3.38) will vanish) and

$$Q_\lambda^\mu(z) = e^{i\mu\pi}\left\{\left(\frac{z-1}{z+1}\right)^{\mu/2}\frac{\Gamma(1+\lambda+\mu)\,\Gamma(-\mu)}{2\Gamma(1+\lambda-\mu)} \times\right.$$

$$\times F\left(-\lambda,\ 1+\lambda;\ 1+\mu;\ \frac{1-z}{2}\right) + \tag{3.3.39}$$

$$\left. + \frac{\Gamma(\mu)}{2}\left(\frac{z+1}{z-1}\right)^{\mu/2} F\left(-\lambda,\ 1+\lambda;\ 1-\mu;\ \frac{1-z}{2}\right)\right\},$$

$|1 - z| < 2$, or

$$Q_\lambda^\mu(z) = e^{i\mu\pi \mp \frac{i\pi}{2}(\lambda+\mu)}\,\Gamma\left(\frac{1}{2}\right)2^\mu(z^2-1)^{\mu/2}\times$$
$$\times\left\{\mp\frac{i}{2}\,\frac{\Gamma\left(\frac{1+\mu+\lambda}{2}\right)}{\Gamma\left(1+\frac{\lambda-\mu}{2}\right)}\times\right.$$
$$\times F\left(\frac{1+\lambda+\mu}{2},\ \frac{\mu-\lambda}{2};\ \frac{1}{2};\ z^2\right)+\frac{\Gamma\left(1+\frac{\lambda+\mu}{2}\right)}{\Gamma\left(\frac{1+\lambda-\mu}{2}\right)}\times \qquad (3.3.40)$$
$$\left.\times zF\left(1+\frac{\lambda+\mu}{2},\ \frac{1-\lambda+\mu}{2};\ \frac{3}{2};\ z^2\right)\right\},$$
$$\mathrm{Im}\,z \gtrless 0,\quad |z|<1,$$

where through $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ we denote, just as before, the adjoint /119 Legendre functions of first and second sort, respectively.

If $z = x + i\varepsilon$, where x is real and $x \in (-1, +1)$, $\varepsilon > 0$, and if we set

$$P(x \pm i0) = \lim_{\varepsilon\to 0} P(x \pm i\varepsilon),$$

then the relationships

$$P_\lambda^\mu(x \pm i0) = \frac{e^{\mp i\mu\pi/2}}{\Gamma(1-\mu)}\left(\frac{1+x}{1-x}\right)^{\mu/2}\times$$
$$\times F\left(-\lambda,\ 1+\lambda;\ \lambda-\mu;\ \frac{1-x}{2}\right) \qquad (3.3.41)$$
$$P_\lambda^\mu(x) = e^{-i\mu\pi/2}P_\lambda^\mu(x-i0) = e^{i\mu\pi/2}P_\lambda^\mu(x+i0), \qquad (3.3.42)$$
$$e^{-i\mu\pi}Q_\lambda^\mu(x \pm i0) = e^{\pm i\mu\pi/2}\left[Q_\lambda^\mu(x) \mp \frac{i\pi}{2}\,P_\lambda^\mu(x)\right]. \qquad (3.3.43)$$

are valid.

In the case of the integral values $\lambda = n$ and $\mu = m$,

$$P_n^m(x \pm i0) = i^{\mp m}P_n^m(x),$$
$$Q_n^m(x \pm i0) = i^{\mp m}\left[Q_n^m(x) \mp \frac{i\pi}{2}\,P_n^m(x)\right]. \qquad (3.3.44)$$

For complex values of z, satisfying the condition $|z| > 1$, the linearly independent solutions to equation (3.3.1) for any values of λ and μ are determined by the expressions:

$$P_\lambda^\mu(z) = \frac{1}{\Gamma(1-\mu)}\left(\frac{z+1}{z-1}\right)^{\mu/2} \times \times F\left(-\lambda,\ 1+\lambda;\ 1-\mu;\ \frac{1-z}{2}\right), \tag{3.3.45}$$

where $\arg \frac{z+1}{z-1} = 0$, if z is real and greater than 1, and

$$Q_\lambda^\mu(z) = \frac{\sqrt{\pi}\,\Gamma(1+\lambda+\mu)\,e^{i\mu\pi}}{2^{\lambda+1}\,\Gamma\left(\lambda+\frac{3}{2}\right)} \cdot \frac{(z^2-1)^{\mu/2}}{z^{\lambda+\mu+1}} \times \times F\left(\frac{1+\lambda+\mu}{2},\ 1+\frac{\lambda+\mu}{2};\ \lambda+\frac{3}{2};\ \frac{1}{z^2}\right). \tag{3.3.46}$$

In the case of integral positive values of $\lambda = n$, $\mu = m$, the adjoint Legendre functions for $|z| > 1$ are determined by the equations, /1

$$P_n^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m P_n(z)}{dz^m},$$
$$Q_n^m(z) = (z^2-1)^{\frac{m}{2}} \frac{d^m Q_n(z)}{dz^m}, \tag{3.3.47}$$

where $P_n(z)$ and $Q_n(z)$ are the Legendre functions determined for $|z| > 1$ in the preceding section. The functions $P_n^m(z)$ and $Q_n^m(z)$ satisfy the functional relationships in (3.3.32) and the recurrent relationships in (3.3.24) in which $1 - z^2$ should be substituted by $z^2 - 1$ ($|z| > 1$). The equations,

$$P_n^{-m}(z) = \frac{(n-m)!}{(n+m)!} P_n^m(z),$$
$$Q_n^{-m}(z) = \frac{(n-m)!}{(n+m)!} Q_n^m(z) \tag{3.3.48}$$

are valid for them as are (3.3.30) and (3.3.31) if in these latter equations we substitute $\ln \frac{1+z}{1-z}$ by $\ln \frac{z+1}{z-1}$ and $1 - z^2$ by $z^2 - 1$ ($|z| > 1$).

For large values of z ($|z| \to \infty$) and arbitrary values of λ and μ we find the following asymptotic representations,

$$P_\lambda^\mu(z) \simeq \frac{\Gamma\left(\lambda+\frac{1}{2}\right)(2z)^\lambda}{\Gamma(1+\lambda-\mu)\,\Gamma\left(\frac{1}{2}\right)}, \tag{3.3.49}$$

$$Q_\lambda^\mu(z) \simeq \frac{e^{i\pi\mu}\,\Gamma\left(\frac{1}{2}\right)\Gamma(1+\lambda+\mu)}{\Gamma\left(\lambda+\frac{3}{2}\right)(2z)^{\lambda+1}}, \tag{3.3.50}$$

which, for the integral positive values $\lambda = n$, $\mu = m$, assume the form,

$$P_n^m(z) \simeq \frac{(2n-1)!!\,z^n}{(n-m)!}, \tag{3.3.51}$$

$$Q_n^m(z) \simeq \frac{(n+m)!}{(2n+1)!!\,z^{n+1}}. \tag{3.3.52}$$

The Wronskian of the functions $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$ (z is complex) is equal to /121

$$W\{P_\lambda^\mu(z), Q_\lambda^\mu(z)\} = \frac{e^{i\mu\pi}\cdot 2^{2\mu}\,\Gamma\left(1+\frac{\lambda+\mu}{2}\right)\Gamma\left(\frac{1+\lambda+\mu}{2}\right)}{(1-z^2)\,\Gamma\left(1+\frac{\lambda-\mu}{2}\right)\Gamma\left(\frac{1+\lambda-\mu}{2}\right)}. \tag{3.3.53}$$

In conclusion, let us mention that for a complex values of z, the relationships

$$Q_\lambda^\mu(z) = e^{2i\mu\pi}\,\frac{\Gamma(1+\lambda+\mu)}{\Gamma(1+\lambda-\mu)}\,Q_\lambda^{-\mu}(z),$$

$$P_\lambda^\mu(z) = \frac{e^{-i\mu\pi}}{\pi\cos(\lambda\pi)} \times$$

$$\times\left[Q_\lambda^\mu(z)\sin[\pi(\lambda+\mu)] - Q_{-\lambda-1}^\mu(z)\sin[\pi(\lambda-\mu)]\right],$$

$$P_\lambda^{-\mu}(z) = \frac{\Gamma(1+\lambda-\mu)}{\Gamma(1+\lambda+\mu)} \times$$

$$\times\left[P_\lambda^\mu(z) - \frac{2}{\pi}\,e^{-i\mu\pi}\sin(\mu\pi)\,Q_\lambda^\mu(z)\right],$$

$$P_\lambda^\mu(-z) = e^{\mp i\lambda\pi} P_\lambda^\mu(z) - \frac{2}{\pi} e^{-i\mu\pi} \sin[\pi(\lambda+\mu)] Q_\lambda^\mu(z),$$
$$Q_\lambda^\mu(-z) = -e^{\pm i\lambda\pi} Q_\lambda^\mu(z), \qquad (3.3.54)$$
$$\operatorname{Im} z \gtrless 0.$$

are valid.

For $z \in [-1, +1]$,

$$|P_n^m(z)| < \frac{(n+m)!}{n!}, \quad m > 1. \qquad (3.3.55)$$

§4. Spherical Functions.

If we seek the particular solution to the Helmholtz equation (1.6.26) by the method of separation of variables on the assumption that $\lambda = n$, $\mu = m$, where n and m are integral positive numbers $n \geq 0$, $m \geq 0$), and if it is sought on a sphere of a given radius, then /1 it is obvious that such a solution, written in the most general form, will be the functions,

$$V(\theta, \varphi) = [c_1 \cos m\varphi + c_2 \sin m\varphi] \times$$
$$\times [c_3 P_n^m(\cos\theta) + c_4 Q_n^m(\cos\theta)].$$

In order that it be restricted to the surface of the sphere, including its poles $\theta = 0$, $\theta = \pi$, we must assume here $c_4 = 0$, as a result of which, such a solution representing the given n and m can be described by the formula,

$$V_n^m(\theta, \varphi) = a_{nm} P_n^m(\cos\theta) \cos m\varphi +$$
$$+ b_{nm} P_n^m(\cos\theta) \sin m\varphi, \qquad (3.4.1)$$

where a, b are arbitrary constants dependent on n and m. The functions in (3.4.1) which depend only on the angular variables θ and ϕ, are called spherical functions (spherical harmonics). As is obvious from (3.4.1) they represent a linear combination of the functions,

$$P_n^m(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix} (m\varphi) \qquad (3.4.2)$$

($m < n$), called the tesseral spherical functions of n-th power of m-th order. The functions

$$P_n^n(\cos\theta) \begin{matrix}\cos \\ \sin\end{matrix} (n\varphi) \qquad (3.4.3)$$

are called sectorial spherical functions, and the functions $P_n(\cos\theta)$ are called zonal spherical functions.

We can show that for each n there only $2n + 1$ independent tesseral spherical functions (3.4.2), by summation of which over m we obtain the function,

$$V_n(\theta, \varphi) = \sum_{m=0}^{n} \{a_{nm} \cos m\varphi + b_{nm} \sin m\varphi\} P_n^m(\cos\theta), \tag{3.4.4}$$

called the surface spherical function of n-th power. In the case of an integral value of m, also assuming negative values, the surface spherical function (3.4.4) may be written in the complex form,

$$V_n(\theta, \varphi) = \sum_{m=-n}^{n} A_{nm} P_n^m(\cos\theta)\, e^{im\varphi}. \tag{3.4.5}$$

The spherical functions (3.4.2), (3.4.3) form on the surface of the /123 of the sphere a complete orthogonal system of functions, and for them,

$$\int_0^{2\pi}\int_0^{\pi} \cos m\varphi \cos k\varphi\, P_n^m(\cos\theta)\, P_n^k(\cos\theta) \sin\theta\, d\theta\, d\varphi =$$

$$= \int_0^{2\pi}\int_0^{\pi} \sin m\varphi \sin k\varphi\, P_n^m(\cos\theta)\, P_n^k(\cos\theta) \sin\theta\, d\theta\, d\varphi =$$

$$= \frac{2\pi}{2n+1} \cdot \frac{(n+m)!}{(n-m)!}\, \delta_{mk}, \qquad m,\ k \neq 0,$$

where δ_{mk} is the Kronecker symbol. On the basis of these properties of the functions in (3.4.5) any function $f(\theta, \phi)$, which is continuous along with the derivatives up to second order of magnitude inclusively, may be expanded into an absolutely and uniformly converging series for the surface spherical functions (3.4.5),

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} A_{mn} P_n^m(\cos\theta)\, e^{im\varphi},$$

the coefficients of which are determined by the formula

$$A_{mn} = \frac{(2n+1)(n-m)!}{4\pi\,(n+m)!} \times$$

$$\times \int_0^{2\pi}\int_0^{\pi} f(\theta, \varphi)\, P_n^m(\cos\theta)\, e^{-im\varphi} \sin\theta\, d\theta\, d\varphi.$$

In the case if expansion of the function $f(\theta, \phi)$ is written in the form,

$$f(\theta, \varphi) = \sum_{n=0}^{\infty} \left[a_{n0} P_n(\cos\theta) + \right.$$

$$\left. + \sum_{m=1}^{n} (a_{nm} \cos m\varphi + b_{nm} \sin m\varphi) P_n^m(\cos\theta) \right]$$

the coefficients of the series may be computed from the formulas:

$$a_{n0} = \frac{2n+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) P_n(\cos\theta) \sin\theta \, d\theta \, d\varphi,$$

$$a_{nm} = \frac{2n+1}{2\pi} \cdot \frac{(n-m)!}{(n+m)!} \times$$

$$\times \int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) P_n^m(\cos\theta) \cos m\varphi \sin\theta \, d\theta \, d\varphi, \quad /1$$

$$b_{nm} = \frac{2n+1}{2\pi} \cdot \frac{(n-m)!}{(n+m)!} \times$$

$$\times \int_0^{2\pi} \int_0^{\pi} f(\theta, \varphi) P_n^m(\cos\theta) \sin m\varphi \sin\theta \, d\theta \, d\varphi.$$

The whole series of useful formulas for the spherical functions may be obtained from results of the group theory of representing the rotation of a three-dimensional space (sphere) in which the so-called generalized spherical functions of n-th order of magnitude occur $T_{km}^{n}(\psi, \theta, \phi)$ (in the future it is assumed that $k, m = -n; n \geq 0$ and is an integral number), determined by the formula (see, for example, [24, 25, 27]),

$$T_{km}^{n}(\psi, \theta, \varphi) = e^{-ik\psi} P_{km}^{n}(\theta) e^{-im\varphi}, \qquad (3.4.6)$$

where

$$P_{km}^{n}(\theta) = \frac{(-1)^{n-k} i^{m-k}}{2^n (n-k)!} \left\{ \frac{(n-k)!\,(n+m)!}{(n+k)!\,(n-m)!} \right\}^{1/2} \times$$

$$\times (1-\mu)^{-\frac{m-k}{2}} (1+\mu)^{-\frac{m+k}{2}} \times$$

$$\times \frac{d^{n-m}}{d\mu^{n-m}} [(1-\mu)^{n-k} (1+\mu)^{n+k}], \quad \mu = \cos\theta \qquad (3.4.7)$$

(here ψ, θ, ϕ are the Euler angles, Figure 16; θ is the angle between the axes Oz and $O\zeta$ of the fixed $Oxyz$ and the movable $O\xi\eta\zeta$ systems of coordinates; ψ is the angle between the axis $O\xi$ and the straight line L, along which the planes Oxy and $O\xi\eta$ intersect; ϕ is the angle between the axis Ox and the straight line L, figured in the directions shown on the figure by the arrows). The functions $P^n_{0m}(\theta)$ are proportional to the adjoint Legendre functions $P^m_n(\cos\theta)$, since /125

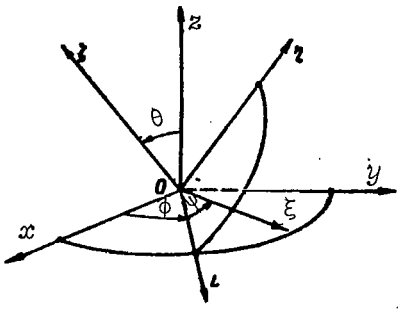

Fig. 16. Euler Angles.

$$P^n_{0m}(\theta) = P^n_{0,-m}(\theta) = i^{-m}\sqrt{\frac{(n-m)!}{(n+m)!}}\,P^m_n(\cos\theta). \qquad (3.4.8)$$

In turn, the generalized spherical functions (3.4.6) are proportional to the spherical functions (3.4.2), since on the basis of (3.4.6) and (3.4.8),

$$T^n_{0m}(\psi,\ \theta,\ \varphi) = P^n_{0m}e^{-im\varphi} =$$
$$= i^{-m}\sqrt{\frac{(n-m)!}{(n+m)!}}\,P^m_n(\cos\theta)\,e^{-im\varphi}.$$

Here we find the relationships,

$$P^n_{-k,\,-m}(\theta) = P^n_{km}(\theta),\quad P^n_{km}(\theta) = P^n_{mk}(\theta),$$
$$\sum_{m=-n}^{n}|P^n_{km}(\theta)|^2 \equiv 1, \qquad (3.4.9)$$

where m, $k = -n, \ldots, n$. The particular values of the function $P^n_{km}(\theta)$ are

$$P^n_{km}(\theta) = \begin{cases} 0, & k \neq m, \quad \theta = 0,\ \theta = \pi, \\ 1, & k = m, \quad \theta = 0, \\ (-1)^n, & k = -m,\ \theta = \pi. \end{cases} \qquad (3.4.10)$$

From the relationship,

$$P^{n_1}_{k_1m_1}(\theta)\,P^{n_2}_{k_2m_2}(\theta) =$$
$$= \sum_{n=|n_1-n_2|}^{n_1+n_2}(n_1n_2k_1k_2\,|\,n,\ k_1+k_2)(n_1n_2m_1m_2\,|\,n, \qquad (3.4.11)$$
$$m_1+m_2)\,P^n_{k_1+k_2,\,m_1+m_2}(\theta)$$

using (3.4.8) we obtain formulas for expansion of the product of two adjoint Legendre functions according to the same functions of the previous argument:

$$P_{n_1}^{m_1}(\cos\theta)\,P_{n_2}^{m_2}(\cos\theta)=\sum_{n=|n_1-n_2|}^{n_1+n_2}a_n^{(n_1m_1n_2m_2)}P_n^{m_1+m_2}(\cos\theta), \tag{3.4.12}$$

$$P_{n_1}^{m_1}(\cos\theta)\,P_{n_2}^{m_2}(\cos\theta)=\sum_{n=|n_1-n_2|}^{n_1+n_2}b_n^{(n_1m_1n_2m_2)}P_n^{m_1-m_2}(\cos\theta), \tag{3.4.13}$$

/1

where we have set[2]

$$a_n^{(n_1m_1n_2m_2)}=\left\{\frac{(n_1+m_1)!\,(n_2+m_2)!\,(n-m_1-m_2)!}{(n_1-m_1)!\,(n_2-m_2)!\,(n+m_1+m_2)!}\right\}^{1/2}\times$$
$$\times(n_1n_200|\,n0)\,(n_1n_2m_1m_2|\,n,\ m_1+m_2), \tag{3.4.14}$$

$$b_n^{(n_1m_1n_2m_2)}=$$
$$=(-1)^{m_2}\left\{\frac{(n_1+m_1)!\,(n_2+m_2)!\,(n-m_1+m_2)!}{(n_1-m_1)!\,(n_2-m_2)!\,(n+m_1-m_2)!}\right\}^{1/2}\times$$
$$\times(n_1n_200|\,n0)\,(n_1n_2m_1,\ -m_2|\,n,\ m_1-m_2). \tag{3.4.15}$$

The symbols $(n_1n_2m_1m_2|\,n,\ m_1+m_2)$ here denote the so-called Clebsch-Gordan coefficients which are often encountered in problems of quantum mechanics (see, for example, [24 - 27]). For them we are familiar with two different forms of their explicit expression. In particular, they are represented in the form,

$$(n_1n_2m_1m_2|n,\ m_1+m_2)=$$
$$=\left\{\frac{2n+1}{(n+n_1+n_2+1)!}\,(n_2+n-n_1)!\,(n+n_1-n_2)!\times\right.$$
$$\times(n_1+n_2-n)!\,(n_1-m_1)!\,(n_1+m_1)!\times$$
$$\left.\times(n_2+m_2)!\,(n_2-m_2)!\,(n+m_1+m_2)\,(n-m_1-m_2)!\right\}^{1/2}\times$$
$$\times\sum_z(-1)^z\,[z!\,(n_1+n_2-n-z)!\,(z+n-n_1-m_2)!\times \tag{3.4.16}$$
$$\times(n_2+m_2-z)!\times$$
$$\times(z+n-n_2+m_1)!\,(n_1-m_1-z)!\,]^{-1}$$

where summation is carried out over z, which covers all non-negative integral numbers no less than m_2+n_2-n, n_2-n-m_1, and no greater than

[2] For computation of a_n and b_n we can use [190]; see also the Appendix.

n_1+n_2-n, n_2+m_2, n_1-m_1. Therefore, the number of terms in (3.4.16) exceeds by one the least of the nine sums n_2-m_2, n_1+m_1, n_2-n_1+n, $n+m_1+m_2$, $n-m_1-m_2$, n_1+n-n_2, n_1+n_2-n, n_2+m_2, n_1-m_1. Formula (3.4.16), /127 when $m_1 = m_2 = 0$, is simplified and assumes the form,

$$(n_1 n_2 00 \mid n0) =$$

$$= \begin{cases} (-1)^{n+l/2} \dfrac{\left(\frac{l}{2}\right)!}{\left(\frac{l}{2}-n_1\right)!\left(\frac{l}{2}-n_2\right)!\left(\frac{l}{2}-n\right)!} \times \\ \times \left[\dfrac{(2n+1)(l-2n_1)!\,(l-2n_2)!\,(l-2n)!}{(l+1)!}\right]^{1/2}, \\ \text{if } l \text{ is even} \\ 0, \quad \text{if } l \text{ is odd, } l = n+n_1+n_2. \end{cases} \tag{3.4.17}$$

The generalized spherical functions (3.4.6) are connected by the relationship,

$$T^n_{km}(\psi_1, \theta_1, \varphi_1) = \sum_{s=-n}^{n} T^n_{ks}(\psi_2, \theta_2, \varphi_2)\, T^n_{sm}(\psi_3, \theta_3, \varphi_3), \tag{3.4.18}$$

representing the addition formula for the generalized spherical functions and containing as the particuler cases the addition formulas for the spherical functions (3.4.2) and for the Legendre polynomials $P_n(\cos\theta)$. Thus, after substituting $k = 0$ into (3.4.18) and using formula (3.4.8), we find

$$P^m_n(\cos\theta_1)\, e^{-im\varphi_1} = i^m \sqrt{\frac{(n+m)!}{(n-m)!}} \times \sum_{s=-n}^{n} T^n_{-s,\,m}(\psi_3, \theta_3, \varphi_3)\, i^{-s} \times \sqrt{\frac{(n-s)!}{(n+s)!}}\, P^s_n(\cos\theta_2)\, e^{is\varphi_2} \tag{3.4.19}$$

or

$$P^m_n(\cos\theta_1)\, e^{im\varphi_1} = i^m \sqrt{\frac{(n+m)!}{(n-m)!}} \times \sum_{s=-n}^{n} T^n_{-s,\,-m}(\psi_3, \theta_3, \varphi_3)\, i^{-s} \times \sqrt{\frac{(n-s)!}{(n+s)!}}\, P^s_n(\cos\theta_2)\, e^{is\varphi_2}, \tag{3.4.20}$$

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on the basis of which we have

$$P_n^m(\cos\theta_1)\cos m\varphi_1 = \sum_{s=-n}^{n} L_{sm}^n(\psi_3, \theta_3, \varphi_3)\, P_n^s(\cos\theta_2)\, e^{is\varphi_2}, \tag{3.4.21}$$

$$P_n^m(\cos\theta_1)\sin m\varphi_1 = \sum_{s=-n}^{n} L_{sm}^{*n}(\psi_3, \theta_3, \varphi_3)\, P_n^s(\cos\theta_2)\, e^{is\varphi_2} \tag{3.4.22}$$

and

$$P_n(\cos\theta_1) = \sum_{s=-n}^{n} L_{sn}(\psi_3, \theta_3, \varphi_3)\, P_n^s(\cos\theta_2)\, e^{is\varphi_2}, \tag{3.4.23}$$

where

$$L_{sm}^n = \frac{i^{m-s}}{2}\left\{\frac{(n+m)!}{(n-m)!}\cdot\frac{(n-s)!}{(n+s)!}\right\}^{1/2}\times \times (T_{-s,-m}^n + T_{-s,m}^n),$$

$$L_{sm}^{*n} = \frac{i^{m-s-1}}{2}\left\{\frac{(n+m)!}{(n-m)!}\cdot\frac{(n-s)!}{(n+s)!}\right\}^{1/2}\times \times (T_{-s,-m}^n - T_{-s,m}^n),$$

$$L_{sn} = (-1)^s\,\frac{(n-s)!}{(n+s)!}\,P_n^s(\cos\theta_3)\,e^{is\psi_3} \tag{3.4.24}$$

ıd where it is assumed that the Euler angles are mutually related y the formulas,

$$\cos\theta_1 = \cos\theta_2\cos\theta_3 - \sin\theta_2\sin\theta_3\cos(\varphi_2+\psi_3),$$

$$e^{i(\varphi_1-\varphi_3)} = [\sin\theta_3\cos\theta_2 + \cos\theta_3\sin\theta_2\times \times\cos(\varphi_2+\psi_3) + i\sin\theta_2\sin(\varphi_2+\psi_3)]/\sin\theta_1,$$

$$e^{i(\varphi_1-\varphi_3+\psi_1-\psi_3)/2} = \left[\cos\frac{\theta_3}{2}\cos\frac{\theta_2}{2}\,e^{i(\varphi_3+\psi_3)/2} - \right. \tag{3.4.25}$$

$$\left. - \sin\frac{\theta_3}{2}\sin\frac{\theta_2}{2}\,e^{-i(\varphi_3+\psi_3)/2}\right] \Big/ \cos\frac{\theta_1}{2}$$

(ψ varies from 0 to 2π, θ varies from 0 to π and ϕ varies from -2π to 2π). Formula (3.4.23) represents the ordinary addition theorem for the Legendre polynomial. It can be given the form,

$$P_n(\cos\theta_1) = P_n(\cos\theta_0)\,P_n(\cos\theta_2) +$$

$$+2\sum_{s=1}^{n}(-1)^s\frac{(n-s)!}{(n+s)!}\times \qquad (3.4.26)$$

$$\times P_n^s(\cos\theta_2)\,P_n^s(\cos\theta_3)\cos s(\varphi_2+\psi_3).$$

§5. Certain Properties of Spherical Bessel Functions.

The properties of the spherical Bessel functions $j_n(x)$, $n_m(x)$, $h_\lambda^{(1),(2)}(x)$, determined in §1 of this chapter, are established on the basis of the properties of the ordinary Bessel functions $J_\lambda(x)$, $N_\lambda(x)$ and $H_\lambda^{(1),(2)}(x)$ through which they are expressed using the formulas in (3.1.9). Thus, for example, the expansion of the functions $j_n(x)$ and $n_n(x)$ into a series in the vicinity of the singular point $x = 0$ are obtained directly from (2.2.), (2.2.3) using (2.2.13) and the relationships $\Gamma(\lambda+1) = \lambda\Gamma(\lambda)$, $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, $\Gamma\left(\lambda+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2\lambda)/\Gamma(\lambda)\,2^{2\lambda-1}$.

For the function $j_n(x)$ the expansion has the form,

$$j_n(x) = 2^n x^n \sum_{m=0}^{\infty}\frac{(-1)^m(n+m)!}{m!\,(2n+2m+1)!}\,x^{2m}, \qquad (3.5.1)$$

and for the function $n_n(x)$,

$$n_n(x) = -\frac{1}{2^n x^{n+1}}\sum_{m=0}^{\infty}\frac{(2n-2m)!}{m!\,(n-m)!}\,x^{2m}. \qquad (3.5.2)$$

From (3.5.1) it is clear that $j_n(x)$ is an integral function of the argument x, whereas the function $n_n(x)$ at the point $x = 0$ has a terminal of $n+1$-th order. From expansions (3.5.1), (3.5.2) we obtain approximate formulas expressing the values of the functions $j_n(x)$, $n_n(x)$ for small values of their argument:

$$j_n(x) \underset{x\to 0}{\to} \frac{2^n n!\,x^n}{(2n+1)!},\quad n_n(x) \underset{x\to 0}{\to} \frac{-(2n)!}{2^n n!\,x^{n+1}}, \qquad (3.5.3)$$

on the basis of which we have /130

$$h_n^{(1),(2)}(x) \underset{x\to 0}{\to} \mp\frac{i(2n)!}{2^n n!\,x^{n+1}}. \qquad (3.5.4)$$

From the asymptotic formulas for the ordinary Bessel functions we find that when $n \gg 1$ and $n \gg x$,

$$j_n(x) \simeq \frac{2^n n!\,x^n}{(2n+1)!},\quad h_n^{(1),(2)}(x) \simeq \mp\frac{i(2n)!}{2^n n!\,x^{n+1}}, \qquad (3.5.5)$$

and when $x \to \infty$,

$$h_n^{(1),(2)}(x) \simeq (\mp i)^{n+1} e^{\pm ix}/x, \tag{3.5.6}$$

$$j_n(x) \simeq \frac{1}{x} \cos\left(x - \frac{n+1}{2}\pi\right) \tag{3.5.7}$$

$$n_n(x) \simeq \frac{1}{x} \sin\left(x - \frac{n+1}{2}\pi\right) \tag{3.5.8}$$

The spherical Bessel functions satisfy a series of recurrent relationships, for example,

$$z_{n+1}(x) = \frac{2n+1}{x} z_n(x) - z_{n-1}(x), \tag{3.5.9}$$

$$z_n'(x) = \frac{1}{2n+1}\{n z_{n-1}(x) - (n+1) z_{n+1}(x)\}, \tag{3.5.10}$$

where through $z_n(x)$ we denote one of the functions $j_n(x)$, $n_n(x)$ or $h_n^{(1),(2)}(x)$.

Proceeding from the common theory of integral representations of the solutions to ordinary differential equations (see, for example, [14]), we may establish that the spherical Bessel functions can be represented by the integrals:

$$j_n(kr) = \frac{i^{-n}}{2} \int_{-1}^{1} e^{ikrx} P_n(x)\, dx =$$

$$= \frac{i^{-n}}{2} \int_0^{\pi} e^{ikr\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta, \tag{3.5.11}$$

$$h_n^{(1)}(kr) = i^{-n} \int_{i\infty}^{1} e^{ikrx} P_n(x)\, dx =$$

$$= i^{-n} \int_0^{\frac{\pi}{2} - i\infty} e^{ikr\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta, \tag{3.5.12}$$

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$$h_n^{(2)}(kr) = i^{-n} \int_{-1}^{i\infty} e^{ikrx} P_n(x)\, dx =$$

$$= i^{-n} \int_{\frac{\pi}{2} - i\infty}^{\pi} e^{ikr\cos\theta} P_n(\cos\theta) \sin\theta\, d\theta. \tag{3.5.13}$$

§6. Expansion of Plane Spherical Waves According to Spherical Wave Functions.

The formulas for expansion of a plane wave (2.3.1) and of a spherical wave (2.5.1) according to the spherical wave functions (3.1.10), (3.1.11) can be easily obtained from expansion of the Green function $G_k(r|r_0)$ (2.5.3) of the Helmholtz scalar wave equation (2.5.2), studied in the entire space according to the same functions. Based on the properties of the Green function $G_k(r|r_0)$, its expansion can be sought in the form,

$$G_k(r|r_0) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} A_{mn} P_n^m(\cos\theta) P_n^m(\cos\theta_0) \cos m(\varphi - \varphi_0) \times \times \begin{cases} j_n(kr)\, h_n^{(1)}(kr_0), & r < r_0, \\ j_n(kr_0)\, h_n^{(1)}(kr), & r > r_0 \end{cases} \qquad (3.6.1)$$

when $r \neq r_0$, the function $G_k(r|r_0)$ satisfies the homogeneous Helmholtz equation). In spherical coordinates, equation (2.5.2) is written in the form,

$$\frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \frac{\partial G_k}{\partial r}\right) + \frac{1}{r^2 \sin\theta} \times \times \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial G_k}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta} \frac{\partial^2 G_k}{\partial \varphi^2} + + k^2 G_k = -\frac{1}{r^2 \sin\theta} \delta(\theta - \theta_0)\, \delta(r - r_0)\, \delta(\varphi - \varphi_0).$$

After multiplying both parts of the equation by r^2 and integrating them over r in the immediate vicinity of the point $r = r_0$ from $r_0 -$ to $r_0 + \varepsilon$ (the Green function remains continuous when $r = r_0$), we find after converting to the limit when $\varepsilon \to 0$, that /132

$$r^2 \frac{\partial G_k(r, \theta, \varphi; r_0, \theta_0, \varphi_0)}{\partial r}\Bigg|_{r=r_0-0}^{r=r_0+0} = = -\frac{1}{\sin\theta} \delta(\theta - \theta_0)\, \delta(\varphi - \varphi_0).$$

Instead of $G_k(r|r_0)$, if we substitute here expansion (3.6.1), we find that

$$\sum_{n,m} A_{mn} P_n^m(\cos\theta) P_n^m(\cos\theta_0) \cos m(\varphi - \varphi_0) = = \frac{ik}{\sin\theta} \delta(\theta - \theta_0)\, \delta(\varphi - \varphi_0), \qquad (3.6.2)$$

since the Wronskian of the functions $j_n(kr)$, $h_n^{(1)}(kr)$ is equal to

$$W[j_n(kr), h_n^{(1)}(kr)] = -\frac{i}{kr^2}. \qquad (3.6.3)$$

Now if we multiply both parts of (3.6.2) by $P_n^{m'}(\cos\theta)\cos m'\phi \times \sin\theta$ and integrate them over θ and ϕ in the range from 0 to π and from 0 to 2π, respectively, we find

$$A_{mn} = ik(2-\delta_{0m})/2\pi N_{mn}, \qquad (3.6.4)$$

where through N_{mn} we denote the expression,

$$N_{mn} = \frac{2}{2n+1}\cdot\frac{(n+m)!}{(n-m)!}. \qquad (3.6.5)$$

As a result, the unknown expansion of the Green function $G_k(r|r_0)$ is represented by a series,

$$\frac{e^{ik|r-r_0|}}{4\pi|r-r_0|} = \frac{ik}{2\pi}\times$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{0m}}{N_{mn}}P_n^m(\cos\theta)P_n^m(\cos\theta_0)\cos m(\varphi-\varphi_0)\times \qquad (3.6.6)$$

$$\times\begin{cases} j_n(kr)h_n^{(1)}(kr_0), & r<r_0,\\ j_n(kr_0)h_n^{(1)}(kr) & r>r_0.\end{cases}$$

If we assume here that $|r - r_0| = R = \sqrt{r^2 + r_0^2 - 2rr_0\cos\gamma}$, where /1
$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)$, we find the formula for expansion of the spherical wave (2.5.1) according to the spherical wave functions,

$$\frac{e^{ikR}}{R} = 2ik\times$$

$$\times\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{2-\delta_{0m}}{N_{mn}}P_n^m(\cos\theta)P_n^m(\cos\theta_0)\cos m(\varphi-\varphi_0)\times \qquad (3.6.7)$$

$$\times\begin{cases} j_n(kr)h_n^{(1)}(kr_0), & r_0>r,\\ j_n(kr_0)h_n^{(1)}(kr), & r>r_0.\end{cases}$$

On the assumption that $r_0 \to \infty$ (the source is removed to infinity) we have $R \to r_0 - r\cos\gamma$. Therefore, after applying the asymptotic formula (3.5.6) to the right side of (3.6.7) and converting to the

limit when $r_0 \to \infty$, we find

$$e^{-ik(nR)} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n}(-i)^n \frac{2-\delta_{0m}}{N_{mn}} \times \\ \times j_n(kr) P_n^m(\cos\theta_0) P_n^m(\cos\theta)\cos m(\varphi-\varphi_0), \tag{3.6.8}$$

where in spherical coordinates $(nR) = r\cos\gamma = r\cos\theta\cos\theta_0 + r\sin\theta\sin\theta_0\cos(\phi-\phi_0)$. Formula (3.6.8) gives us the expansion of the plane wave (2.3.1), propagating in a negative direction with respect to the direction from the source to the origin, according to the spherical wave functions (3.1.10). After substitution into (3.6.8) of $\alpha = \pi - \theta_0$, $\beta = \phi_0 - \pi$, we find the formula for expansion of the plane wave (2.3.1)

$$e^{ik(nR)} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n} i^n \frac{2-\delta_{0m}}{N_{mn}} \times \\ \times j_n(kr) P_n^m(\cos\theta) P_n^m(\cos\alpha)\cos m(\varphi-\beta), \tag{3.6.9}$$

corresponding to the positive direction of propagation of the wave. In another form,

$$e^{ik(nR)} = 2\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n \frac{j_n(kr)}{N_{mn}} P_n^m(\cos\theta) P_n^m(\cos\alpha) e^{im(\varphi-\beta)}. \tag{3.6.10}$$

With these relationships between r and r_0, the series (3.6.7) and (3.6.9) converge absolutely and uniformly.

Formula (3.6.9) is suitable for obtaining an integral representation of the wave functions (3.1.10). Thus, after multiplying both parts of (3.6.9) by $P_n^{m'}(\cos\alpha)\begin{matrix}\cos\\ \sin\end{matrix}(m'\beta)\sin\alpha d\alpha d\beta$ and integrating them over α from 0 to π and over α from 0 to 2π, we find, /134

$$j_n(kr) P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{i^{-n}}{4\pi} \times \\ \times \int_0^{2\pi}\int_0^{\pi} e^{ik(nR)} P_n^m(\cos\alpha)\begin{matrix}\cos\\ \sin\end{matrix}(m\beta)\sin\alpha\, d\alpha\, d\beta, \tag{3.6.11}$$

or

$$j_n(kr) P_n^m(\cos\theta) e^{im\varphi} = \frac{i^{-n}}{4\pi} \times \\ \times \int_0^{2\pi}\int_0^{\pi} e^{ik(nR)} P_n^m(\cos\alpha) e^{im\beta}\sin\alpha\, d\alpha\, d\beta, \tag{3.6.12}$$

if we proceed from the expansion formula (3.6.10). After setting here $\phi = 0$ and carrying out integration over β, we find the formula,

$$j_n(kr)\,P_n^m(\cos\theta) = \frac{i^{m-n}}{2} \times$$
$$\times \int_0^{\pi} e^{ikr\cos\theta\cos\alpha}\, J_m(kr\sin\theta\sin\alpha)\, P_n^m(\cos\alpha)\sin\alpha\, d\alpha, \qquad (3.6.13)$$

from which when $m = 0$, $\theta = 0$ we arrive at the representation (3.5.11) for the function $j_n(kr)$. The integral representations of the functions (3.1.11) are given by the formulas[3]

$$h_n^{(1)}(kr)\,P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{i^{-n}}{2\pi} \times$$
$$\int_0^{2\pi}\int_0^{\pi/2-i\infty} e^{ik(nR)}\, P_n^m(\cos\alpha)\begin{matrix}\cos\\ \sin\end{matrix}(m\beta)\sin\alpha\, d\alpha\, d\beta, \qquad (3.6.14)$$

$$h_n^{(2)}(kr)\,P_n^m(\cos\theta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \frac{i^{-n}}{2\pi} \times$$
$$\times \int_0^{2\pi}\int_{\pi/2-i\infty}^{\pi} e^{ik(nR)}\, P_n^m(\cos\alpha)\begin{matrix}\cos\\ \sin\end{matrix}(m\beta)\sin\alpha\, d\alpha\, d\beta,$$

or

$$h_n^{(1)}(kr)\,P_n^m(\cos\theta)\,e^{im\varphi} = \frac{i^{-n}}{2\pi} \times$$
$$\times \int_0^{2\pi}\int_0^{\pi/2-i\infty} e^{ik(nR)}\, P_n^m(\cos\alpha)\, e^{im\beta}\sin\alpha\, d\alpha\, d\beta, \qquad (3.6.15)$$

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$$h_n^{(2)}(kr)\,P_n^m(\cos\theta)\,e^{im\varphi} = \frac{i^{-n}}{2\pi} \times$$
$$\times \int_0^{2\pi}\int_{\pi/2-i\infty}^{\pi} e^{ik(nR)}\, P_n^m(\cos\alpha)\, e^{im\beta}\sin\alpha\, d\alpha\, d\beta.$$

If we set here $\phi = 0$ and carry out integration over β, we find on the basis of (2.3.9) that

$$h_n^{(1)}(kr)\,P_n^m(\cos\theta) = i^{m-n} \times$$
$$\times \int_0^{\pi/2-i\infty} e^{ikr\cos\theta\cos\alpha}\, J_m(kr\sin\theta\sin\alpha)\, P_n^m(\cos\alpha)\sin\alpha\, d\alpha,$$

---

[3] Formulas (3.6.14) - (3.6.16) are valid when $r \neq 0$, $0 \leq \theta < \pi/2$, $\mathrm{Re}\, k > 0$. If $\theta \in (\pi/2, \pi]$, then in (3.6.14) - (3.6.16) we substitute $\exp[ik(nR)]$ by $\exp[-ik(nR)]$.

$$h_n^{(2)}(kr)\,P_n^m(\cos\theta) = i^{m-n} \times$$

$$\times \int_{\pi/2-i\infty}^{\pi} e^{ikr\cos\theta\cos\alpha}\, J_m(kr\sin\theta\sin\alpha)\, P_n^m(\cos\alpha)\sin\alpha\, d\alpha, \qquad (3.6.16)$$

whence when $m = 0$ and $\theta = 0$ we find formulas (3.5.12), (3.5.13) for the functions $h_n^{(1),(2)}(kr)$.

§7. Addition Theorems for Spherical Wave Functions.

The formula for expansion of a plane wave (3.6.10) according to spherical wave functions along with the formulas of the integral representation of the wave functions permits obtaining the addition theorems for spherical wave functions (3.1.10) and (3.1.11) with relative ease.

Let $O_1x_1y_1z_1$ and $O_2x_2y_2z_2$ be the two Cartesian rectangular coordinate systems x_1, y_1, z_1 and x_2, y_2, z_2 with identically oriented and parallel respective axes Oz, Oy and Ox, and let r_1, θ_1, ϕ_1 and /136 r_2, θ_2, ϕ_2 be connected with them by the formulas $x_s = r_s \sin\theta_s \cos\phi_s$, $y_s = r_s \sin\theta_s \sin\phi_s$, $z_s = r_s \cos\theta_s$ ($s = 1, 2$) of the system of spherical coordinates (Fig. 17). Let us give in each of the coordinate systems the spherical wave functions (3.1.10) and (3.1.11) by the formulas

$$j_n(kr_s)\,P_n^m(\cos\theta_s)\,e^{im\varphi_s},\ n=0,\ 1,\ \ldots\ ;\ -n\leqslant m\leqslant n, \qquad (3.7.1)$$

$$h_n^{(1),(2)}(kr_s)\,P_n^m(\cos\theta_s)\,e^{im\varphi_s},\ n=0,\ 1,\ \ldots\ ;\ -n\leqslant m\leqslant n, \qquad (3.7.2)$$

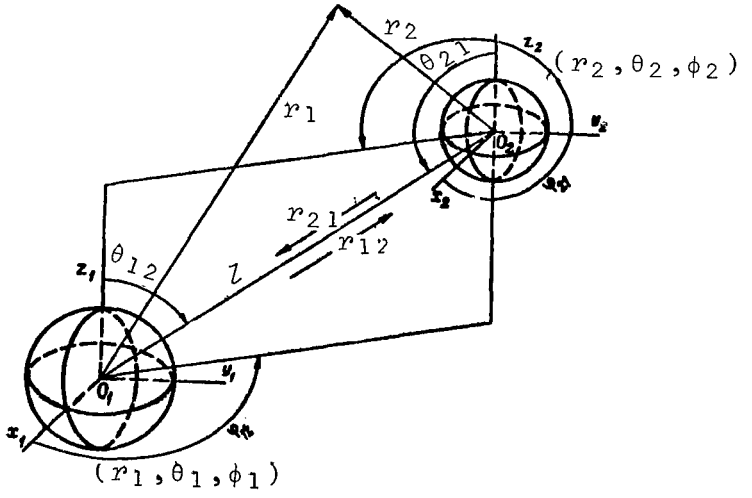

Fig. 17. Addition Theorem for Spherical Wave Functions.

where r_s, θ_s, ϕ_s are spherical coordinates of an arbitrary point in space, $s = 1$ or $s = 2$, and let us establish the form of the addition theorems which will permit us to write functions (3.7.1) and (3.7.2) pertaining to the s-th coordinate system, and through the same functions written in the j-th coordinate system ($s, j = 1, 2$; $s \neq j$). For this purpose, let us introduce into study the plane wave (2.3.1) which in the s-th system of coordinates is written in the form,

$$\psi = e^{ik(nR_s)} \tag{3.7.3}$$

where $(nR_s) = r_s[\cos\theta_s \cos\theta_0 + \sin\theta_s \cos(\phi_s - \phi_0)]$. Since $R_s = R_j + R_{sj}$, it is then obvious that

$$e^{ik(nR_s)} = e^{ik(nR_j)}\, e^{ik(nR_{sj})}$$

(through r_{sj}, θ_{sj}, ϕ_{sj} we denote the spherical coordinates of the origin O_j in the system with the origin at O_s; $j, s = 1, 2$; $j \neq s$ or on the basis of (3.6.10) /13

$$e^{ik(nR_s)} = 2\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n \frac{j_n(kr_j)}{N_{mn}} \times \times P_n^m(\cos\theta_j) P_n^m(\cos\alpha)\, e^{im(\varphi_j-\beta)}\, e^{ik(nR_{sj})}. \tag{3.7.4}$$

If we multiply both parts of (3.7.4) by $P_q^p(\cos\alpha)e^{ip\beta}\sin\alpha d\alpha d\beta$ and integrate them then over β from 0 to 2π and over α along the boundaries c_1, c_2, c_3, respectively (ends of boundary c_1 lie at points 0 and π; boundary c_2 at points 0 and $\pi/2 - i\infty$; boundary c_3 at points $\pi/2 - i\infty$ and π), we find on the basis of formulas (3.4.13), (3.6.12), (3.6.15) the unknown addition theorems for the functions (3.1.10), (3.1.11):

$$j_q(kr_s) P_q^p(\cos\theta_s) e^{ip\varphi_s} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Q_{mnpq}^{(0)}(r_{sj}, \theta_{sj}, \varphi_{sj})\, j_n(kr_j) P_n^m(\cos\theta_j) e^{im\varphi_j}, \tag{3.7.5}$$

$$h_q^{(1)}(kr_s) P_q^p(\cos\theta_s) e^{ip\varphi_s} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Q_{mnpq}^{(1)}(r_{sj}, \theta_{sj}, \varphi_{sj})\, j_n(kr_j) P_n^m(\cos\theta_j) e^{im\varphi_j}, \tag{3.7.6}$$

$$h_q^{(2)}(kr_s) P_q^p(\cos\theta_s) e^{ip\varphi_s} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Q_{mnpq}^{(2)}(r_{sj}, \theta_{sj}, \varphi_{sj})\, j_n(kr_j) P_n^m(\cos\theta_j) e^{im\varphi_j}, \tag{3.7.7}$$

where

$$Q_{mnpq}^{(0)} = \frac{2i^{n-q}}{N_{mn}} \times$$

$$\times \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} b_{\sigma}^{(qpnm)} j_{\sigma}(kr_{sj}) P_{\sigma}^{p-m}(\cos\theta_{sj}) e^{i(p-m)\varphi_{sj}} \qquad (3.7.8)$$

$$Q_{mnpq}^{(1),(2)} = \frac{2i^{n-q}}{N_{mn}} \times$$

$$\times \sum_{\sigma=|n-q|}^{n+q} \tau^{\sigma} b_{\sigma}^{(qpnm)} h_{\sigma}^{(1),(2)}(kr_{sj}) P_{\sigma}^{p-m}(\cos\theta_{sj}) e^{i(p-m)\varphi_{sj}}. \qquad (3.7.9)$$

In the case when the origins of the coordinate systems (points O_s and O_j) lie on one straight line and their respective axes Ox and Oy are identically oriented and lie in identical planes, the azimuth angles ϕ_s and ϕ_j will be equal and then in (3.7.5) - (3.7.9) their azimuth numbers p and m will also be equal, by virtue of which the addition theorems are written in the form, /138

$$Z_q^{(l)}(kr_s) P_q^p(\cos\theta_s) = \qquad (3.7.10)$$

$$= \sum_{n=p}^{\infty} Q_{pnpq}^{(l)}(r_{sj}, \theta_{sj}) j_n(kr_j) P_n^p(\cos\theta_j),$$

where

$$Q_{pnpq}^{(l)} = \frac{2i^{n-q}}{N_{pn}} \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} b_{\sigma}^{(qpnp)} Z_{\sigma}^{(l)}(kr_{sj}) P_{\sigma}(\cos\theta_{sj}) \qquad (3.7.11)$$

and where $Z_q^{(l)}$ is one of the spherical Bessel functions, $l = 0, 1, 2$. Furthermore, here now we have $\theta_{sj} = 0$ or $\theta_{sj} = \pi$.

When $p = 0$, from (3.7.10) and (3.7.11) we obtain the formulas,

$$Z_q^{(l)}(kr_s) P_q(\cos\theta_s) = \qquad (3.7.12)$$

$$= \sum_{n=0}^{\infty} Q_{0n0q}^{(l)}(r_{sj}, \theta_{sj}) j_n(kr_j) P_n(\cos\theta_j),$$

where

$$Q_{0n0q}^{(l)} = \frac{2i^{n-q}}{N_{0n}} \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} b_{\sigma}^{(q0n0)} Z_{\sigma}^{(l)}(kr_{sj}) P_{\sigma}(\cos\theta_{sj}). \qquad (3.7.13)$$

In expansions (3.7.6), (3.7.7), (3.7.10) and (3.7.12) it is assumed that $r_{sj} > r_j$. If $r_{sj} < r_j$, then all the cited addition theorems become unsuitable in the applications because of the divergence of the expansions. The only exceptions are the formulas which are set up for the spherical wave functions (3.1.10). They remain suitable for $r_{sj} < r_j$ as well.

When $r_{sj} < r_j$, the addition theorems for the functions (3.1.11) are also obtained by means of expanding $\exp(ik\ \mathrm{n}R_{sj})$ (instead of $\exp(ik\ (\mathrm{n}R_j))$ into a series of spherical wave functions. Here we find the formulas,

$$Z_q^{(l)}(kr_s)\,P_q^{\nu}(\cos\theta_s)\,e^{ip\varphi_s} = \\ = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} R_{mnpq}^{(l)}(r_{sj},\ \theta_{sj},\ \varphi_{sj}) \times \\ \times h_n^{(l)}(kr_j)\,P_n^{p-m}(\cos\theta_j)\,e^{i(p-m)\varphi_j}, \qquad (3.7.14)$$

where /139

$$R_{mnpq}^{(l)} = 2i^{n-q}\sum_{\sigma=|n-q|}^{n+q} i^{\sigma}\,\frac{b_n^{(qp\sigma m)}}{N_{m\sigma}} \times \\ \times j_{\sigma}(kl)\,P_{\sigma}^{m}(\cos\theta_{sj})\,e^{im\varphi_{sj}},\quad l = 1,\ 2. \qquad (3.7.15)$$

If the axes Oz of the coordinate systems agree and $\phi_s = \phi_j$, then $\theta_{sj} = 0$ or $\theta_{sj} = \pi$ and in (3.7.15) we will have $P_{\sigma}^{m}(\pm 1) = 0$, if $m \neq 0$, and as a result formula (3.7.14) is written in the form,

$$Z_q^{(l)}(kr_s)\,P_q^{p}(\cos\theta_s) = \\ = \sum_{n=0}^{\infty} R_{0npq}^{(l)}(r_{sj},\ \theta_{sj})\,h_n^{(l)}(kr_j)\,P_n^{p}(\cos\theta_j), \qquad (3.7.16)$$

where

$$= 2i^{n-q}\sum_{\sigma=|n-q|}^{n+q} i^{\sigma}\,\frac{b^{(qp\sigma 0)}}{N_{0\sigma}}\,j_{\sigma}(kl)\,P_{\sigma}(\cos\theta_{sj}). \qquad (3.7.17)$$

If, in addition we still have $p = 0$, then

$$Z_q^{(l)}(kr_s)\,P_q(\cos\theta_s) = \\ = \sum_{n=0}^{\infty} R_{0n0q}^{(l)}(r_{sj},\ \theta_{sj})\,h_n^{(l)}(kr_j)\,P_n(\cos\theta_j), \qquad (3.7.18)$$

where

$$R_{0n0q}^{(l)} = 2i^{n-q} \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} \frac{b_{\sigma}^{(q0\sigma0)}}{N_{0\sigma}} j_{\sigma}(kl) P_{\sigma}(\cos\theta_{sj}). \qquad (3.7.19)$$

The addition theorems may be assigned another form if, as the original expansion, we take formula (3.6.9) rather than (3.6.10). In this case, for example when $r_{sj} > r_j$, we find

$$Z_q^{(l)}(kr_s) P_q^l(\cos\theta_s) \begin{matrix}\cos\\ \sin\end{matrix}(p\varphi_s) =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \Big\{ A_{mnpq} j_n(kr_j) P_n^m(\cos\theta_j) \times$$

$$\times \begin{matrix}\cos\\ \sin\end{matrix}[-m\varphi_j + (m+p)\varphi_{sj}] + B_{mnpq} j_n(kr_j) P_n^m(\cos\theta_j) \times \qquad (3.7.20)$$

$$\times \begin{matrix}\cos\\ \sin\end{matrix}[m\varphi_j - (m-p)\varphi_{sj}] \Big\},$$

where

$$A_{mnpq} = \frac{i^{n-q}}{N_{mn}} \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} a_{\sigma}^{(nmqp)} Z_{\sigma}^{(l)}(kr_{sj}) P_{\sigma}^{p+m}(\cos\theta_{sj}),$$

$$B_{mnpq} = \frac{i^{n-q}}{N_{mn}} \sum_{\sigma=|n-q|}^{n+q} i^{\sigma} b_{\sigma}^{(nmqp)} Z_{\sigma}^{(l)}(kr_{sj}) P_{\sigma}^{m-p}(\cos\theta_{sj}) \qquad (3.7.21)$$

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and where, just as before, $Z_{\sigma}^{(l)}$ is one of the spherical Bessel functions, but $l = 0, 1, 2$. When $r_{sj} < r_j$, addition theorems of analogous type are obtained from (3.7.20) by substituting r_{sj}, θ_{sj}, ϕ_{sj}, respectively by r_j, θ_j, ϕ_j.

Let us note that the basis for the change in the order of summation and integration in deriving formulas (3.7.6), (3.7.7), (3.7.20) assumes that for any values of θ_j and $r_{sj} > r_j$ we will still have $0 < \theta_s, \theta_{sj} < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta_s, \theta_{sj} \leq \pi$, which is always valid for the coaxial case (Re $k > 0$). This comment also pertains to (3.7.14) when $r_{sj} < r_j$.

The addition theorems studied in this section along with the addition theorems for spherical functions studied in §4 of this chapter permit expressing the spherical wave functions (3.1.10), (3.1.11) written in one of the local coordinate systems through the same functions written in another local coordinate system with an arbitrary direction of the coordinate axes with respect to the first.

The addition theorems are cited in another form, for example in [154, 157]. They may be obtained as part of the results in references [40, 155, 156]. It should be noted however that in these

works they have been derived only for the case of identically directed and parallel respective axes of the two local coordinate systems.

CHAPTER 4

WAVE FUNCTIONS OF AN ELLIPTICAL CYLINDER AND CERTAIN OF THEIR PROPERTIES.

§1. Helmholtz Equation in Elliptical Coordinates. Wave Functions of an Elliptical Cylinder.

The Helmholtz scalar wave equation in the system of cylindrical elliptical coordinates ξ, η, z has the form of (1.6.21). If we seek the particular solution to equation (1.6.21) in the form of the product of three functions, /141

$$u(\xi, \eta, z) = R(\xi)\Phi(\eta)Z(z) \not\equiv 0, \tag{4.1.1}$$

each of which is a function of only one coordinate, then after substituting (4.1.1) into (1.6.21) and subsequent separation of variables in the equation we come to the conclusion that for constructing the particular solution to equation (1.6.21) in the form of (4.1.1) we should take as the functions Z, Φ and R any solutions to the respective equations,

$$\frac{d^2Z}{dz^2} + h^2Z = 0, \tag{4.1.2}$$

$$\frac{d^2\Phi}{d\eta^2} + (a - 2q\cos 2\eta)\Phi = 0, \tag{4.1.3}$$

$$\frac{d^2R}{d\xi^2} - (a - 2q\operatorname{ch} 2\xi)R = 0, \tag{4.1.4}$$

in which a and h are separation constants, and $q = \frac{f^2}{4}(k^2 - h^2)$. The particular solutions to equation (4.1.2) are the functions $Z = \begin{matrix}\cos\\ \sin\end{matrix}(hz)$ or in complex form $Z = e^{ihz}$, where h remains an arbitrary constant. Equation (4.1.3) is a Mathieu function written in the conventional canonical form. For a given q its unique solutions /142 (periodic with the period π or 2π relative to the argument η) are the periodic Mathieu functions (Mathieu functions of the first sort) corresponding to the computed sequence $\{a_n\}$ with the properly chosen numbers a (eigenvalues of the parameter a). There exist four types of periodic solutions to the Mathieu equation (4.1.3):

$ce_{2n}(\eta, q)$ is the even Mathieu function with the period π; it

corresponds to the eigen number $a = a_{2n}$, $n = 0, 1, \ldots$;

$ce_{2n+1}(\eta, q)$ is the even Mathieu function with the period 2π; it corresponds to the eigen number $a = a_{2n+1}$, $n = 0, 1, \ldots$;

$se_{2n+1}(\eta, q)$ is the odd Mathieu function with the period 2π; it corresponds to the eigen number $a = b_{2n+1}$, $n = 0, 1, \ldots$;

$se_{2n+2}(\eta, q)$ is the odd Mathieu function with the period π; it corresponds to the eigen number $a = b_{2n+2}$, $n = 0, 1, \ldots$.

Here to each eigenvalue of the parameter a there corresponds only one periodic solution to the Mathieu equation (4.1.3) with a fixed q. Any second solution is nonperiodic.

Equation (4.1.4) coincides in form with the Mathieu equation and may be obtained from a subsequent substitution in it of η by $\pm i\xi$. Equation (4.1.4) is called a modified Mathieu equation and its solution, representing values of the parameter a for which the Mathieu equation (4.1.3) allows periodic solutions for a given q, are called modified Mathieu functions. The first particular solutions to equation (4.1.4), representing the eigen numbers a_n and b_n, are denoted through $Ce_n(\xi, q)$ and $Se_n(\xi, q)$, respectively. These functions are largely similar to the Bessel functions $J_n(x)$. They are regular everywhere inside the ellipse in the plane z = const, including the lines of the focal points of the ellipse. As the second solutions to equation (4.1.4) we take the functions which agree in their properties with the Neumann functions $N_n(x)$. They are denoted through $Fey_n(\xi, q)$ and $Gey_n(\xi, q)$. Here $Fey_n(\xi, q)$ is the second solution to equation (4.1.4) when as its first solution we take the function $Ce_n(\xi, q)$, but $Gey_n(\xi, q)$ is the second solution to equation (4.1.4) when as the first solution we take the function $Se_n(\xi, q)$. The functions $Fey_n(\xi, q)$ and $Gey_n(\xi, q)$ are regular everywhere outside the ellipse on the plane z = const and inside it. In application, as the second solution to equation (4.1.4), we ordinarily take rather than the functions $Fey_n(\xi, q)$ and $Gey_n(\xi, q)$ the linear combinations:

$$Me_n^{(1),(2)}(\xi, q) = Ce_n(\xi, q) \pm i\,Fey_n(\xi, q), \quad a = a_n, \tag{4.1.5}$$

$$Ne_n^{(1),(2)}(\xi, q) = Se_n(\xi, q) \pm i\,Gey_n(\xi, q), \quad a = b_n. \tag{4.1.6}$$

The functions $Me_n^{(1),(2)}(\xi, q)$, $Ne_n^{(1),(2)}(\xi, q)$, called the combined Mathieu functions, are similar to the Hankel functions $H_n^{(1),(2)}(x)$, /14 respectively. They satisfy the condition of radiation at infinity.

Thus, the unique particular solutions to equation (1.6.21), represented in the form of (4.1.1), will be

$$Ce_n(\xi, q)\,ce_n(\eta, q)\,e^{ihz}, \quad a = a_n, \quad n = 0, 1, \ldots, \tag{4.1.7}$$

$$\mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q)\,e^{ihz}, \quad a = b_n, \quad n = 1, 2, \ldots, \tag{4.1.8}$$

$$\mathrm{Me}_n^{(1),(2)}(\xi, q)\,\mathrm{ce}_n(\eta, q)\,e^{ihz}, \quad a = a_n, \quad n = 0, 1, \ldots, \tag{4.1.9}$$

$$\mathrm{Ne}_n^{(1),(2)}(\xi, q)\,\mathrm{se}_n(\eta, q)\,e^{ihz}, \quad a = b_n, \quad n = 1, 2, \ldots \tag{4.1.10}$$

The functions (4.1.7) - (4.1.10) are called wave functions of an elliptical cylinder or simply elliptical wave functions. By virtue of the homogeneity and linearity of equation (1.6.21), its general solution, that is regular inside the cylinder, may be represented in the form of a series of Fourier integrals such as[1]

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \{\bar{a}_n(h)\,\mathrm{Ce}_n(\xi, q)\,\mathrm{ce}_n(\eta, q) + \bar{b}_n(h)\,\mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q)\}\,e^{ihz}dh, \tag{4.1.11}$$

and the general solution which satisfies the condition of radiation at infinity can be given in the form of the series,

$$\sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \{\bar{c}_n(h)\,\mathrm{Me}_n^{(1),(2)}(\xi, q)\,\mathrm{ce}_n(\eta, q) + \bar{d}_n(h)\,\mathrm{Ne}_n^{(1),(2)}(\xi, q)\,\mathrm{se}_n(\eta, q)\}\,e^{ihz}\,dh. \tag{4.1.12}$$

The expansion coefficients $\bar{a}$, $\bar{b}$, $\bar{c}$ and $\bar{d}$ depend on n and h and are subject to determination from the boundary conditions.

In the particular case when equation (1.6.21) is independent of the space coordinate z, as the elliptical wave functions we have the functions

$$\mathrm{Ce}_n(\xi, q)\,\mathrm{ce}_n(\eta, q), \quad a = a_n, \tag{4.1.13}$$

$$\mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q), \quad a = b_n, \tag{4.1.14}$$

$$\mathrm{Me}_n^{(1),(2)}(\xi, q)\,\mathrm{ce}_n(\eta, q), \quad a = a_n, \tag{4.1.15}$$

$$\mathrm{Ne}_n^{(1),(2)}(\xi, q)\,\mathrm{se}_n(\eta, q), \quad a = b_n, \tag{4.1.16}$$

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[1] Here and everywhere below the terms with $\mathrm{Se}_0(\xi, q)\mathrm{se}_0(\eta, q)$ or with $\mathrm{Ne}_0^{(1),(2)}(\xi, q)\mathrm{se}_0(\eta, q)$ should be omitted in the sums.

and the general solution to the equation may be represented by the /11
expansions:

$$\sum_{n=0}^{\infty} \{\bar{a}_n \mathrm{Ce}_n(\xi;\ q)\,\mathrm{ce}_n(\eta,\ q) + \bar{b}_n \mathrm{Se}_n(\xi,\ q)\,\mathrm{se}_n(\eta,\ q)\}, \qquad (4.1.17)$$

$$\sum_{n=0}^{\infty} \{\bar{c}_n \mathrm{Me}_n^{(1),\,(2)}(\xi,\ q)\,\mathrm{ce}_n(\eta,\ q) + \bar{d}_n \mathrm{Ne}_n^{(1),\,(2)}(\xi,\ q)\,\mathrm{se}_n(\eta,\ q)\}, \qquad (4.1.18)$$

where now $q = f^2k^2/4$.

§2. Mathieu Functions and Certain of Their Properties.

The Mathieu equation,

$$\frac{d^2y}{dz^2} + (a - 2q\cos 2z)\,y = 0 \qquad (4.2.1.)$$

with the parameters a and q represents a particular case of the more common linear differential equation of second order with periodic coefficients. Equation (4.2.1.) has solutions for any values of the parameters a and q. Since any point z (excluding infinity) is a regular point of the Mathieu equation, then the solutions to the Mathieu equation are also regular functions of the argument z. Depending on the values of the parameters a and q the solutions to the Mathieu equation may be expressed in different forms. Not all the solutions to equation (4.2.1) are most important for application, but only the periodic solutions relative to the argument z. In particular, when $q = 0$ and when equation (4.2.1) converts to the equation $y'' + ay = 0$, such solutions will be the trigonometric functions $\begin{matrix}\cos\\ \sin\end{matrix}(nz)$, if $a = n^2$, where $n = 0, \pm 1, \ldots$ In the general case the Mathieu equation does not have periodic solutions for each pair of values of the parameters a and q. For a given real value of the parameter q there exists a denumerable set of such values of the parameter a (eigenvalues of the parameter a), for which the Mathieu equation has periodic solutions with the period 2π relative to the argument z. In studying such solutions to the Mathieu equation it is sufficient to confine ourselves to an examination of only the even and odd periodic solutions relative to z, which are not identically zero, and which are called Mathieu functions (in the broadest sense by the Mathieu functions we mean any solution to the Mathieu equation). In practice, the Mathieu functions are usually given in the form of trigonometric Fourier series ($q > 0$): /11

$$\mathrm{ce}_{2n}(z,\ q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \cos 2rz, \cdot \qquad a_{2n}, \qquad (4.2.2)$$

$$\mathrm{ce}_{2n+1}(z, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \cos(2r+1)z, \quad a_{2n+1} \tag{4.2.3}$$

$$\mathrm{se}_{2n+2}(z, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \sin(2r+2)z, \quad b_{2n+2}, \tag{4.2.4}$$

$$\mathrm{se}_{2n+1}(z, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \sin(2r+1)z, \quad b_{2n+1}, \tag{4.2.5}$$

whose coefficients satisfy the recurrent relationships:

$$\left.\begin{aligned} aA_0^{(2n)} - qA_2^{(2n)} &= 0 \\ (a-4)A_2^{(2n)} - q(A_4^{(2n)} + 2A_0^{(2n)}) &= 0 \\ (a-4r^2)A_{2r}^{(2n)} - q(A_{2r+2}^{(2n)} + A_{2r-2}^{(2n)}) &= 0 \end{aligned}\right\} \quad \text{for } \mathrm{ce}_{2n}(z, q), \quad r \geqslant 2, \tag{4.2.6}$$

$$\left.\begin{aligned} (a-1-q)A_1^{(2n+1)} - qA_3^{(2n+1)} &= 0 \\ [a-(2r+1)^2]A_{2r+1}^{(2n+1)} - q(A_{2r+3}^{(2n+1)} + A_{2r-1}^{(2n+1)}) &= 0 \end{aligned}\right\} \quad \text{for } \mathrm{ce}_{2n+1}(z, q), \quad r \geqslant 1, \tag{4.2.7}$$

$$\left.\begin{aligned} (a-4)B_2^{(2n+2)} - qB_4^{(2n+2)} &= 0 \\ (a-4r^2)B_{2r}^{(2n+2)} - q(B_{2r+2}^{(2n+2)} + B_{2r-2}^{(2n+2)}) &= 0 \end{aligned}\right\} \quad \text{for } \mathrm{se}_{2n+2}(z, q), \quad r \geqslant 2, \tag{4.2.8}$$

$$\left.\begin{aligned} (a-1+q)B_1^{(2n+1)} - qB_3^{(2n+1)} &= 0 \\ [a-(2r+1)^2]B_{2r+1}^{(2n+1)} - q(B_{2r+3}^{(2n+1)} + B_{2r-1}^{(2n+1)}) &= 0 \end{aligned}\right\} \quad \text{for } \mathrm{se}_{2n+1}(z, q), \quad r \geqslant 1. \tag{4.2.9}$$

Here it is assumed that, for example in (4.2.2), the coefficients of the series satisfy the condition,

$$|v_{2r}| \sim \frac{q}{4(r+1)^2} \underset{r\to\infty}{\to} 0, \qquad v_{2r} = \frac{A_{2r+2}^{(2n)}}{A_{2r}^{(2n)}}, \tag{4.2.10}$$

which guarantees an absolute and uniform convergence of the series for all real values of the argument z (a and q are assumed finite). We find a similar conclusion also for the other series (4.2.3) - (4.2.5). We can show that the series (4.2.2) - (4.2.5) allow a term-by-term differentiation and integration and that the series obtained here converge absolutely and uniformly. /146

Unlike the Bessel functions, those of Mathieu have not been studied completely and fully, although many papers have been devoted to them (see, for example, [30, 32, 36, 37, 40, 43, 44, 183-185, 188]. Let us cite certain of their properties.

1. The Mathieu functions, which correspond to one and the same value of the parameter q and to different eigen numbers, are orthogonal in the interval (o, 2π), and

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{ce}_{2n}^2(z, q)\,dz = 2[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty}[A_{2r}^{(2n)}]^2, \tag{4.2.11}$$

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{ce}_{2n+1}^2(z, q)dz = \sum_{r=0}^{\infty}[A_{2r+1}^{(2n+1)}]^2, \tag{4.2.12}$$

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{se}_{2n+2}^2(z, q)\,dz = \sum_{r=0}^{\infty}[B_{2r+2}^{(2n+2)}]^2, \tag{4.2.13}$$

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{se}_{2n+1}^2(z, q)\,dz = \sum_{r=0}^{\infty}[B_{2r+1}^{(2n+1)}]^2. \tag{4.2.14}$$

On the basis of (4.2.11) - (4.2.14), the Mathieu functions for all real values of the parameter q are usually normalized by the conditions,

$$\frac{1}{\pi}\int_0^{2\pi} \mathrm{ce}_n^2(z, q)\,dz = \frac{1}{\pi}\int_0^{2\pi} \mathrm{se}_n^2(z, q)\,dz = 1, \tag{4.2.15}$$

if we assume

$$2[A_0^{(2n)}]^2 + \sum_{r=1}^{\infty}[A_{2r}^{(2n)}]^2 = 1, \quad \sum_{r=0}^{\infty}[A_{2r+1}^{(2n+1)}]^2 = 1,$$
$$\sum_{r=0}^{\infty}[B_{2r+2}^{(2n+2)}]^2 = 1, \quad \sum_{r=0}^{\infty}[B_{2r+1}^{(2n+1)}]^2 = 1. \tag{4.2.16}$$

From the condition of normalization (4.2.15) it follows that any Mathieu function has in the interval (0, 2π) a mean square value 1/2, equal to the same value for the respective trigonometric function. From this condition it also follows that when $q = 0$, /147

$$A_0^{(0)} = 2^{-1/2}, \quad \text{such that } \lim_{q\to 0}\mathrm{ce}_0(z, q) = 2^{-1/2}. \tag{4.2.17}$$

With substitution of q by -q the Mathieu equation can be rewritten in the form $\frac{d^2y}{dz^2} + (a + 2q\cos 2z)y = 0$. It takes such a

form also with the substitution in it of the argument z by the argument $\pm(\pi/2 \pm z)$. Consequently, in the case of negative values of the parameter q the periodic solutions to the Mathieu equations may be obtained from (4.2.2) - (4.2.5) by substitution of the argument z with the argument $\pm(\pi/2 \pm z)$. In particular, if instead of z we set $\pi/2 - z$, then

$$\begin{aligned}
\mathrm{ce}_{2n}(z,-q) &= (-1)^n \mathrm{ce}_{2n}\left(\frac{\pi}{2} - z,\ q\right),\\
\mathrm{ce}_{2n+1}(z,-q) &= (-1)^n \mathrm{se}_{2n+1}\left(\frac{\pi}{2} - z,\ q\right),\\
\mathrm{se}_{2n+2}(z,-q) &= (-1)^n \mathrm{se}_{2n+2}\left(\frac{\pi}{2} - z,\ q\right),\\
\mathrm{se}_{2n+1}(z,\ q) &= (-1)^n \mathrm{ce}_{2n+1}\left(\frac{\pi}{2} - z,\ q\right),
\end{aligned} \qquad (4.2.18)$$

where the existence of the factor $(-1)^n$ guarantees conversion of the Mathieu functions into the respective trigonometric functions when $q = 0$.

3. On the basis of the property of orthogonality of the Mathieu functions and of the trigonometric functions we can establish:

$$\begin{aligned}
\cos 2rz &= \sum_{n=0}^{\infty} A_{2r}^{(2n)} \mathrm{ce}_{2n}(z,\ q), && a_{2n},\\
\cos(2r+1)z &= \sum_{n=0}^{\infty} A_{2r+1}^{(2n+1)} \mathrm{ce}_{2n+1}(z,\ q), && a_{2n+1},\\
\sin(2r+1)z &= \sum_{n=0}^{\infty} B_{2r+1}^{(2n+1)} \mathrm{se}_{2n+1}(z,\ q), && b_{2n+1},\\
\sin(2r+2)z &= \sum_{n=0}^{\infty} B_{2r+2}^{(2n+2)} \mathrm{se}_{2n+2}(z,\ q), && b_{2n+2},
\end{aligned} \qquad (4.2.19)$$

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and

$$\sum_{n=0}^{\infty} [A_{2r}^{(2n)}]^2 = \sum_{n=0}^{\infty} [A_{2r+1}^{(2n+1)}]^2 =$$

$$= \sum_{n=0}^{\infty} [B_{2r+1}^{(2n+1)}]^2 = \sum_{n=0}^{\infty} [B_{2r+2}^{(2n+2)}]^2 = 1,$$

$$\sum_{n=0}^{\infty}[A_{2r}^{(2n)}A_{2s}^{(2n)}]=\sum_{n=0}^{\infty}[A_{2r+1}^{(2n+1)}A_{2s+1}^{(2n+1)}]=$$

$$=\sum_{n=0}^{\infty}[B_{2r+1}^{(2n+1)}B_{2s+1}^{(2n+1)}]=\sum_{n=0}^{\infty}[B_{2r+2}^{(2n+2)}B_{2s+2}^{(2n+2)}]=0,\ r\neq s. \qquad (4.2.20)$$

4. From the recurrent relationships (4.2.6) - (4.2.9), we can obtain the convergent infinite continued fractions:

$$a=\cfrac{-q^2/2}{1-\frac{1}{4}a-\cfrac{q^2/64}{1-\frac{a}{16}-\cfrac{q^2/576}{1-\frac{a}{36}-\cfrac{q^2/2304}{1-\frac{a}{64}-}}}} \qquad (a_{2n}) \qquad (4.2.21)$$

with the common term,

$$[q^2/16r^2(r-1)^2]/[1-(a-qv_{2r})/4r^2],\ r\geqslant 2,\quad v_{2r}=\frac{A_{2r+2}^{(2n)}}{A_{2r}^{(2n)}};$$

$$a=1+q-\cfrac{q^2/9}{1-\frac{a}{9}-\cfrac{q^2/225}{1-\frac{a}{25}-\cfrac{q^2/1225}{1-\frac{a}{49}-}}} \qquad (a=a_{2n+1}) \qquad (4.2.22)$$

with the common term, /149

$$[q^2/(2r+1)^2(2r-1)^2]/[1-a/(2r+1)^2+$$

$$+qv_{2r+1}/(2r+1)^2,\ r\geqslant 1,\ v_{2r+1}=\frac{A_{2r+3}^{(2n+1)}}{A_{2r+1}^{(2n+1)}};$$

$$b=1+q-\cfrac{q^2/9}{1-\frac{a}{9}-\cfrac{q^2/225}{1-\frac{a}{25}-\cfrac{q^2/1225}{1-\frac{a}{49}-}}} \qquad (a=b_{2n+1}) \qquad (4.2.23)$$

with the common term

$$[q^2/(2r+1)^2(2r-1)^2]/[1-a/(2r+1)^2+$$
$$+qv_{2r+1}/(2r+1)^2],\ r\geqslant 1,\ v_{2r+1}=\frac{B_{2r+3}^{(2n+1)}}{B_{2r+1}^{(2n+1)}};$$
$$b=4-\cfrac{q^2/16}{1-\cfrac{a}{16}-\cfrac{q^2/576}{1-\cfrac{a}{36}-\cfrac{q^2/2304}{1-\cfrac{a}{64}-\ddots}}}\qquad(a=b_{2n+2}) \tag{4.2.24}$$

and with the common term,

$$[q^2/(2r+2)^2(2r)^2]/[1-a/4(r+1)^2+$$
$$+qv_{2r}/4(r+1)^2],\quad r\geqslant 2,\quad v_{2r}=\frac{B_{2r+4}^{(2n+2)}}{B_{2r+2}^{(2n+2)}}.$$

Each of the fractions is a transcendental equation relative to the eigenvalues of the Mathieu functions. If we use (4.2.21) - (4.2.24), according to the given q we can compute the eigenvalues with any degree of accuracy (see, for example, [32], Ch. III), and through them, from the recurrent relationships (4.2.6) - (4.2.9), the coefficients of the Mathieu functions as well.

It is known that to the infinite continued fraction there corresponds an infinite sequence of convergents, each of which may be represented in the form of the ratio of two integral functions (polynomials) of its own elements.

Therefore, if we study the sequential convergents, corresponding to the infinite continued fractions (4.2.21) - (4.2.24), we /150
can establish that each of the infinite continued fractions allows representation in the form of a certain rational function of the parameters a, q having the form of the ratio of two integral functions whose denominator does not vanish if a is an eigen number of the Mathieu function. Consequently, the eigen numbers of the Mathieu functions are continuous and unique functions of the parameter q. The coefficients A, B of the Mathieu functions are also continuous and unique functions of q.

5. If we use the recurrent formulas (4.2.6) - (4.2.9) we can show that when $q \to 0$ all coefficients A, $B \to 0$, with the exception only of the coefficients $A_n^{(n)}$, $B_n^{(n)}$ which tend to unity. More complete investigations of the behavior of the coefficients $A_m^{(n)}$, $B_m^{(n)}$, when $q \to 0$, show that with a small value of q they have orders

of smallness of $O\left(q^{\frac{|n-m|}{2}}\right)$ and can be represented by the formulas:

$$\left.\begin{matrix} A_{n-2r}^{(n)} \\ B_{n-2r}^{(n)} \end{matrix}\right\} \approx \frac{(n-r-1)!}{r!\,(n-1)!}\left(\frac{q}{4}\right)^r, \tag{4.2.25}$$

$$\left.\begin{matrix} A_{n+2r}^{(n)} \\ B_{n+2r}^{(n)} \end{matrix}\right\} \approx \frac{(-1)^r n!}{r!\,(n+r)!}\left(\frac{q}{4}\right)^r, \tag{4.2.26}$$

from which it is clear that the coefficients A, B of an order greater than n have alternating signs, and all coefficients of an order less than n are positive.

When $q \to 0$, approximately,

$$\begin{aligned} &\mathrm{ce}_n(\eta, q) \to \cos n\eta = \cos n\varphi, \quad n \geqslant 1, \\ &\mathrm{ce}_0(\eta, q) \to A_0^{(0)} = 2^{-1/2}, \\ &\mathrm{se}_n(\eta, q) \to \sin n\eta = \sin n\varphi, \quad n \geqslant 1, \end{aligned} \tag{4.2.27}$$

where ϕ is a polar coordinate. Here $a \to n^2$.

6. From the general theory of ordinary differential equations of Sturm-Liouville type to which the Mathieu equation belongs, it follows that when $n \to \infty$ the eigenvalues of the equation will have the asymptotic expression,

$$a_n = n^2 + O(1), \tag{4.2.28}$$

and the Mathieu functions corresponding to them are written in the form,

$$\begin{aligned} &\mathrm{ce}_n(z, q) = \cos nz + 0\left(\frac{1}{n}\right), \\ &\mathrm{se}_n(z, q) = \sin nz + 0\left(\frac{1}{n}\right). \end{aligned} \tag{4.2.29}$$

If we again use the recurrent relationships (4.2.6) - (4.2.9), we can establish that when $n \to \infty$ all A, $B \to 0$, with the exception of $A_n^{(n)}$, $B_n^{(n)} \to 1$. A more complete analysis will lead to asymptotic representations of the coefficients A, B: /151

$$\left.\begin{matrix} A_{n-2r}^{(n)} \\ B_{n-2r}^{(n)} \end{matrix}\right\} \to \frac{(n-r-1)!}{r!\,(n-1)!}\left(\frac{q}{4}\right)^r, \quad 0 \leqslant 2r \leqslant n, \tag{4.2.30}$$

$$\left.\begin{matrix} A_{n+2r}^{(n)} \\ B_{n+2r}^{(n)} \end{matrix}\right\} \to \frac{(-1)^r n!}{r!\,(n+r)!}\left(\frac{q}{4}\right)^r, \quad r \geqslant 0 \tag{4.2.31}$$

(a and q are fixed), from which it follows that when $n \to \infty$ the coefficients will be $0(n^{-r})$ such that for them,

$$|A_m^{(n)}|, \quad |B_m^{(n)}| \leqslant \text{const} \left(\frac{q}{n}\right)^{\frac{|n-m|}{2}} \tag{4.2.32}$$

7. The Mathieu functions form a complete system of functions in the interval $(0, 2\pi)$. This property of theirs, along with the property of orthogonality, plays an important role in the theory of the expansion of arbitrary functions into Fourier series according to the Mathieu functions.

§3. Modified Mathieu Functions and Certain of Their Properties.

An equation such as

$$\frac{d^2y}{dz^2} - (a - 2q\operatorname{ch} 2z)\, y = 0 \tag{4.3.1}$$

is called an equation of the modified Mathieu functions (as already noted, it may be obtained from (4.2.1) by substitution there of z by iz). The solutions to equation (4.3.1), possessing a purely imaginary period πi or $2\pi i$ relative to the argument z, are called modified Mathieu functions (in the broad sense any solution to equation (4.3.1) is called a modified Mathieu function). For values of the parameter a, represented by the ordinary Mathieu functions, the modified functions are obtained from (4.2.2) - (4.2.5) with the substitution there of iz instead of z:

$$\begin{aligned}
\mathrm{Ce}_{2n}(z, q) &= \mathrm{ce}_{2n}(iz, q) = \sum_{r=0}^{\infty} A_{2r}^{(2n)} \operatorname{ch} 2rz, \quad a_{2n}, \\
\mathrm{Ce}_{2n+1}(z, q) &= \mathrm{ce}_{2n+1}(iz, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2n+1)} \operatorname{ch}(2r+1)z, \quad a_{2n+1}, \\
\mathrm{Se}_{2n+1}(z, q) &= -i\,\mathrm{se}_{2n+1}(iz, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2n+1)} \operatorname{sh}(2r+1)z, \quad b_{2n+1}, \\
\mathrm{Se}_{2n+1}(z, q) &= -i\,\mathrm{se}_{2n+2}(iz, q) = \sum_{r=0}^{\infty} B_{2r+2}^{(2n+2)} \operatorname{sh}(2r+2)z, \quad b_{2n+2},
\end{aligned} \tag{4.3.2}$$

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where $q > 0$ and where the coefficients of the series A, B satisfy the recurrent relationships (4.2.6) - (4.2.9). In the case of a negative q the expressions of the modified Mathieu functions are obtained from (4.2.18) by substituting iz for z there. The series (4.3.2) converge absolutely and uniformly in any finite region of the plane of the complex variable z or in any closed interval of the real axis.

In application, other forms of representing the modified Mathieu functions are more suitable which have the form of series of Bessel functions or their products. The latter are preferable for a number of reasons. Through the products of the Bessel functions the modified Mathieu functions of the first sort are determined in the following manner:

$$\mathrm{Ce}_{2n+1}(z,\ q) = \frac{i}{c_{2n+1}A_1^{(2n+1)}} \times$$

$$\times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} [J_r(v_1) J_{r+1}(v_2) + J_{r+1}(v_1) J_r(v_2)],$$

$$\mathrm{Ce}_{2n}(z,\ q) = \frac{1}{c_{2n} A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(v_1) J_r(v_2),$$

$$\mathrm{Se}_{2n+1}(z,q) = \frac{i}{d_{2n+1} B_1^{(2n+1)}} \times \qquad (4.3.3)$$

$$\times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} [J_r(v_1) J_{r+1}(v_2) - J_{r+1}(v_1) J_r(v_2)],$$

$$\mathrm{Se}_{2n+2}(z,q) = \frac{-1}{d_{2n+2} B_2^{(2n+2)}} \times$$

$$\times \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} [J_r(v_1) J_{r+2}(v_2) - J_{r+2}(v_1) J_r(v_2)],$$

where

$$v_1 = q^{1/2} e^{-z}, \quad v_2 = q^{1/2} e^{z}, \text{ a}$$

$$c_{2n} = c_{2n}(q) = \frac{A_0^{(2n)}}{\mathrm{ce}_{2n}(0,\ q)\,\mathrm{ce}_{2n}\left(\frac{\pi}{2},\ q\right)},$$

$$c_{2n+1} = c_{2n+1}(q) = \frac{-iq^{1/2} A_1^{(2n+1)}}{\mathrm{ce}_{2n+1}(0,\ q)\,\mathrm{ce}_{2n+1}\left(\frac{\pi}{2},\ q\right)}, \qquad (4.3.4) \qquad /15$$

$$d_{2n+2} = d_{2n+2}(q) = \frac{qB_2^{(2n+2)}}{\mathrm{se}'_{2n+2}(0,\ q)\,\mathrm{se}'_{2n+2}\left(\frac{\pi}{2},\ q\right)},$$

$$d_{2n+1} = d_{2n+1}(q) = \frac{iq^{1/2} B_1^{(2n+1)}}{\mathrm{se}'_{2n+1}(0,\ q)\,\mathrm{se}_{2n+1}\left(\frac{\pi}{2},\ q\right)}.$$

By substituting $N_\eta(v_2)$ for $J_r(v_2)$ in (4.3.3) we find expressions of the second solutions to the modified Mathieu equation (the Mathieu functions of the second sort) which correspond to the same values of the parameter a:

$$\mathrm{Fey}_{2n}(z,\ q) = \frac{1}{c_{2n}A_0^{(2n)}} \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2n)} J_r(v_1) N_r(v_2),$$

$$\mathrm{Fey}_{2n+1}(z,\ q) = \frac{i}{c_{2n+1}A_1^{(2n+1)}} \times$$

$$\times \sum_{r=0}^{\infty} (-1)^r A_{2r+1}^{(2n+1)} [J_r(v_1)N_{r+1}(v_2) + J_{r+1}(v_1) N_r(v_2)],$$

$$\mathrm{Gey}_{2n+1}(z,\ q) = \frac{i}{d_{2n+1}B_1^{(2n+1)}} \times \qquad (4.3.5)$$

$$\times \sum_{r=0}^{\infty} (-1)^r B_{2r+1}^{(2n+1)} [J_r(v_1) N_{r+1}(v_2) - J_{r+1}(v_1) N_r(v_2)],$$

$$\mathrm{Gey}_{2n+2}(z,\ q) = \frac{-1}{d_{2n+2}B_2^{(2n+2)}} \times$$

$$\times \sum_{r=0}^{\infty} (-1)^r B_{2r+2}^{(2n+2)} [J_r(v_1) N_{r+2}(v_2) - J_{r+2}(v_1) N_r(v_2)],$$

where c_n and d_n are defined just as before. From (4.1.5), (4.1.6) on the basis of (4.3.3) and (4.3.5) we obtain expressions of the combined Mathieu functions. The series (4.3.3) and (4.3.5) converge absolutely and uniformly in any closed region of the plane z, by virtue of which the functions representing them are regular in any finite region of the plane including the origin. The rapidity of the convergence of the series increases with increase in z (z is real). /154

Let us look at certain properties of the modified Mathieu functions.

1. When q is substituted by $-q$, the modified Mathieu equation assumes the form $\frac{d^2y}{dz^2} - (a + 2q \operatorname{ch} 2z)y = 0$, at which we may arrive also by substituting in it $\frac{\pi}{2} i + z$ for the argument z. Therefore,

$$\mathrm{Ce}_{2n}(z, -q) = (-1)^n \mathrm{Ce}_{2n}\left(\frac{\pi}{2} i + z,\ q\right),$$

$$\mathrm{Ce}_{2n+1}(z, -q) = (-1)^{n+1} i\, \mathrm{Se}_{2n+1}\left(\frac{\pi}{2} i + z,\ q\right),$$

$$\mathrm{Fey}_{2n}(z, -q) = (-1)^n \mathrm{Fey}_{2n}\left(\frac{\pi}{2} i + z,\ q\right),$$

$$\mathrm{Fey}_{2n+1}(z, -q) = (-1)^{n+1} i\, \mathrm{Gey}_{2n+1}\left(\frac{\pi}{2} i + z,\ q\right),$$

$$\mathrm{Se}_{2n+1}(z, -q) = (-1)^{n+1} i\, \mathrm{Ce}_{2n+1}\left(\frac{\pi}{2} i + z,\ q\right),$$

$$\mathrm{Se}_{2n+2}(z, -q) = (-1)^{n+1}\,\mathrm{Se}_{2n+2}\left(\frac{\pi}{2}i + z,\ q\right),$$
$$\mathrm{Gey}_{2n+1}(z, -q) = (-1)^{n+1} i\,\mathrm{Fey}_{2n+1}\left(\frac{\pi}{2}i + z,\ q\right), \qquad (4.3.6)$$
$$\mathrm{Gey}_{2n+2}(z, -q) = (-1)^{n+1}\,\mathrm{Gey}_{2n+2}\left(\frac{\pi}{2}i + z,\ q\right).$$

On the basis of relationships (4.3.6) we may obtain expansions of the modified Mathieu functions with a negative value of the parameter q according to the Bessel functions of the purely imaginary argument or according to their products in a form that is analogous to expansion (4.3.3), (4.3.5).

2. If we study the series (4.3.3), (4.3.5) for large values of the parameter n ($n \to \infty$), on the basis of (4.2.30) - (4.2.32) and formulas (2.2.26) - (2.2.28) we can establish that with fixed values of q and z the dominant terms of the expansions will be /15
$\left(q = \frac{f^2k^2}{4}\right)$

$$\mathrm{Ce}_n(z, q) \sim \frac{i^n}{c_n} J_n(kv), \qquad \mathrm{Se}_n(z, q) \sim \frac{i^n}{d_n} J_n(kv),$$
$$\mathrm{Fey}_n(z, q) \sim \frac{i^n}{c_n} N_n(kv), \qquad \mathrm{Gey}_n(z, q) \sim \frac{i^n}{d_n} N_n(kv), \qquad (4.3.7)$$
$$\mathrm{Me}_n^{(1),(2)}(z, q) \sim \frac{i^n}{c_n} H_n^{(1),(2)}(kv), \quad \mathrm{Ne}_n^{(1),(2)}(z, q) \sim \frac{i^n}{d_n} H_n^{(1),(2)}(kv),$$
$$v = \frac{f}{2} e^z.$$

3. If $q > 0$, then with large values of the argument z the dominant terms of the asymptotic expansions of the modified Mathieu functions will be

$$\mathrm{Ce}_{2n}(z, q) \sim \frac{1}{c_{2n}} \left(\frac{2}{\pi kv}\right)^{1/2} \sin\left(kv + \frac{\pi}{4}\right),$$
$$\mathrm{Ce}_{2n+1}(z, q) \sim \frac{-i}{c_{2n+1}} \left(\frac{2}{\pi kv}\right)^{1/2} \cos\left(kv + \frac{\pi}{4}\right),$$
$$\mathrm{Se}_{2n+1}(z, q) \sim \frac{-i}{d_{2n+1}} \left(\frac{2}{\pi kv}\right)^{1/2} \cos\left(kv + \frac{\pi}{4}\right),$$
$$\mathrm{Se}_{2n+2}(z, q) \sim \frac{1}{d_{2n+2}} \left(\frac{2}{\pi kv}\right)^{1/2} \sin\left(kv + \frac{\pi}{4}\right),$$

$$\begin{aligned}
\mathrm{Fey}_{2n}(z,\ q) &\sim \frac{-1}{c_{2n}}\left(\frac{2}{\pi kv}\right)^{1/2}\cos\left(kv+\frac{\pi}{4}\right),\\
\mathrm{Fey}_{2n+1}(z,\ q) &\sim \frac{-i}{c_{2n+1}}\left(\frac{2}{\pi kv}\right)^{1/2}\sin\left(kv+\frac{\pi}{4}\right),\\
\mathrm{Gey}_{2n+1}(z,\ q) &\sim \frac{-i}{d_{2n+1}}\left(\frac{2}{\pi kv}\right)^{1/2}\sin\left(kv+\frac{\pi}{4}\right),\\
\mathrm{Gey}_{2n+2}(z,\ q) &\sim \frac{-1}{d_{2n+2}}\left(\frac{2}{\pi kv}\right)^{1/2}\cos\left(kv+\frac{\pi}{4}\right),
\end{aligned} \tag{4.3.8}$$

where, just as earlier, $v = fe^{z}/2$. The asymptotic expressions for the combined Mathieu functions are obtained directly from (4.1.5), (4.1.6) on the basis of (4.3.8); here /156

$$\begin{aligned}
c_{2n}\,\mathrm{Me}_{2n}^{(1),(2)}(z,\ q) &\sim \left(\frac{2}{\pi kv}\right)^{1/2} e^{\pm i\left(kv-\frac{\pi}{4}\right)},\\
c_{2n+1}\,\mathrm{Me}_{2n+1}^{(1),(2)}(z,\ q) &\sim \pm\left(\frac{2}{\pi kv}\right)^{1/2} e^{\pm i\left(kv-\frac{\pi}{4}\right)},\\
d_{2n+1}\,\mathrm{Ne}_{2n+1}^{(1),(2)}(z,\ q) &\sim \pm\left(\frac{2}{\pi kv}\right)^{1/2} e^{\pm i\left(kv-\frac{\pi}{4}\right)},\\
d_{2n+2}\,\mathrm{Ne}_{2n+2}^{(1),(2)}(z,\ q) &\sim \left(\frac{2}{\pi kv}\right)^{1/2} e^{\pm i\left(kv-\frac{\pi}{4}\right)}.
\end{aligned} \tag{4.3.9}$$

From the representations (4.3.8), it is clear that with an increase in z ($z > 0$) all modified functions fluctuate at an ever increasing frequency and with decreasing amplitudes. When $z \to \infty$ they all tend to zero according to the law of exponential functions.

In the case of negative values of the parameter q the dominant terms of the asymptotic expansions of the modified Mathieu functions are equal to

$$\begin{aligned}
c_{2n}\mathrm{Ce}_{2n}(z,-q) &\sim (-1)^n(2\pi kv)^{-1/2}e^{kv},\\
d_{2n+1}\mathrm{Se}_{2n+1}(z,-q) &\sim (-1)^n i(2\pi kv)^{-1/2}e^{kv},\\
c_{2n}\mathrm{Fey}_{2n}(z,-q) &\sim (-1)^n i(2\pi kv)^{-1/2}e^{kv},\\
d_{2n+1}\mathrm{Gey}_{2n+1}(z,-q) &\sim (-1)^n i\ (2\pi kv)^{-1/2}e^{kv},\\
c_{2n+1}\mathrm{Ce}_{2n+1}(z,-q) &\sim (-1)^n i(2\pi kv)^{-1/2}e^{kv},\\
d_{2n+2}\mathrm{Se}_{2n+2}(z,-q) &\sim (-1)^{n+1}(2\pi kv)^{-1/2}e^{kv},\\
c_{2n+1}\mathrm{Fey}_{2n+1}(z,-q) &\sim (-1)^n i(2\pi kv)^{-1/2}e^{kv},\\
d_{2n+2}\mathrm{Gey}_{2n+2}(z,-q) &\sim i(-1)^{n+1}(2\pi kv)^{-1/2}e^{kv}.
\end{aligned} \tag{4.3.10}$$

As is obvious from (4.3.10) they all differ from one another only by the constant factors, and each of the modified functions of the first sort tends monotonically to $+\infty$ when $z \to \infty$, and each of the functions of the second sort tends to $+i\infty$ when $z \to \infty$.

4. If $q \to 0$, but $z \to \infty$ such that $\frac{kfe^z}{2} \to kr$ remains constant (finite), then we find the formulas for degeneration of the modified Mathieu functions to Bessel functions: /15

$$\begin{gathered} c_n \operatorname{Ce}_n(z, q) \to i^n J_n(kr), \quad d_n \operatorname{Se}_n(z, q) \to i^n J_n(kr), \\ c_n \operatorname{Fey}_n(z, q) \to i^n N_n(kr), \quad d_n \operatorname{Gey}_n(z, q) \to i^n N_n(kr), \end{gathered} \tag{4.3.11}$$

$$\begin{gathered} c_n \operatorname{Me}_n^{(1),(2)}(z, q) \to i^n H_n^{(1),(2)}(kr), \\ d_n \operatorname{Ne}_n^{(1),(2)}(z, q) \to i^n H_n^{(1),(2)}(kr) \end{gathered}$$

and

$$\begin{gathered} \frac{d}{dz} \operatorname{Ce}_n(z, q) \to \frac{i^n}{c_n} (kr) J_n'(kr), \quad n \geqslant 0, \\ \frac{d}{dz} \operatorname{Se}_n(z, q) \to \frac{i^n}{d_n} (kr) J_n'(kr), \quad n \geqslant 1, \end{gathered}$$

$$\begin{gathered} \frac{d}{dz} \operatorname{Fey}_n(z, q) \to \frac{i^n}{c_n} (kr) N'(kr), \quad n \geqslant 0, \\ \frac{d}{dz} \operatorname{Gey}_n(z, q) \to \frac{i^n}{d_n} (kr) N_n'(kr), \quad n \geqslant 1 \end{gathered} \tag{4.3.12}$$

in which $q > 0$. If $q < 0$, then

$$\begin{gathered} c_{2n} \operatorname{Ce}_{2n}(z, -q) \to (-1)^n I_{2n}(kr), \\ d_{2n+1} \operatorname{Ce}_{2n+1}(z, -q) \to (-1)^n i I_{2n+1}(kr), \\ c_{2n+1} \operatorname{Se}_{2n+1}(z, -q) \to (-1)^n i I_{2n}(kr), \\ d_{2n+2} \operatorname{Se}_{2n+2}(z, -q) \to (-1)^{n+1} I_{2n+1}(kr), \end{gathered}$$

$$\begin{gathered} c_{2n} \operatorname{Me}_{2n}^{(1)}(z, -q) \to \frac{2i(-1)^{n+1}}{\pi} K_{2n}(kr), \\ d_{2n+1} \operatorname{Me}_{2n+1}^{(1)}(z, -q) \to \frac{2(-1)^{n+1}}{\pi} K_{2n+1}(kr), \\ c_{2n+1} \operatorname{Ne}_{2n+1}^{(1)}(z, -q) \to \frac{2(-1)^{n+1}}{\pi} K_{2n+1}(kr), \\ d_{2n+2} \operatorname{Ne}_{2n+2}^{(1)}(z, -q) \to \frac{2i(-1)^n}{\pi} K_{2n+1}(kr) \end{gathered} \tag{4.3.13}$$

nd

$$\frac{d}{dz}\mathrm{Se}_{2n+1}(z,-q)\to\frac{(-1)^n kri}{c_{2n+1}}I_{2n+1}(kr)$$

$$\frac{d}{dz}\mathrm{Se}_{2n+2}(z,-q)\to\frac{(-1)^{n+1}kr}{d_{2n+2}}I'_{2n+2}(kr),$$

$$\frac{d}{dz}\mathrm{Ce}_{2n}(z,-q)\to\frac{(-1)^n kr}{c_{2n}}I'_{2n}(kr), \qquad (4.3.14)$$

$$\frac{d}{dz}\mathrm{Ce}_{2n+1}(z,-q)\to\frac{(-1)^n kri}{d_{2n+1}}I'_{2n+1}(kr),$$

where $I_n(x)$, $K_n(x)$ are modified Bessel functions, connected with the respective Bessel functions of the purely imaginary argument by the relationships: /158

$$I_\nu(z)=e^{-i\pi\nu/2}J_\nu(ze^{i\pi/2}),$$

$$-\pi<\arg z<\frac{\pi}{2}, \qquad (4.3.15)$$

$$K_\nu(z)=\frac{\pi i}{2}e^{i\pi\nu/2}H_\nu^{(1)}(ze^{i\pi/2}),$$

$$I_\nu(z)=e^{i\pi\nu/2}J_\nu(ze^{-i\pi/2}),$$

$$-\frac{\pi}{2}<\arg z<\pi. \qquad (4.3.16)$$

$$K_\nu(z)=-\frac{\pi i}{2}e^{-i\pi\nu/2}H_\nu^{(2)}(ze^{-i\pi/2}),$$

5. If $q\to 0$, but z is finite, then as the limiting forms of the modified Mathieu functions we have the expressions,

$$\mathrm{Fey}_0(z,q)\simeq\frac{\sqrt{2}}{\pi}\left(z+\frac{1}{2}\ln q\right),$$

$$\left.\begin{matrix}\mathrm{Fey}_n\\ \mathrm{Gey}_n\end{matrix}\right\}(z,q)\simeq-2^{2n-1}(n-1)!\,n!\,\frac{e^{-nz}}{\pi q^n}, \quad n\geqslant 1, \qquad (4.3.17)$$

on the assumption that $e^{\pm z}$ is sufficiently small.

6. The Wronskian of the modified Mathieu functions is equal to

$$W\left[\mathrm{Ce}_n(z,q),\ \mathrm{Me}_n^{(1),(2)}(z,q)\right]=\pm\frac{2i(-1)^n}{\pi c_n^2},\ n\geqslant 0, \qquad (4.3.18)$$

$$W\left[\mathrm{Se}_n(z,q),\ \mathrm{Ne}_n^{(1),(2)}(z,q)\right]=\pm\frac{2i(-1)^n}{\pi d_n^2},\ n\geqslant 1.$$

7. If q is positive and large, then for the eigen numbers a we have the asymptotic formulas,

$$\left.\begin{matrix} a_{2n}(q) \\ b_{2n+1}(q) \end{matrix}\right\} \sim -2q + (8n+2)\,q^{1/2}, \qquad (4.3.19)$$

$$\left.\begin{matrix} a_{2n+1}(q) \\ b_{2n+2}(q) \end{matrix}\right\} \sim -2q + (8n+6)\,q^{1/2}.$$

If we use the recurrent relationships for the coefficients of the Mathieu functions, we can show that when $q \to \infty$ all coefficients A, $B \to 0$, and then /15

$$\begin{aligned} A_{2r}^{(2n)}/A_0^{(2n)} &\to (-1)^r 2, \\ A_{2r+1}^{(2n+1)}/A_1^{(2n+1)} &\to (-1)^r (2r+1), \\ B_{2r+1}^{(2n+1)}/B_1^{(2n+1)} &\to (-1)^r, \\ B_{2r+2}^{(2n+2)}/B_2^{(2n+2)} &\to (-1)^r (r+1). \end{aligned} \qquad (4.3.20)$$

The asymptotic representations of the ordinary Mathieu functions have the form[2]

$$\left.\begin{matrix} \mathrm{ce}_n(z,\,q) \\ \mathrm{se}_{n+1}(z,\,q) \end{matrix}\right\} \sim \left.\begin{matrix} C_n \\ S_{n+1} \end{matrix}\right\} \left\{ e^{2h\sin z}\left[\cos\left(\frac{z}{2}+\frac{\pi}{4}\right)\right]^{2n+1} \pm \right. \qquad (4.3.21)$$

$$\left. \pm\, e^{-2h\sin z}\left[\sin\left(\frac{z}{2}+\frac{\pi}{4}\right)\right]^{2n+1}\right\} \Big/ \cos^{n+1} z,\ h = q^{1/2} = \frac{kf}{2},$$

on the assumption that z varies in the interval $(-\pi/2,\ \pi/2)$. If $z \in (\pi/2,\ 3\pi/2)$, then the signs $\pm$ in the right hand side of (4.3.21) should be substituted by $\mp$, respectively. The constants C_n S_{n+1} are determined here by the formulas,

$$C_{2n} = \frac{(-1)^n 2^{2n}}{\sqrt{2\pi h}\, c_{2n}}, \quad C_{2n+1} = \frac{(-1)^n i 2^{2n+1/2}}{\sqrt{\pi h}\, c_{2n+1}}, \qquad (4.3.22)$$

$$S_{2n+1} = \frac{(-1)^n 2^{2n} i}{\sqrt{2\pi h}\, d_{2n+1}}, \quad S_{2n+2} = \frac{(-1)^{n+1} 2^{2n+1/2}}{\sqrt{\pi h}\, d_{2n+2}}$$

---

[2] In (4.3.21) when $n = 0$ in both cases we take only C_0. In the superscript the interval $-\pi/2<z<\pi/2$ corresponds to the function ce_0, but the interval $\pi/2<z<3\pi/2$ corresponds to the function se_1; in the subscript the situation is reversed.

which with large values of $q > 0$ convert into the expressions

$$C_n \sim S_{n+1} \sim (2\pi h)^{1/4} \frac{2^{3n+1/2}}{\sqrt{n!}} h^{n/2} e^{-2h} \qquad (4.3.23)$$

The asymptotic expressions for the modified Mathieu functions $\mathrm{Ce}_n(z, q)$ and $\mathrm{Se}_n(z, q)$ have the form,

$$\left.\begin{matrix} \mathrm{Ce}_n(z, q) \\ \mathrm{Se}_{n+1}(z, q) \end{matrix}\right\} \sim$$

$$\sim \left.\begin{matrix} \frac{C_n}{2^{n-1/2}\mathrm{ch}^{1/2} z} \cos \\ \frac{S_{n+1}}{2^{n-1/2}\mathrm{ch}^{1/2} z} \sin \end{matrix}\right\} \left[2h \operatorname{sh} z - (2n+1) \arctan \left(\operatorname{th} \frac{z}{2} \right) \right], \qquad (4.3.24)$$

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where it is assumed that $\arg z \in (-\pi/2, \pi/2)$. In application we usually encounter the products $c_n \mathrm{Ce}_n(z, q)$ and $d_n \mathrm{Se}_n(z, q)$, where c_n and d_n are determined by formulas (4.3.4). For them

$$\left.\begin{matrix} c_n \mathrm{Ce}_n(z, q) \\ d_n \mathrm{Se}_n(z, q) \end{matrix}\right\} \sim$$

$$\sim \frac{i^n}{\sqrt{\pi h \operatorname{ch} z}} \left.\begin{matrix} \cos \\ \sin \end{matrix}\right\} \left[2h \operatorname{sh} z - (2n+1) \arctan \left(\operatorname{th} \frac{z}{2} \right) \right]. \qquad (4.3.25)$$

On the basis of (4.3.24), from the integral representations of the combined Mathieu functions we obtain the asymptotic expressions of the functions $\mathrm{Me}_n^{(1),(2)}(z, q)$ and $\mathrm{Ne}_n^{(1),(2)}(z, q)$ [171 - 174]

$$c_n \mathrm{Me}_n^{(1),(2)}(z, q) \sim \frac{i^n}{\sqrt{\pi h \operatorname{ch} z}} e^{\pm i[2h \operatorname{sh} z \mp (2n+1) \arctan (\operatorname{th} z/2)]}, \qquad (4.3.26)$$

$$d_n \mathrm{Ne}_n^{(1),(2)}(z, q) \sim \mp i c_n \mathrm{Me}_n^{(1),(2)}(z, q). \qquad (4.3.27)$$

In the case when q is large in absolute value and negative, the asymptotic expressions for the Mathieu functions are obtained on the basis of formulas (4.3.21) - (4.3.27) from relationships (4.2.18), (4.3.6).

§4. Expansion of Plane and Spherical Waves According to Elliptical Wave Functions. Integral Representations of Wave Functions.

Let us represent function (2.3.1) in the form,

$$\psi = \psi_0 e^{ikz \cos \theta_0}, \qquad (4.4.1)$$

after setting

$$\psi_0 = e^{ik(nr)}, \tag{4.4.2}$$

where $(nr) = n_x x + n_y y$ or in elliptical coordinates $(nr) = f(\operatorname{ch}\xi \cos\eta \cos\phi_0 + \operatorname{sh}\xi \sin\eta \sin\phi_0)$. Then the expansion of the plane wave (4.4.2) (the function ψ_0 is the solution to the two-dimensional Helmholtz equation that is independent of the coordinate z) may be found from the expansion of the two-dimensional Green function $G_k(r|r_0)$ of the equation, /1

$$\Delta G_k(r|r_0) + k^2 G_k(r|r_0) = -\delta(r - r_0), \tag{4.4.3}$$

given on the plane Oxy according to the wave functions (4.1.13) - (4.1.16). The solution to equation (4.4.3), as we know, is the function,

$$G_k(r|r_0) = \pm \frac{i}{4} H_0^{(1),(2)}(k|r - r_0|), \tag{4.4.4}$$

which is symmetrical relative to the vector coordinates r and r_0 and which satisfies the condition of radiation at infinity. When $r \neq r_0$, function (4.4.4) will be the solution to the homogeneous Helmholtz equation, therefore it may be sought in the form of the expansion,

$$\begin{aligned} G_k(r|r_0) = &\sum_{n=0}^{\infty} A_n \operatorname{ce}_n(\eta, q) \operatorname{ce}_n(\eta_0, q) \begin{cases} \operatorname{Me}_n^{(1),(2)}(\xi, q)\operatorname{Ce}_n(\xi_0, q) \\ \operatorname{Me}_n^{(1),(2)}(\xi_0, q)\operatorname{Ce}_n(\xi, q) \end{cases} + \\ &+ \sum_{n=1}^{\infty} B_n \operatorname{se}_n(\eta, q) \operatorname{se}_n(\eta_0, q) \times \\ &\times \begin{cases} \operatorname{Ne}_n^{(1),(2)}(\xi, q) \operatorname{Se}_n(\xi_0, q), & \xi_0 < \xi, \\ \operatorname{Ne}_n^{(1),(2)}(\xi_0, q) \operatorname{Se}_n(\xi, q), & \xi_0 > \xi, \end{cases} \end{aligned} \tag{4.4.5}$$

whose coefficients are determined as usual by the nonhomogeneous term of equation (4.4.3). In elliptical coordinates we have,

$$\begin{aligned} &\frac{1}{f^2(\operatorname{ch}^2\xi - \cos^2\eta)} \left(\frac{\partial^2 G_k}{\partial \xi^2} + \frac{\partial^2 G_k}{\partial \eta^2} \right) + k^2 G_k = \\ &= -\frac{1}{f^2(\operatorname{ch}^2\xi - \cos^2\eta)} \delta(\xi - \xi_0)\delta(\eta - \eta_0). \end{aligned} \tag{4.4.6}$$

If we multiply both parts of (4.4.6) by $f^2(\mathrm{ch}^2\xi - \cos^2\eta)$ and integrate them relative to ξ in the immediate vicinity of the point $\xi = \xi_0$ from $\xi_0 - \varepsilon$ to $\xi_0 + \varepsilon$ and convert to the limit when $\varepsilon \to 0$, we find

$$\frac{\partial G_k(\eta, \xi; \eta_0, \xi_0)}{\partial \xi}\Bigg|_{\xi=\xi_0-0}^{\xi=\xi_0+0} = -\delta(\eta - \eta_0)$$

(function (4.4.5) remains continuous when $\xi = \xi_0$), whence after substitution here of (4.4.5) and converting to (4.3.18), we find /162

$$\sum_{n=0}^{\infty} A_n \mathrm{ce}_n(\eta, q)\mathrm{ce}_n(\eta_0, q)\frac{(-1)^n}{c_n^2} + \sum_{n=1}^{\infty} B_n \mathrm{se}_n(\eta, q)\mathrm{se}(\eta_0, q)\frac{(-1)^n}{d_n^2} = \pm\frac{i\pi}{2}\delta(\eta - \eta_0). \tag{4.4.7}$$

If we multiply successively both parts of (4.4.7) by $\mathrm{ce}_n(\eta, q)$ and $\mathrm{se}_n(\eta, q)$ and integrate them over η in the range from 0 to 2π, we obtain on the basis of the property of orthogonality of the angular Mathieu functions,

$$A_n = \pm\frac{i(-1)^n}{2}c_n^2, \qquad B_n = \pm\frac{i(-1)^n}{2}d_n^2, \tag{4.4.8}$$

such that

$$H_0^{(1),(2)}(k|\mathbf{r} - \mathbf{r}_0|) = 2\sum_{n=0}^{\infty}(-1)^n\Big[c_n^2\mathrm{ce}_n(\eta, q)\mathrm{ce}_n(\eta_0, q)\times \begin{cases}\mathrm{Me}_n^{(1),(2)}(\xi, q)\mathrm{Ce}_n(\xi_0, q)\\ \mathrm{Me}_n^{(1),(2)}(\xi_0, q)\mathrm{Ce}_n(\xi, q)\end{cases} + d_n^2\mathrm{se}_n(\eta, q)\mathrm{se}_n(\eta_0, q)\times \begin{cases}\mathrm{Ne}_n^{(1),(2)}(\xi, q)\mathrm{Se}_n(\xi_0, q)\\ \mathrm{Ne}_n^{(1),(2)}(\xi_0, q)\mathrm{Se}_n(\xi, q)\end{cases}\Big], \quad \begin{matrix}\xi_0 < \xi,\\ \xi_0 > \xi.\end{matrix} \tag{4.4.9}$$

If we set here $\xi_0 \to \infty$ (the source is removed to infinity), then $|\mathbf{r} - \mathbf{r}_0| \to r_0 - r\cos(\phi - \phi_0)$, and on the basis of the asymptotic formulas for the Bessel function $H_n^{(1)}(x)$ and the modified Mathieu functions we obtain the unknown formula of expansion of the plane wave (4.4.2) according to the elliptical wave functions:

$$e^{ik(\mathbf{nr})} = 2\sum_{n=0}^{\infty}\left[c_n\mathrm{Ce}_n(\xi, q)\mathrm{ce}_n(\eta, q)\mathrm{ce}_n(\varphi_0, q) + d_n\mathrm{Se}_n(\xi, q)\mathrm{se}_n(\eta, q)\mathrm{se}_n(\varphi_0, q)\right], \tag{4.4.10}$$

where $(nr) = f(\operatorname{ch}\xi\cos\eta\cos\phi_0 + \operatorname{sh}\xi\sin\eta\sin\phi_0)$. After substituting (4.4.10) into (4.4.1), we obtain the expansion of the space wave ψ according to the same functions [unlike (4.4.10), where with $z = 0$, $q = \frac{f^2k^2}{4}$, in the general case we will have $q = \frac{f^2}{4}(k^2 - h^2)$].

If we introduce into study the Green function (2.5.3), then in the usual manner we obtain the expansion of the spherical wave (2.5.1) according to the cylindrical elliptical functions (4.1.7) - (4.1.10) and then /163

$$\frac{e^{ikR}}{R} = i\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}(-1)^n\left[c_n^2\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\times\right.$$
$$\times\begin{cases}\mathrm{Me}_n^{(1),(2)}(\xi, q)\,\mathrm{Ce}_n(\xi_0, q)\\ \mathrm{Me}_n^{(1),(2)}(\xi_0, q)\,\mathrm{Ce}_n(\xi, q)\end{cases} + d_n^2\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)\times \tag{4.4.11}$$
$$\left.\times\begin{cases}\mathrm{Ne}_n^{(1),(2)}(\xi, q)\,\mathrm{Se}_n(\xi_0, q), & \xi_0 < \xi\\ \mathrm{Ne}_n^{(1),(2)}(\xi_0, q)\,\mathrm{Se}_n(\xi, q), & \xi_0 > \xi\end{cases}\right]e^{ih(z-z_0)}dh,$$

where $q = \frac{f^2}{4}(k^2 - h^2)$.

After multiplying both sides of the expansion (4.4.10) successively by $\mathrm{ce}_n(\eta_0, q)$ and by $\mathrm{se}_n(\eta_0, q)$ and using the property of orthogonality of the Mathieu functions, after integrating both parts relative to η_0 in the range from 0 to 2π, we obtain formulas for the integral representation of the functions (4.1.7), (4.1.8):

$$\mathrm{Ce}_n(\xi, q)\,\mathrm{ce}_n(\eta, q) = \frac{1}{2\pi c_n}\int_0^{2\pi}e^{ik(nr)}\,\mathrm{ce}_n(\eta_0, q)\,d\eta_0,$$
$$\mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q) = \frac{1}{2\pi d_n}\int_0^{2\pi}e^{ik(nr)}\,\mathrm{se}_n(\eta_0, q)\,d\eta_0. \tag{4.4.12}$$

For the functions (4.1.9), (4.1.10) we have the formulas[3]

$$\mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{ce}_n(\eta, q) = \frac{1}{\pi c_n}\int_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty}e^{ik(nr)}\,\mathrm{ce}_n(\eta_0, q)\,d\eta_0,$$

[3] In (4.4.13), (4.4.14) $|\eta| < \pi/2$ and $\operatorname{ch}\xi\cos\eta > 1$. If $\pi/2 < \eta < 3\pi/2$ and $-\operatorname{ch}\xi\cos\eta > 1$, in (4.4.13), (4.4.14) we should substitute $\exp[-ik(nr)]$ for $\exp[ik(nr)]$

$$\mathrm{Ne}_n^{(1)}(\xi,\ q)\,\mathrm{se}_n(\eta,\ q) = \frac{1}{\pi d_n}\int\limits_{-\frac{\pi}{2}+i\infty}^{\frac{\pi}{2}-i\infty} e^{ik(nr)}\,\mathrm{se}_n(\eta_0,\ q)\,d\eta_0, \qquad (4.4.13)$$

$$\mathrm{Me}_n^{(2)}(\xi,\ q)\,\mathrm{ce}_n(\eta,\ q) = \frac{1}{\pi c_n}\int\limits_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{ik(nr)}\,\mathrm{ce}_n(\eta,\ q)\,d\eta_0,$$

$$\mathrm{Ne}_n^{(2)}(\xi,\ q)\,\mathrm{se}(\eta,\ q) = \frac{1}{\pi d_n}\int\limits_{\pi/2-i\infty}^{3\pi/2+i\infty} e^{ik(nr)}\,\mathrm{se}_n(\eta_0,\ q)\,d\eta_0, \qquad (4.4.14)$$

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which just as (4.4.12), may be derived on the basis of arguments analogous to those given in §3 of Chapter 2. In formulas (4.4.12) - (4.4.14), (nr) = f(ch ξ cos η cos η_0+ sh ξ sin η sin η_0). In particular, from (4.4.12) - (4.4.14) it is easy to obtain integral representations for the modified Mathieu functions.

§5. Addition Theorems for Elliptical Wave Functions.

Let there be given on the plane Oxy two systems of Cartesian rectangular coordinate systems x_1, y_1 and x_2, y_2 and the systems of elliptical coordinates ξ_j, η_j, j = 1, 2 that correspond to them and are associated with them by the transformation formulas x_j = $f_j \mathrm{ch}\xi_j \cos\eta_j$, y_j = $f_j \mathrm{sh}\xi_j \sin\eta_j$. Let us assume that the coordinate

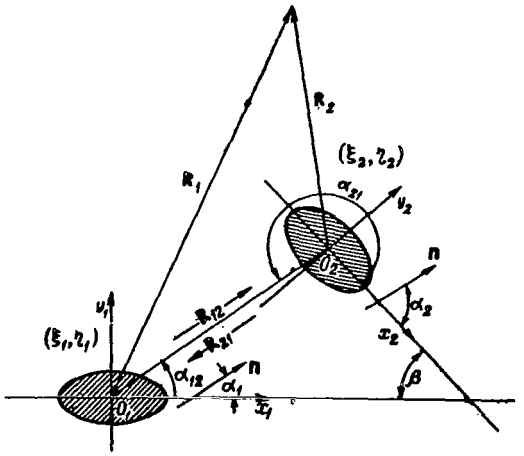

Fig. 18. Addition Theorem for Elliptical Wave Functions.

axes $O_j x_j$ of the systems are directed with respect to one another at an angle β (see Fig. 18). Furthermore let there be given elliptical wave functions (4.1.13) - (4.1.16) written in each of the local coordinate systems as the expressions,

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$$\mathrm{Ce}_n(\xi_j, q_j)\,\mathrm{ce}_n(\eta_j, q_j), \qquad \mathrm{Se}_n(\xi_j, q_j)\,\mathrm{se}_n(\eta_j, q_j)$$
$$j = 1,2. \qquad (4.5.1)$$
$$\mathrm{Me}_n^{(1),(2)}(\xi_j, q_j)\,\mathrm{ce}_n(\eta_j, q_j), \quad \mathrm{Ne}_n^{(1),(2)}(\xi_j, q_j)\,\mathrm{se}_n(\eta_j, q_j),$$

Let us find formulas which will permit writing the functions (4.5.1) belonging to the j-th system of coordinates through the wave functions written in the s-th system of coordinates, $j \neq s$, $j, s = 1, 2$. It is obvious that

$$\mathrm{ce}_n(q_j, \alpha \pm \beta)\,\mathrm{ce}_m(q_s, \alpha) =$$
$$= \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_j) A_p^{(m)}(q_s) \cos r\beta\,[\cos(r-p)\alpha + \cos(r+p)\alpha] \mp$$
$$\mp \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_j) A_p^{(m)}(q_s) \sin r\beta\,[\sin(r-p)\alpha + \sin(r+p)\alpha],$$
$$\mathrm{ce}_n(q_j, \alpha \pm \beta)\,\mathrm{se}_m(q_s, \alpha) =$$
$$= \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_j) B_p^{(m)}(q_s) \cos r\beta\,[\sin(p-r)\alpha + \sin(p+r)\alpha] \mp \qquad (4.5.2)$$
$$\mp \frac{1}{2}\sum_{r\ p}^{\infty} A_r^{(n)}(q_j) B_p^{m}(q_s) \sin r\beta\,[\cos(p-r)\alpha - \cos(p+r)\alpha],$$
$$\mathrm{se}_n(q_j, \alpha \pm \beta)\,\mathrm{ce}_m(q_s, \alpha) =$$
$$= \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_j) A_p^{(m)}(q_s) \cos r\beta\,[\sin(r-p)\alpha + \sin(r+p)\alpha] \pm$$
$$\pm \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_j) A_p^{(m)}(q_s) \sin r\beta\,[\cos(r-p)\alpha + \cos(r+p)\alpha],$$
$$\mathrm{se}_n(q_j, \alpha \pm \beta)\,\mathrm{se}_m(q_s, \alpha) =$$
$$= \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_j) B_p^{(m)}(q_s) \cos r\beta\,[\cos(r-p)\alpha - \cos(r+p)\alpha] \pm$$
$$\pm \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_j) B_p^{(m)}(q_s) \sin r\beta\,[\sin(p-r)\alpha + \sin(p+r)\alpha]$$

(summation here and below is carried out over even or odd numbers of r and p depending on the parity or nonparity of n and m, respectively). Let us introduce into study the plane wave (2.3.1) which, for example in the j-th system of coordinates, may be written in the form, /16

$$e^{ik(\mathbf{nR}_j)} = e^{ik(\mathbf{nR}_s)} e^{ik(\mathbf{nR}_{js})}, \qquad (4.5.3)$$

since $R_j = R_s + R_{js}$, where R_{js} is the radius-vector which connects the origin O_s with the origin O_j of the system $O_j x_j y_j$, but R_j, R_s is the radius-vector of an arbitrary point of observation in the respective system of coordinates. Here the direction of wave propagation determined by the unit vector n, forms arbitrary angles α_j and α_s with the positive direction of the coordinate axes $O_j x_j$ and $O_s x_s$ and we assume that $\alpha_j = \alpha_s + \beta$. On the basis of (4.4.10), function (4.5.3) may be represented in the form of an expansion,

$$e^{ik(nR_j)} = 2\sum_{n=0}^{\infty} [c_n \mathrm{Ce}_n(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s)\,\mathrm{ce}_n(\alpha_s, q_s) + d_n \mathrm{Se}_n(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\,\mathrm{se}_n(\alpha_s, q_s)]\, e^{ik(nR_{js})}. \tag{4.5.4}$$

After multiplying both parts of (4.5.4) successively by $\mathrm{ce}_m(\alpha_j, q_j)$ and by $\mathrm{se}_m(\alpha_j, q_j)$ and integrating them relative to α_j along the boundaries $c_{1,2,3}$ (c_1 is the boundary with ends at the points 0 and 2π; c_2 is the boundary with ends at the points $-\frac{\pi}{2} + i\infty$ and $\frac{\pi}{2} - i\infty$; c_3 is the boundary with ends at the points $\frac{\pi}{2} - i\infty$ and $\frac{3\pi}{2} + i\infty$), we find on the basis of the formulas for the integral representation of the elliptical wave functions (4.4.12) - (4.4.14), formulas (4.5.2) and the formulas for the integral representation of the wave functions of a circular cylinder (2.3.15) the desired addition theorems for the elliptical wave functions:

$$\mathrm{Ce}_m(\xi_j, q_j)\,\mathrm{ce}_m(\eta_j, q_j) = \frac{2}{c_m}\sum_{n=0}^{\infty} \{Q_{mn}(j, s; \beta)\, c_n \mathrm{Ce}_n(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + R_{mn}(j, s; \beta)\, d_n \mathrm{Se}_n(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}, \quad m = 0, 1, \ldots, \tag{4.5.5}$$

$$\mathrm{Se}_m(\xi_j, q_j)\,\mathrm{se}_m(\eta_j, q_j) = \frac{2}{d_m}\sum_{n=0}^{\infty} \tilde{Q}_{mn}(j, s; \beta)\, c_n \mathrm{Ce}_n(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + \tilde{R}_{mn}(j, s; \beta)\, d_n \mathrm{Se}_n(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}, \quad m = 1, 2, \ldots, \tag{4.5.6}$$

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$$\mathrm{Me}_m^{(1),(2)}(\xi_j, q_j)\,\mathrm{ce}_m(\eta_j, q_j) = \frac{2}{c_m}\sum_{n=0}^{\infty} \{Q_{mn}^{(1),(2)}(j, s; \beta)\, c_n \mathrm{Ce}_n(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + R_{mn}^{(1),(2)}(j, s; \beta)\, d_n \mathrm{Se}_n(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}, \quad m = 0, 1, \ldots, \tag{4.5.7}$$

$$\mathrm{Ne}_m^{(1),(2)}(\xi_j, q_j)\,\mathrm{se}_m(\eta_j, q_j) =$$

$$= \frac{2}{d_m}\sum_{n=0}^{\infty}\{\tilde{Q}_{mn}^{(1),(2)}(j, s; \beta)\,\mathrm{Ce}_n(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) +$$

$$+ \tilde{R}_{mn}^{(1),(2)}(j, s; \beta)\,d_n\,\mathrm{Se}_n(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}, \quad m = 1, 2, \ldots, \qquad (4.5.8)$$

where

$$Q_{mn}(j, s; \beta) = \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\,A_p^{(m)}(q_j)\cos r\beta \times$$

$$\times\{i^{r-p}J_{r-p}(kl)\cos(r-p)\alpha_{js} + i^{r+p}J_{r+p}(kl)\cos(r+p)\alpha_{js}\} \pm$$

$$\pm\frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\,A_p^{(m)}(q_j)\sin r\beta\{i^{r-p}J_{r-p}(kl)\sin(r-p)\alpha_{js} +$$

$$+ i^{r+p}J_{r+p}(kl)\sin(r+p)\alpha_{js}\}, \qquad (4.5.9)$$

$$R_{mn}(j, s; \beta) = \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\,A_p^{(m)}(q_j)\cos r\beta \times$$

$$\times\{i^{r-p}J_{r-p}(kl)\sin(r-p)\alpha_{js} + i^{r+p}J_{r+p}(kl)\sin(r+p)\alpha_{js}\} \mp$$

$$\mp\frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\,A_p^{(m)}(q_j)\sin r\beta\{i^{r-p}J_{r-p}(kl)\cos(r-p)\alpha_{js} +$$

$$+ i^{r+p}J_{r+p}(kl)\cos(r+p)\alpha_{js}\}, \qquad (4.5.10)$$

$$\tilde{Q}_{mn}(j, s; \beta) = -\frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\,B_p^{(m)}(q_j)\cos r\beta \times$$

$$\times\{i^{r-p}J_{r-p}(kl)\sin(r-p)\alpha_{js} - i^{r+p}J_{r+p}(kl)\sin(r+p)\alpha_{js}\} \pm$$

$$\pm\frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\,B_p^{(m)}(q_j)\sin r\beta\{i^{r-p}J_{r-p}(kl)\cos(r-p)\alpha_{js} -$$

$$- i^{r+p}J_{r+p}(kl)\cos(r+p)\alpha_{js}\}, \qquad (4.5.11)$$

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$$\tilde{R}_{mn}(j, s; \beta) = \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\,B_p^{(m)}(q_j)\cos r\beta \times$$

$$\times\{i^{r-p}J_{r-p}(kl)\cos(r-p)\alpha_{js} - i^{r-p}J_{r+p}(kl)\cos(r+p)\alpha_{js}\} \mp$$

$$\mp\frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\,B_p^{(m)}(q_j)\sin r\beta \times$$

$$\times\{i^{p-r}J_{p-r}(kl)\sin(p-r)\alpha_{js} + i^{p+r}J_{p+r}(kl)\sin(p+r)\alpha_{js}\}, \qquad (4.5.12)$$

$$Q_{mn}^{(1),(2)}(j,\ s;\ \beta) = \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\, A_p^{(m)}(q_j)\cos r\beta \times$$

$$\times \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\cos(r-p)\alpha_{js} + i^{r+p} H_{r+p}^{(1),(2)}(kl)\cos(r+p)\alpha_{js} \right\} \pm$$

$$\pm \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\, A_p^{(m)}(q_j)\sin r\beta \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\sin(r-p)\alpha_{js} + \right.$$

$$\left. + i^{r+p} H_{r+p}^{(1),(2)}(kl)\sin(r+p)\alpha_{js} \right\}, \qquad (4.5.13)$$

$$R_{mn}^{(1),(2)}(j,\ s;\ \beta) = \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\, A_p^{(m)}(q_j)\cos r\beta \times$$

$$\times \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\sin(r-p)\alpha_{js} + i^{r+p} H_{r+p}^{(1),(2)}(kl)\sin(r+p)\alpha_{js} \right\} \mp$$

$$\mp \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\, A_p^{(m)}(q_j)\sin r\beta \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\cos(r-p)\alpha_{js} + \right.$$

$$\left. + i^{r+p} H_{r+p}^{(1),(2)}(kl)\cos(r+p)\alpha_{js} \right\}, \qquad (4.5.14)$$

$$\tilde{Q}_{mn}^{(1),(2)}(j,\ s;\ \beta) = \pm \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\, B_p^{(m)}(q_j)\sin r\beta \times$$

$$\times \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\cos(r-p)\alpha_{js} - i^{r+p} H_{r+p}^{(1),(2)}(kl)\cos(r+p)\alpha_{js} \right\} -$$

$$- \frac{1}{2}\sum_{r,p}^{\infty} A_r^{(n)}(q_s)\, B_p^{(m)}(q_j)\cos r\beta \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\sin(r-p)\alpha_{js} - \right.$$

$$\left. - i^{r+p} H_{r+p}^{(1),(2)}(kl)\sin(r+p)\alpha_{js} \right\}, \qquad (4.5.15) \qquad /169$$

$$\tilde{R}_{mn}^{(1),(2)}(j,\ s;\ \beta) = \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\, B_p^{(m)}(q_j)\cos r\beta \times$$

$$\times \left\{ i^{r-p} H_{r-p}^{(1),(2)}(kl)\cos(r-p)\alpha_{js} - i^{r+p} H_{r+p}^{(1),(2)}(kl)\cos(r+p)\alpha_{js} \right\} \mp$$

$$\mp \frac{1}{2}\sum_{r,p}^{\infty} B_r^{(n)}(q_s)\, B_p^{(m)}(q_j)\sin r\beta \left\{ i^{p-r} H_{p-r}^{(1),(2)}(kl)\sin(p-r)\alpha_{js} + \right.$$

$$\left. + i^{p+r} H_{p+r}^{(1),(2)}(kl)\sin(p+r)\alpha_{js} \right\}. \qquad (4.5.16)$$

In these formulas $l = r_{js}$ and α_{js} are polar coordinates of the point O_s (origin of the s-th coordinate system) in the coordinates of the j-th system ($l = r_{js} = r_{sj}$ is the distance between the origins of the coordinate systems). Formulas (4.5.5), (4.5.6) may be used under the condition that $\xi_{js} \gtrless \xi_s$ (ξ_{js} is the elliptical radial coordinate of the point O_s in the j-th system of coordinates, determining the length $l = r_{js}$), whereas formulas (4.5.7), (4.5.8) are valid only under the condition that $l > f_s \mathrm{ch}\xi_s$ (proof of the substitution of the order of summation and integration in deriving

(4.5.7), (4.5.8) requires that when Re $k > 0$, $l \geq f_j[\operatorname{ch}\xi_{js}\cos\eta_{js}\cos\eta_j + \operatorname{sh}\xi_{js}\sin\eta_{js}\sin\eta_j] >> f_j + f_s$ [formulas (4.5.7), (4.5.8) are analogous to those given in [135], where their validity was established for the case when $\xi_s < A = \min\{\xi_{s,f_j}, \xi_{s,-f_j}\}$. Here $\xi_{s,\pm f_j}$ is the radial elliptical coordinate of the focal points of the coordinate ellipses of the j-th system figured in the coordinates of the s-th system].

Expressions (4.5.9) - (4.5.16) are simplified in the case if $\beta = 0$ (the coordinate axes Ox of the local systems are parallel and identically directed) because the factor $\sin r\beta$ vanishes. They are substantially simplified also in the case when the coordinate axes Ox of both systems are found to lie on a single straight line, since then $\alpha_{js} = 0$ or $\alpha_{js} = \pi$, by virtue of which $\sin(p \pm r)\alpha_{js} = 0$ for all indices p, r and in (4.5.9) - (4.5.16) the terms containing $\sin r\beta$ and $\sin(p \pm r)\alpha_{js}$ will disappear.

For q_j ($j = 1,2$) when the elliptical coordinates are degenerated into polar coordinates, from formulas (4.5.5) - (4.5.8) addition theorems are obtained for wave functions of a circular cylinder. /17
In particular, if only the j-th system of coordinates is degenerate ($j = 1$ or $j = 2$; $\xi_j \to 0$, $q_j \to \infty$ such that $f_j\frac{e^{\xi_j}}{2} = \rho_j = \text{const}$),

then from (4.5.5) - (4.5.8) we obtain the theorems for expansion of the wave functions of a circular cylinder according to the wave functions of the elliptical cylinder.

If $q_s \to 0$ ($s = 1$ or $s = 2$, $f_s\frac{e^{\xi_s}}{2} = \rho_s = \text{const}$) then, on the contrary, from (4.5.5) - (4.5.8) we obtain formulas for expansion of the wave functions of the elliptical cylinder according to the wave functions of the circular cylinder.

In formulas (4.5.9) - (4. 5.16) where the parity of the numbers $p \pm r$ always agrees with the parity of $m \pm n$, the superscript is taken if $\alpha_s = \alpha_j - \beta$, and the subscript if $\alpha_s = \alpha_j + \beta$.

The addition theorems which represent the case $l < f_s\operatorname{ch}\xi_s$, have been given in [40].

CHAPTER 5

SPHEROIDAL WAVE FUNCTIONS AND CERTAIN OF THEIR PROPERTIES

§1. Helmholtz Equation in Spheroidal Coordinates. Spheroidal Wave Functions.

The Helmholtz scalar wave equation in prolate spheroidal coordinates ξ, η, ϕ has the form, /171

$$\frac{\partial}{\partial \xi}(\xi^2-1)\frac{\partial u}{\partial \xi}+\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial u}{\partial \eta}+ \\ +\frac{\xi^2-\eta^2}{(\xi^2-1)(1-\eta^2)}\cdot\frac{\partial^2 u}{\partial \varphi^2}+c^2(\xi^2-\eta^2)u=0, \tag{5.1.1}$$

where $c = kf$. If we seek the particular solutions to equation (5.1.1) in the form of the product of three functions,

$$u(\xi, \eta, \varphi)=R(\xi)S(\eta)\Phi(\varphi)\neq 0, \tag{5.1.2}$$

each of which depends only on one coordinate, then after substituting (5.1.2) into (5.1.1) and subsequent separation of variables in the equation we come to the conclusion that for representation of the particular solution to equation (5.1.1) in the form of (5.1.2) as the functions R, S and Φ it follows to take any solutions of the respective equations,

$$\frac{d^2\Phi}{d\varphi^2}+\mu^2\Phi=0, \tag{5.1.3}$$

$$\frac{d}{d\eta}(1-\eta^2)\frac{dS}{d\eta}+\left(\lambda-c^2\eta^2-\frac{\mu^2}{1-\eta^2}\right)S=0, \tag{5.1.4}$$

$$\frac{d}{d\xi}(\xi^2-1)\frac{dR}{d\xi}-\left(\lambda-c^2\xi^2+\frac{\mu^2}{\xi^2-1}\right)R=0, \tag{5.1.5}$$

in which μ and λ are separation constants. In application the basic role is not played by any solutions to equation (5.1.1) having the form of (5.1.2), but only those which in the region under analysis satisfy the requirements of singularity and finiteness. Based on this, in equation (5.1.3) it follows to set $\mu = m$, where m is an integral number. Then the independent solutions to equation (5.1.3) will be the functions $\begin{matrix}\cos\\ \sin\end{matrix}(m\phi)$ or in complex form $\Phi = e^{im\phi}$. With the proper choice of values of the parameter λ (the eigenvalues of the parameter λ), equation (5.1.4) allows solutions /172

which are bound in the interval (-1, +1) and at its ends, that are called prolate spheroidal angular functions of the first sort of m-order of magnitude and n power (n is an integral and non-negative number) and are denoted through $S_{mn}^{(1)}(c, \eta)$. The solutions to equation (5.1.5) with those values of the parameter λ, for which equation (5.1.4) allows existence of the solutions bound in [-1, 1], are denoted by the symbol $R_{mn}(c, \xi)$ and are called prolate radial spheroidal functions. In physical problems, four types of radial functions are usually satisfied, that are denoted respectively through $R_{mn}^{(1)}$, $R_{mn}^{(2)}$, $R_{mn}^{(3)}$ and $R_{mn}^{(4)}$. Below they are defined such that when $f \to 0$ and the spheroidal coordinates are degenerated into spherical ones, they are all reduced to the Bessel spherical functions $j_n(kr)$, $n_n(kr)$, $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$, respectively. In analogy with the spherical Bessel functions, the functions $R_{mn}^{(3),(4)}(c, \xi)$ are defined through $R_{mn}^{(1)}(c, \xi)$ and $R_{mn}^{(2)}(c, \xi)$ in the form of the relationships,

$$R_{mn}^{(3),(4)}(c, \xi) = R_{mn}^{(1)}(c, \xi) \pm iR_{mn}^{(2)}(c, \xi). \qquad (5.1.6.)$$

Through $R_{mn}^{(1)}(c, \xi)$ we denote the radial functions that are regular inside the spheroid, and through $R_{mn}^{(2)}(c, \xi)$ those functions which have logarithmic singularity at the points $\xi = \pm 1$ inside the spheroid. The functions $R_{mn}^{(3),(4)}(c, \xi)$ satisfy the condition of radiation at infinity.

Thus, the unique and continuous solutions to (5.1.1) when $\eta = \pm 1$ and which are represented in the form of (5.1.2) are the functions,

$$R_{mn}^{(1)}(c, \xi) S_{mn}^{(1)}(c, \eta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) \qquad (5.1.7)$$

and

$$R_{mn}^{(3),(4)}(c, \xi) S_{mn}^{(1)}(c, \eta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi). \qquad (5.1.8)$$

The first of these are regular at each point inside the spheroid, including the lines of the focal points of the spheroid, and the second are regular at each point outside the spheroid and satisfy the condition of radiation at infinity (the superscript (3) represents the time dependence $\exp(-i\omega t)$). In determining the spheroidal functions when $f \to 0$, /17

$$R_{mn}(c, \xi) S_{mn}^{(1)}(c, \eta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) \to Z_n(kr) P_n^m(\cos\theta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi), \qquad (5.1.9)$$

where $Z_n(kr)$ is one of the spherical Bessel functions.

By virtue of the linearity and homogeneity of equation (5.1.1) its general solution in the respective region may be constructed in the form of a superposition of all particular solutions to (5.1.7) or (5.1.8) with expansion coefficients that are determined from the boundary conditions.

In oblate spheroidal coordinates the Helmholtz equation is written in the form,

$$\frac{\partial}{\partial \xi}(1+\xi^2)\frac{\partial u}{\partial \xi}+\frac{\partial}{\partial \eta}(1-\eta^2)\frac{\partial u}{\partial \eta}+ \\ +\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}\cdot\frac{\partial^2 u}{\partial \varphi^2}+c^2(\xi^2+\eta^2)u=0. \qquad (5.1.10)$$

Equation (5.1.10) may be obtained from equation (5.1.1) by substituting into it $-ic$ for c and $i\xi$ for ξ (with such a substitution, as noted earlier, the systems of prolate spheroidal coordinates convert into systems of oblate spheroidal coordinates). Consequently, the solutions to equation (5.1.10) may be obtained from the solutions to equation (5.1.1) if in this latter equation we make the indicated substitution. Here the partial solutions to equation (5.1.10), represented in the form of the product of three functions, each of which depends only on one coordinate, are the functions

$$R_{mn}^{(1)}(-ic, i\xi)\,S_{mn}^{(1)}(-ic, \eta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi), \qquad (5.1.11)$$

corresponding to the functions in (5.1.7) and

$$R_{mn}^{(3),\,(4)}(-ic, i\xi)\,S_{mn}^{(1)}(-ic, \eta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi), \qquad (5.1.12)$$

corresponding to the functions in (5.1.8). The general solution to equation (5.1.10) may also be represented in the form of a superposition of solutions to (5.1.11) or (5.1.12), depending on the region being studied.

The functions $S_{mn}^{(1)}(-ic, \eta)$ and $R_{mn}^{(j)}(-ic, i\xi)$ are called, respectively, oblate spheroidal angular functions of the first sort and oblate spheroidal radial functions of the first, second, third and fourth sort, $j = 1, 2, 3, 4$.

The functions (5.1.7), (5.1.8) and (5.1.11), (5.1.12) are called prolate spheroidal wave functions and oblate spheroidal wave functions, respectively. /174

The properties of the angular and radial spheroidal functions have been studied in [37, 39-43].

§2. Angular Spheroidal Functions and Certain of Their Properties.

Equation (5.1.4), written in the form,

$$\frac{d}{dz^2}(1-z^2)\frac{dy}{dz}+\left(\lambda-c^2z^2-\frac{\mu^2}{1-z^2}\right)y=0, \qquad (5.2.1)$$

is characterized by three singular points: regular singular points $z = \pm 1$ and substantially by the singular point $z = \infty$. When $c = 0$, equation (5.2.1) converts to the equation of adjoint Legendre functions (3.3.1), if in (5.2.1) instead of λ we take $\lambda(\lambda + 1)$. Consequently, the solutions to equation (5.2.1) may be studied as generalizations of the adjoint Legendre functions $P_\lambda^\mu(z)$ and $Q_\lambda^\mu(z)$. Equation (5.2.1) allows solution for any values of the parameters λ, μ as well as values of the argument z that are arbitrary, including complex values. Each solution may be called a spheroidal function. However solutions to the equation that are restricted to the interval [-1, +1] do not exist for all values of the parameters λ and μ, but only for certain of them: when $\mu = m$, where m is an integral number, and when $\lambda = \lambda_{mn}(c)$ where $\lambda_{mn}(c)$ from a denumerable sequence of values of the parameter λ (eigenvalues of the parameter λ), n is whole and $n \geq 0$ (it is obvious that if $c = 0$, then we must have $\lambda_{mn} = n(n + 1)$ and the restricted solution to equation (5.2.1) will be the adjoint Legendre function $P_n^m(z)$). The solutions to equation (5.2.1) which are restricted to [-1, +1] are denoted through $S_{mn}^{(1)}(c, z)$ and are called prolate spheroidal angular functions of the first sort. They are determined by a series of adjoint Legendre functions,

$$S_{mn}^{(1)}(c, z)=\sum_{r=0}^{\infty} d_r^{mn}(c)\,P_{m+r}^m(z), \quad -1 \leqslant z \leqslant 1, \qquad (5.2.2)$$

for the coefficients of which, if we substitute (5.2.2) into (5.2.1) when $\mu = m$ and $\lambda = \lambda_{mn}(c)$, we obtain the recurrent relationships,

$$\left[-\lambda_{mn}(c)+\frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}\,c^2\right]d_r^{mn}(c)+$$
$$+\frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}\,d_{r-2}^{mn}(c)=0,\ r\geqslant 0, \qquad (5.2.3)$$

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from which they are also computed. Relationships (5.2.3) connect the coefficents d_r with the indices r of one and the same parity. Therefore there exist two sequences of solutions to equation (5.2.1)

which are restricted when $z = \pm 1$: one sequence is represented by the even numbers of r and the other by the odd. In this respect expansion (5.2.2) is usually written in the form,

$$S_{mn}^{(1)}(c, z) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) P_{m+r}^m(z), \quad -1 \leqslant z \leqslant 1, \tag{5.2.4}$$

where the prime over the sign of the sum indicates that summation is carried out over even numbers of r, if $n - m$ is even, and over odd numbers of r only if $n - m$ is odd.

Relationship (5.2.3) may be studied as a homogeneous difference equation of second order of magnitude. As its solution we take one for which

$$|v_r| \underset{r \to \infty}{\to} 0, \text{ as } \left|\frac{c^2}{4r^2}\right|, \tag{5.2.5}$$

where $v_r = d_{r+2}^{mn}/d_r^{mn}$. Such a choice of expansion coefficients (5.2.4) guarantees absolute and uniform convergence of the series for all finite values of the argument z.

The second (which is linearly independent of $S_{mn}^{(1)}(cz)$ in [-1, +1]) solution to equation (5.2.1) when $\mu = m$ and $\lambda = \lambda_{mn}(c)$ is constructed analogously through the adjoint functions of second sort $Q_n^m(z)$. This solution, having logarithmic singularity at the points $z = \pm 1$, in application is encountered extremely rarely, and therefore its properties are not studied here. Much more often encountered are the spheroidal angular functions of the second sort, which are denoted through $S_{mn}^{(2)}(c, z)$ determined for real or complex values of the argument z ($|z| > 1$) by the series

$$S_{mn}^{(2)}(c, z) = \sum_{r=-\infty}^{\infty}{}' d_r^{mn}(c) Q_{m+r}^m(z), \tag{5.2.6}$$

where $Q_n^m(z)$ are adjoint Legendre functions of the second sort, $|z| > 1$. The coefficients $d_r^{mn}(c)$ of series (5.2.6) satisfy the recurrent relationship (5.2.3), expanded also to the case of negative values of r. When $r \geq 0$ the coefficients of series (5.2.6) agree with the coefficients of series (5.2.4). When $r < 0$, the coefficients $d_r^{mn}(c)$ may be computed from the continued fraction, /176

$$\frac{d_{-r}^{mn}}{d_{-r+2}^{mn}} = \frac{-A_{-r+2,\,m}}{B_{-r,\,m}^n - \dfrac{C_{-r-2,\,m} A_{-r,\,m}}{B_{-r-2,\,m}^n - \ddots}}. \tag{5.2.7}$$

obtained from (5.2.3). Here

$$\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)}\,d_{r+2}^{mn}(c)+\Big[(m+r)(m+r+1)-$$

$$A_{r+2,\,m}=\frac{(2m+r+2)(2m+r+1)}{(2m+2r+3)(2m+2r+5)}\,c^2,$$

$$B_{rm}^{n}=(m+r)(m+r+1)-\lambda_{mn}(c)+ \qquad (5.2.8)$$

$$+\frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}\,c^2,$$

$$C_{r-2,\,m}=\frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}.$$

It is obvious that $A_{rm} = 0$ for $r = -2m$ or for $r = 1 - 2m$. Therefore the uninterrupted continued fraction (5.2.7) for d_r^{mn}/d_{r+2}^{mn} contains only a finite number of terms for which $-2m \leq r < 0$. Consequently, all coefficients d_r^{mn} are equal to zero when $r \leq -2m$. However, we can show that when $r < -2m$ the functions $Q_{m+r}^{m}(z)$ from (5.2.6) become infinitely large. In fact, from the functional relationship (3.3.12)

$$Q_{-\lambda-1}^{m}(z)=Q_{\lambda}^{m}(z)-\pi\cot\lambda\pi\,P_{\lambda}^{m}(z)$$

it follows that

$$Q_{m-r+\varepsilon}^{m}(z)=Q_{r-m-1-\varepsilon}^{m}(z)+\frac{\cos\varepsilon\pi}{\varepsilon}P_{r-m-1-\varepsilon}^{m}(z), \qquad (5.2.9)$$

if we set $\lambda = r - m - 1 - \varepsilon$, where $r - m - 1$ is a whole number, and $\varepsilon \to 0$. Since $Q_{m-r}^{m}(z) = \lim Q_{m-r+\varepsilon}^{m}(z)$, then there follows hence the unboundedness of $Q_{m+r}^{m}(z)$ when $r \leq -2m$. Along with this, the limit of the product of the value $d_{-r+\varepsilon}^{mn}$, tending to zero when $\varepsilon \to 0$, $-r < -2m$, and the functions $Q_{m-r+\varepsilon}^{m}(z)$, unboundedly increasing when $\varepsilon \to 0$, remains finite when $r > 2m$ and $\varepsilon \to 0$, and

$$d_{-r}^{mn}Q_{m-r}^{m}(z)=\lim_{\varepsilon\to 0}d_{-r+\varepsilon}^{mn}Q_{m-r+\varepsilon}^{m}(z)=d_{\varepsilon|r}^{mn}P_{r-m-1}^{m}(z), \qquad (5.2.10)$$

where $d_{\varepsilon|r}^{mn}=\lim_{\varepsilon\to 0}d_{-r+\varepsilon/\varepsilon}^{mn}$.

The angular function of the second sort $S_{mn}^{(2)}(c, z)$, based on the above instead of (5.2.6), may be determined by the expansion /177

$$S_{mn}^{(2)}(c,z)=\sum_{r=-2m,-2m-1}^{\infty}{}'\,d_r^{mn}(c)\,Q_{m+r}^{m}(z)+$$

$$+\sum_{r=2m+2,\,2m+1}^{\infty}{}'\,d_{\varepsilon|r}^{mn}(c)\,P_{r-m-1}^{m}(z), \qquad (5.2.11)$$

where $P_n^m(z)$ is the first solution to the equation of the adjoint Legendre functions in the region $|z| > 1$. The coefficients $d_{\varepsilon l r}^{mn}$ may be, with any degree of accuracy, computed from (5.2.7) (see, for example, [39], Ch.3). For a unique determination of the function $S_{mn}^{(2)}(c, z)$ in the case of complex values of z, it is necessary to make a branch cut along the real axis in the plane z from $-\infty$ to +1.

The eigenvalues $\lambda_{mn}(c)$ of the parameter λ are the solutions to the transcendental equation relative to $\lambda_{mn}(c)$ having the form of a continued fraction

$$\lambda_{mn} = \gamma_{n-m}^m - \cfrac{\beta_{n-m}^m}{\gamma_{n-m-2}^m - \lambda_{mn} - \cfrac{\beta_{n-m-2}^m}{\gamma_{n-m-4}^m - \lambda_{mn} - \cfrac{-\beta_{n-m+2}^m}{\gamma_{n-m+2} - \lambda_{mn} - \cfrac{\beta_{n-m+4}^m}{\gamma_{n-m+4} - \lambda_{mn} - \cdots}}}}. \qquad (5.2.12)$$

where

$$\gamma_r^m = (m+r)(m+r+1) + \frac{c^2}{2}\left[1 - \frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right], \quad r \geqslant 0,$$

$$\beta_r^m = \frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r-3)(2m+2r+1)}, \quad r \geqslant 2.$$

For a unique determination of the coefficients d_r^{mn}, as the normalizing condition we take the relationships,

$$S_{mn}^{(1)}(c, 0) = P_n^m(0) = \frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^n\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}, \qquad (5.2.13)$$

$(n-m)$ is even

$$S_{mn}^{(1)'}(c, 0) = P_n^{m'}(0) = \frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^n\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!},$$

$(n-m)$ is odd

on the basis of which in the first case we have

$$\sum_{r=0}^{\infty}{}' \frac{(-1)^{r/2}(r+2m)!}{2^r\left(\frac{r}{2}\right)!\left(\frac{r+2m}{2}\right)!} d_r^{mn}(c) = \frac{(-1)^{\frac{n-m}{2}}(n+m)!}{2^{n-m}\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}, \tag{5.2.14}$$

and in the second, when $n - m$ is odd,

$$\sum_{r=1}^{\infty}{}' \frac{(-1)^{\frac{r-1}{2}}(r+2m+1)!}{2^r\left(\frac{r-1}{2}\right)!\left(\frac{r+2m+1}{2}\right)!} d_r^{mn}(c) = \frac{(-1)^{\frac{n-m-1}{2}}(n+m+1)!}{2^{n-m}\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}. \tag{5.2.15}$$

Conditions (5.2.14), (5.2.15), along with the recurrent relationships (5.2.3), fully determine the coefficients of expansion $d_r^{mn}(c)$.

Since the form of equation (5.2.1) does not vary by substituting $-\mu$ for μ in it, it is then obvious that along with the angular functions $S_{mn}(c, z)$ its solutions will also be the functions $S_{-m,n}(c, z)$ ($\mu = m$, $\lambda = \lambda_{-m,n}(c)$). If we determine, for example, the functions $S^{(1)}_{m,n}(c, z)$ by the series,

$$S^{(1)}_{-m,n}(c, z) = \sum_{r=0,1}^{\infty}{}' d_r^{-m,n}(c) P_{m+r}^{-m}(z), \quad -1 \leqslant z \leqslant 1, \tag{5.2.16}$$

then on the basis of (3.3.22) we will have the expansion,

$$S^{(1)}_{-m,n}(c, z) = \sum_{r=0,1}^{\infty} (-1)^m \frac{r!}{(2m+r)!} d_r^{-m,n}(c) P_{m+r}^{m}(z), \tag{5.2.17}$$

whose coefficients $(-1)^m r! d_r^{-m,n}(c)/(2m + r)!$ satisfy the recurrent relationships in (5.2.3). Consequently, /17

$$\lambda_{-m,n}(c) = \lambda_{mn}(c). \tag{5.2.18}$$

If we set

$$d_r^{-m,\,n}(c) = \frac{(n-m)!}{(n+m)!} \cdot \frac{(2m+r)!}{r!} d_r^{mn}(c), \tag{5.2.19}$$

then

$$S^{(1)}_{-m,\,n}(c,\ z) = (-1)^m \frac{(n-m)!}{(n+m)!} S^{(1)}_{mn}(c,\ z) \tag{5.2.20}$$

and

$$S^{(1)}_{-m,\,n}(0,\ z) = P_n^{-m}(z). \tag{5.2.21}$$

Let us mention several of the properties of the angular functions.

1. The angular spheroidal functions $S^{(1)}_{mn}(c,\ z)$ form in [-1, +1] a complete and orthogonal system of functions, and

$$\int_{-1}^{1} S^{(1)}_{mn}(c,\ z)\, S^{(1)}_{mn'}(c,\ z)\, dz = \delta_{nn'} N_{mn}(c), \tag{5.2.22}$$

where $\delta_{nn'}$ is the Kronecker symbol, and

$$N_{mn}(c) = 2 \sum_{r=0,\,1}^{\infty}{}' \frac{(r+2m)!\,[d_r^{mn}(c)]^2}{(2r+2m+1)\,r!}. \tag{5.2.23}$$

2. When $c \to 0$, the eigenvalues $\lambda_{mn}(c) \to \lambda_{mn}(0) = n(n+1)$, and $S^{(1)}_{mn}(c,\ z) \to P_n^m(\cos\theta)$; all coefficients $d_r^{mn}(c)$, with the exception of $d_{n-m}^{mn}(c)$, tend to zero, but $d_{n-m}^{mn}(c) \to 1$. More complete investigations of the behavior of the coefficients $d_r^{mn}(c)$ for small values of c will lead to the approximate formulas:

$$d^{mn}_{n-m-2k}(c) \approx \frac{(n+m)!\,(2n-4k+1)!!\ (2n-2k-1)!!}{(n+m-2k)!\,(2n+1)!!\,2k!!\,(2n-1)!!} c^{2k}, \quad 0 \leqslant k \leqslant \frac{n-m}{2}, \tag{5.2.24}$$

$$d^{mn}_{n-m+2k}(c) \approx \frac{(-1)^k (n-m+2k-1)!\,(2n-1)!!\,(2n+1)!!}{(n-m)!\,(2n+4k-1)!!\,2k!!\,(2n+2k+1)!!} c^{2k}, \quad k \geqslant 1. \tag{5.2.25}$$

Hence it is clear that all coefficients $d_r^{mn}(c)$ for $r < n-m$ remain positive, and for $r > n - m$ their signs alternate, and all of them, /180 with the exception of the coefficient $d_{n-m}^{mn}(c)$ drop toward zero when

$c \to 0$, as $c^{|n-m-r|}$. When $c \to 0$,

$$N_{mn}(c) \to N_{mn} = \frac{2}{2n+1} \cdot \frac{(n+m)!}{(n-m)!}. \tag{5.2.26}$$

3. When $n \to \infty$ we have the asymptotic approximations,

$$\begin{gathered} \lambda_{mn}(c) \to n(n+1), \\ S_{mn}(c, \eta) = P_n^m(\eta) + O(n^{-1/2}), \ \eta \in [-1, 1]. \end{gathered} \tag{5.2.27}$$

Here all coefficients $d_r^{mn}(c)$, with the exception of $d_{n-m}^{mn}(c)$, tend to zero, but $d_{n-m}^{mn}(c) \to 1$. From the recurrent relationships in (5.2.3) we can establish that asymptotically,

$$d_{n-m-2k}^{mn}(c) \sim \frac{(n+m)!\,(2n-4k+1)!!\,(2n-2k-1)!!\,c^{2k}}{(n+m-2k)!\,(2n+1)!!\,(2k)!!\,(2n-1)!!},$$

$$0 \leqslant k \leqslant \frac{n-m}{2}, \tag{5.2.28}$$

$$d_{n-m+2k}^{mn}(c) \sim \frac{(-1^k(n-m+2k-1)!\,(2n-1)!!\,(2n+1)!!\,c^{2k}}{(n-m)!\,(2n+4k-1)!!\,(2k)!!\,(2n+2k+1)!!},$$

$$k \geqslant 1. \tag{5.2.29}$$

4. Different methods exist for representing the angular spheroidal functions which differ from (5.2.4). In particular, they may be determined by expansions in powers of $1 - z^2$:

$$S_{mn}^{(1)}(c, z) = (1-z^2)^{\frac{m}{2}} \sum_{k=0}^{\infty} c_{2k}^{mn} (1-z^2)^k,$$

$$(n-m) \text{ is even} \tag{5.2.30}$$

$$S_{mn}^{(1)}(c, z) = z(1-z^2)^{\frac{m}{2}} \sum_{k=0}^{\infty} c_{2k}^{mn} (1-z^2)^k,$$

$$(n-m) \text{ is odd} \tag{5.2.31}$$

where the coefficients of the series c_{2k}^{mn} are connected with the coefficients d_r^{mn} by the relationships,

$$c_{2k} = \frac{1}{2^m k!\,(m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r)!}{(2r)!} (-r)_k \left(m+r+\frac{1}{2}\right)_k d_{2r}^{mn},$$

$$(n-m) \text{ is even} \tag{5.2.32}$$

$$c_{2k} = \frac{1}{2^m k!\,(m+k)!} \sum_{r=k}^{\infty} \frac{(2m+2r+1)!}{(2r+1)!} \times$$

$$\times (-r)_k \left(m+r+\frac{3}{2}\right)_k d_{2r+1}^{mn}, \ (n-m) \text{ is odd} \qquad (5.2.32)$$ /181

From (5.2.30, (5.2.31) the character of the parity of the angular functions $S_{mn}^{(1)}(c, z)$ is quite clear: when $(n - m)$ is even the functions $S_{mn}^{(1)}(c, z)$ are even relative to the argument z, and with an odd $(n - m)$ the functions $S_{mn}^{(1)}(c,z)$ are odd relative to z.

5. The partial solutions to the prolate angular functions are

$$S_{0n}^{(1)}(c,1) = \sum_{r=0,1}^{\infty}{}' d_r^{0n}(c),$$

$$S_{0n}^{(1)\prime}(c,\ 1) = \frac{1}{2} \sum_{r=0,1}^{\infty}{}' r(r+1)\, d_r^{0n}(c), \qquad (5.2.33)$$

$$\lim_{z\to 1} (1-z^2)^{1/2} \frac{d}{dz} S_{1n}^{(1)}(c,\ z) =$$

$$= -\frac{1}{2} \sum_{r=0,\ 1}^{\infty}{}' (r+1)(r+2)\, d_r^{1n}(c), \qquad (5.2.34)$$

$$\lim_{z\to 1} (1-z^2)^{-\frac{1}{2}m} S_{mn}^{(1)}(c,\ z) = \sum_{r=0,\ 1}^{\infty}{}' d_r^{mn}(c)\ \frac{(2m+r)!}{2^m m!\, r!}. \qquad (5.2.35)$$

In the past we have been concerned with solutions to equation (5.2.1) written in prolate spheroidal coordinates. It is easy to see that by substituting $-ic$ for c in it we obtain an equation for the oblate angular functions which, unlike the previous ones, are denoted through $S_{mn}(-ic, z)$. They represent the oblate spheroidal eigenvalues $\lambda_{mn}(-ic)$, and in the case of real values of the argument z the functions $S_{mn}^{(1)}(-ic, z)$ are determined by the expansion,

$$S_{mn}^{(1)}(-ic,\ z) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(-ic)\, P_{m+r}^m(z), \ -1 \leqslant z \leqslant 1, \qquad (5.2.36)$$

and are restricted to [-1, +1] by the solutions to the equation /182

$$\frac{d}{dz}(1-z^2)\frac{dy}{dz} +$$

$$+\left(\lambda_{mn}(-ic) + c^2z^2 - \frac{m^2}{1-z^2}\right) y = 0. \qquad (5.2.37)$$

The coefficients $d_r^{mn}(-ic)$ satisfy the recurrent relationships in (5.2.3) where c is substituted by $-ic$.

All the relationships and formulas, obtained above for the function $S_{mn}^{(1)}(c, z)$, are valid also for the oblate spheroidal angular functions if in them we substitute $-ic$ for c. Here their properties are also retained and there is no necessity for a separate investigation of the oblate angular functions.

§3. Radial Spheroidal Functions and Certain of Their Properties.

The radial spheroidal functions in prolate or oblate spheroidal coordinates are solutions to equation (5.1.5) when $\mu = m$ and $\lambda = \lambda_{mn}(c)$ or solutions to the equation

$$\frac{d}{d\xi}(\xi^2+1)\frac{dR}{d\xi} - \left(\lambda_{mn}(-ic) - c^2\xi^2 - \frac{m^2}{\xi^2+1}\right)R = 0, \tag{5.3.1}$$

obtained from (5.1.5) when $\mu = m$, $\lambda = \lambda_{mn}(-ic)$ by substituting $i\xi$ for ξ and $-ic$ for c in it (here it is assumed that $\lambda_{mn}(c)$ and $\lambda_{mn}(-ic)$ are eigenvalues of the parameter λ, for which there exist the angular functions $S_{mn}(c, \eta)$ and $S_{mn}(-ic, \eta)$, respectively that are restricted to $[-1, +1]$. As already noted, the prolate radial functions are denoted through $R_{mn}^{(j)}$, $j = 1, 2, 3, 4$, where $R_{mn}^{(1)}$, $R_{mn}^{(2)}$ are radial functions, respectively, of the first and second sort, and $R_{mn}^{(3),(4)}$ are their linear combinations (5.1.6). They are so defined that with degeneration of the spheroidal coordinates into spherical ones, the functions $R_{mn}^{(1)}(c, \xi)$, $R_{mn}^{(2)}(c, \xi)$, $R_{mn}^{(3)}(c, \xi)$, and $R_{mn}^{(4)}(c, \xi)$ are reduced to the spherical Bessel functions $j_n(kr)$, $n_n(kr)$, $h_n^{(1)}(kr)$ and $h_n^{(2)}(kr)$, respectively. The prolate spheroidal radial functions are given by series such as: /183

$$R_{mn}^{(1)}(c, \xi) = \frac{1}{\sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\times$$
$$\times \sum_{r=0,1}^{\infty}{}' i^{r+m-n} d_r^{mn}(c)\frac{(2m+r)!}{r!} j_{m+r}(c\xi); \tag{5.3.2}$$

$$R_{mn}^{(2)}(c,\ \xi) = \frac{1}{\sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)\,\frac{(2m+r)!}{r!}}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\times$$

$$\times\sum_{r=0,1}^{\infty}{}' i^{r+m-n}\,d_r^{mn}(c)\,\frac{(2m+r)!}{r!}\,n_{m+r}(c\,\xi), \qquad (5.3.3)$$

$$R_{mn}^{(3),(4)}(c,\ \xi) = \frac{1}{\sum_{r=0,1}^{\infty}{}' d_r^{mn}(c)\,\frac{(2m+r)!}{r!}}\left(\frac{\xi^2-1}{\xi^2}\right)^{m/2}\times$$

$$\times\sum_{r=0,1}^{\infty}{}' i^{r+m-n}\,d_r^{mn}(c)\,\frac{(2m+r)!}{r!}\,h_{m+r}^{(1),(2)}(c\,\xi), \qquad (5.3.4)$$

constructed in such a manner that when $c\xi \to \infty$, we obtain from them the asymptotic representations,

$$R_{mn}^{(1)}(c,\ \xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi}\cos\left(c\xi - \frac{1}{2}(n+1)\pi\right),$$

$$R_{mn}^{(2)}(c,\ \xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi}\sin\left(c\xi - \frac{1}{2}(n+1)\pi\right), \qquad (5.3.5)$$

$$R_{mn}^{(3),\ (4)}(c,\ \xi) \xrightarrow[c\xi\to\infty]{} \frac{1}{c\xi}\,e^{\pm i\left[c\xi - \frac{1}{2}(n+1)\pi\right]}.$$

If we substitute $i\xi$ for ξ and $-ic$ for c in (5.3.2) - (5.3.4) we obtain series which determine the oblate spheroidal radial functions $R_{mn}^{(j)}(-ic,\ i\xi)$, $j = 1, 2, 3, 4$, respectively.

Series (5.3.2) converges absolutely and uniformly in any finite interval of variation in $c\xi$, and its convergence is improved with decrease in $c\xi$. Series (5.3.3) converges very slowly or even does not converge with small values of $c\xi$. Series (5.3.4) for radial functions of the third and fourth sort are also only slightly feasible for small values of $c\xi$. /184

From comparison of equations (5.1.4) and (5.1.5) it is clear that they are equations of one and the same type, by virtue of which between the angular and radial spheroidal functions there exists a relationship that is expressed in their proportionality to one another ($\mu = m$, $\lambda = \lambda_{mn}$). Thus, the angular and radial functions of the first sort in prolate spheroidal coordinates are connected among themselves by the relationship of proportionality,

$$S_{mn}(c,\ z) = k_{mn}^{(1)}(c)\,R_{mn}^{(1)}(c,\ z), \qquad (5.3.6)$$

where the coefficient of proportionality is determined by the expressions,

$$k_{mn}^{(1)}(c) = \frac{(2m+1)(n+m)!\sum_{r=0}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}}{2^{n+m}d_0^{mn}(c)c^m m!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}, \quad (n-m) \text{ is even} \qquad (5.3.7)$$

$$k_{mn}^{(1)}(c) = \frac{(2m+3)(n+m+1)!\sum_{r=1}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}}{2^{n+m}d_1^{mn}(c)c^{m+1}m!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}, \quad (n-m) \text{ is odd} \qquad (5.3.8)$$

(it is assumed that with transition from the region $|z| > 1$ to the region $|z| < 1$ in the expression of the adjoint functions and in formula (5.3.2) the factor $(z^2 - 1)^{m/2}$ is substituted by the factor $(1 - z^2)^{m/2}$ and vice versa). For the prolate functions,

$$S_{mn}(-ic, iz) = k_{mn}^{(1)}(-ic)R_{mn}^{(1)}(-ic, iz), \qquad (5.3.9)$$

where

$$k_{mn}^{(1)}(-ic) = \frac{i^m(2m+1)(n+m)!\sum_{r=0}^{\infty}{}' d_r^{mn}(-ic)\frac{(2m+r)!}{r!}}{2^{n+m}d_0^{mn}(-ic)c^m m!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}, \quad (n-m) \text{ is even} \qquad (5.3.10)$$

$$k_{mn}^{(1)}(-ic) = \frac{i^{m+1}(2m+3)(n+m+1)!\sum_{r=1}^{\infty}{}' d_r^{mn}(-ic)\frac{(2m+r)!}{r!}}{2^{n+m}d_1^{mn}(-ic)c^{m+1}m!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)!}, \quad (n-m) \text{ is odd} \qquad (5.3.11)$$

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The angular and radial functions of the second sort are connected among themselves by the relationship of proportionality,

$$S_{mn}^{(2)}(c, z) = k_{mn}^{(2)}(c)R_{mn}^{(2)}(c, z), \qquad (5.3.12)$$

in which

$$k_{mn}^{(2)}(c) = \frac{2^{n-m}(2m)!\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)! d_{-2m}^{mn}(c)}{(2m-1)m!\,(n+m)!\,c^{m-1}} \times \sum_{r=0}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}, \quad (n-m) \text{ is even} \tag{5.3.13}$$

$$k_{mn}^{(2)}(c) = -\frac{2^{n-m}(2m)!\left(\frac{n-m-1}{2}\right)!\left(\frac{n+m+1}{2}\right)! d_{-2m+1}^{mn}(c)}{(2m-3)(2m-1)m!\,(n+m+1)!\,c^{m-2}} \times \sum_{r=1}^{\infty}{}' d_r^{mn}(c)\frac{(2m+r)!}{r!}, \quad (n-m) \text{ is odd} \tag{5.3.14}$$

In oblate spheroidal coordinates,

$$S_{mn}^{(2)}(-ic, iz) = k_{mn}^{(2)}(-ic)\,R_{mn}^{(2)}(-ic, iz), \tag{5.3.15}$$

where the coefficients are determined, as in formulas (5.3.13), (5.3.14), by substituting $-ic$ for c.

On the basis of (5.2.11) from (5.3.12), we find that

$$R_{mn}^{(2)}(c, \xi) = \frac{1}{k_{mn}^{(2)}(c)}\left\{\sum_{r=-2m,\ -2m+1}^{\infty}{}' d_r^{mn}(c)\,Q_{m+r}^{m}(\xi) + \sum_{r=2m+2,\ 2m+1}^{\infty}{}' d_{\varepsilon|r}^{mn}(c)\,P_{r-m-1}^{m}(\xi)\right\}, \tag{5.3.16}$$

and analogously in the case of oblate coordinates, /186

$$R_{mn}^{(2)}(-ic, i\xi) = \frac{1}{k_{mn}^{(2)}(-ic)} \times \left\{\sum_{r=-2m,\ -2m+1}^{\infty}{}' d_r^{mn}(-ic)\,Q_{m+r}^{m}(i\xi) + \sum_{r=2m+2,\ 2m+1}^{\infty}{}' d_{\varepsilon|r}^{mn}(-ic)\,P_{r-m-1}^{m}(i\xi)\right\}. \tag{5.3.17}$$

Series (5.3.16) converges well for not too large values of c and for values of ξ near unity ($\xi \neq 1$). Series (5.3.17) converges well for small values of c and for values of ξ near zero ($\xi \neq 0$). These properties of series (5.3.16), (5.3.17) make them suitable for computing the radial functions of the second sort in the case of small values of $c\xi$, when series (5.3.3), (5.3.4) and the series obtained from them for $R_{mn}^{(3),(4)}(-ic, i\xi)$ become practically unsuitable for such purposes.

From (5.3.6) and (5.3.16) it follows that the prolate radial spheroidal functions of the first sort have no singularity at the points $\xi = \pm 1$, whereas the radial functions of the second sort at these points have logarithmic singularity.

The Wronskian for the prolate radial functions is equal to

$$W\left[R_{mn}^{(1)}, R_{mn}^{(2)}\right] = \frac{1}{c(\xi^2 - 1)}, \tag{5.3.18}$$

$$W\left[R_{mn}^{(1)}, R_{mn}^{(3),(4)}\right] = \frac{\pm i}{c(\xi^2 - 1)}, \tag{5.3.19}$$

and for the oblate radial functions,

$$W\left[R_{mn}^{(1)}, R_{mn}^{(2)}\right] = \frac{1}{c(\xi^2 + 1)}, \tag{5.3.20}$$

$$W\left[R_{mn}^{(1)}, R_{mn}^{(3),(4)}\right] = \frac{\pm i}{c(\xi^2 + 1)}. \tag{5.3.21}$$

The particular solutions to the radial spheroidal functions are:

$$R_{0n}^{(1)}(c, 1) = \begin{cases} \dfrac{2^n \left(\dfrac{n}{2}!\right)^2 d_0^{0n}(c)}{n!}, & n \text{ is even} \\ \dfrac{2^n c \left(\dfrac{n-1}{2}\right)! \left(\dfrac{n+1}{2}\right)! d_1^{0n}(c)}{3(n+1)!}, & n \text{ is odd} \end{cases} \tag{5.3.22}$$

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$$R_{0n}^{(1)'}(c,\ 1) =$$

$$= \begin{cases} \dfrac{2^{n-1}\left(\dfrac{n}{2}!\right)^2 d_0^{0n}(c) \sum\limits_{r=0}^{\infty}{}' r(r+1) d_r^{0n}(c)}{n! \sum\limits_{r=0}^{\infty}{}' d_r^{0n}(c)}, \\ n \text{ is even} \\ \dfrac{2^{n-1} c \left(\dfrac{n-1}{2}\right)! \left(\dfrac{n+1}{2}\right)! d_1^{0n}(c) \sum\limits_{r=0}^{\infty}{}' r(r+1) d_r^{0n}(c)}{3(n+1)! \sum\limits_{r=0}^{\infty}{}' d_r^{0n}(c)}, \\ n \text{ is odd} \end{cases} \qquad (5.3.23)$$

$$\lim_{\xi \to 1} (\xi^2 - 1)^{1/2} \frac{d}{d\xi} R_{1n}^{(1)}(c, \xi) =$$

$$= \begin{cases} \dfrac{2^n c \left(\dfrac{n-1}{2}\right)! \left(\dfrac{n+1}{2}\right)! d_0^{1n}(c)}{3(n+1)!}, & (n-1) \text{ is even} \\ \dfrac{2^n c^2 \left(\dfrac{n-2}{2}\right)! \left(\dfrac{n+2}{2}\right)! d_1^{1n}(c)}{5(n+2)!}, & (n-1) \text{ is odd} \end{cases} \qquad (5.3.24)$$

$$\lim_{\xi \to 1} (\xi^2 - 1)^{-m/2} R_{mn}^{(1)}(c, \xi) = \qquad (5.3.25) \quad /188$$

$$= \begin{cases} \dfrac{2^n c^m \left(\dfrac{n-m}{2}\right)! \left(\dfrac{n+m}{2}\right)! d_0^{mn}(c)}{(2m+1)(n+m)!}, & (n-m) \text{ is even} \\ \dfrac{2^n c^{m+1} \left(\dfrac{n-m-1}{2}\right)! \left(\dfrac{n+m+1}{2}\right)!}{(2m+3)(n+m+1)!}, & (n-m) \text{ is odd} \end{cases}$$

$$\left.\begin{aligned} R_{mn}^{(2)}(c,\ \xi) &\xrightarrow[\xi \to 1]{} \infty, \\ (\xi^2 - 1)^{1/2} \frac{d}{d\xi} R_{mn}^{(2)}(c,\ \xi) &\xrightarrow[\xi \to 1]{} \infty, \end{aligned}\right\} \qquad (5.3.26)$$

$$R_{mn}^{(1)}(-ic,\ i0) =$$

$$= \begin{cases} \dfrac{i^{n-m}\,2^m\,m!\,c^m\,d_0^{mn}(-ic)}{(2m+1)\sum\limits_{r=0,1}^{\infty'} d_r^{mn}(-ic)\,\dfrac{(2m+r)!}{r!}}, & (n-m) \text{ is even} \\ 0, & (n-m) \text{ is odd} \end{cases} \qquad (5.3.27)$$

$$R_{mn}^{(1)'}(-ic,\ i0) =$$

$$= \begin{cases} 0, & (n-m) \text{ is even} \\ \dfrac{i^{n-m-1}\,2^m\,m!\,c^{m+1}\,d_1^{mn}(-ic)}{(2m+3)\sum\limits_{r=0,1}^{\infty'} d_r^{mn}(-ic)\,\dfrac{(2m+r)!}{r!}}, & (n-m) \text{ is odd} \end{cases} \qquad (5.3.28)$$

$$R_{mn}^{(2)}(-ic,\ i0) =$$

$$= \frac{i^{n-m}\,(2m-1)!!\,m!\,c^{m-1}\,\pi}{2^{2n-m+1}\,(2m)!\,d_{-2m}^{mn}(-ic)\sum\limits_{r=0}^{\infty'} d_r^{mn}(-ic)\,\dfrac{(2m+r)!}{r!}} \times$$

$$\times \left[\frac{(n+m)!}{\left(\frac{n-m}{2}\right)!\left(\frac{n+m}{2}\right)!}\right]^2, \quad (n-m) \text{ is even} \qquad (5.3.29)$$

$$R_{mn}^{(2)}(-ic,\ i0) = -\frac{1}{cR_{mn}^{(1)'}(-ic,\ i0)} =$$

$$= \frac{i^{m-n-1}(2m+3)\sum\limits_{r=1}^{\infty'} d_r^{mn}(-ic)\,\dfrac{(2m+r)!}{r!}}{2^m\,m!\,c^{m+2}\,d_1^{mn}(-ic)} = \qquad (5.3.30)$$

$$= \left[i^{-m}\,k_{mn}^{(2)}(-ic)\right]^{-1}\left[\sum_{r=-2m+1}^{\infty'} d_r^{mn}(-ic)\,Q_{m+r}^m(0) + \right.$$

$$\left. + \sum_{r=2m+1}^{\infty'} d_{e\,|\,r}^{mn}(-ic)\,P_{r-m-1}^m(0)\right], \quad (n-m) \text{ is odd}$$

$$R_{mn}^{(2)'}(-ic,\ i0) = \frac{1}{cR_{mn}^{(1)}(-ic,\ i0)} =$$

$$= \frac{i^{n-m}(2m+1)\sum\limits_{r=0}^{\infty'} d_r^{mn}(-ic)\,\dfrac{(2m+r)!}{r!}}{2^m\,m!\,c^{m+1}\,d_0^{mn}(-ic)} =$$

$$= \left[i^{-m-1} k_{mn}^{(2)}(-ic)\right]^{-1} \left[\sum_{r=-2m}^{\infty}{}' d_r^{mn}(-ic) Q_{m+r}^{m'}(0) + \right.$$

$$\left. + \sum_{r=2m+2}^{\infty}{}' d_{e\,|\,r}^{mn}(-ic) P_{r-m-1}^{m'}(0) \right], \quad (n-m) \text{ is even} \tag{5.3.31}$$

$$R_{mr}^{(2)'}(-ic, i0) =$$

$$= \frac{i^{n-m-1}(2m-3)(2m-1)\, m!\, c^{m-2}\pi}{2^{2n-m+1}(2m)!\, d_{-2m+1}^{mn}(-ic) \sum_{r=1}^{\infty}{}' d_r^{mn}(-ic) \frac{(2m+r)!}{r!}} \times$$

$$\times \left[\frac{(n+m+1)!}{\left(\frac{n-m-1}{2}\right)! \left(\frac{n+m+1}{2}\right)!} \right]^2, \quad (n-m) \text{ is odd} \tag{5.3.32}$$

§4. Expansion of Plane and Spherical Waves According to Spheroidal Wave Functions. Integral Representations of Spheroidal Wave Functions.

If we expand the Green function (2.5.3) of the Helmholtz equation (2.5.2) according to the prolate spheroidal wave functions (5.1.7), (5.1.8) by the same manner as in the previous cases, we find that /190

$$\frac{e^{ikR}}{R} = 2ik \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{2-\delta_{0m}}{N_{mn}(c)} S_{mn}(c, \eta_1) S_{mn}(c, \eta_0) \cos m(\varphi - \varphi_0) \times$$

$$\times \begin{cases} R_{mn}^{(1)}(c, \xi) R_{mn}^{(3)}(c, \xi_0), & \xi_0 > \xi, \\ R_{mn}^{(1)}(c, \xi_0) R_{mn}^{(3)}(c, \xi), & \xi_0 < \xi, \end{cases} \tag{5.4.1}$$

where δ_{0m} is the Kronecker symbol. If we bear in mind the discussion given in §2 of this chapter, relative to the angular functions with integral negative values of the index m, then expansion (5.4.1) may be written in the form,

$$\frac{e^{ikR}}{R} = 2ik \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{S_{mn}(c, \eta_0)}{N_{mn}(c)} S_{mn}(c, \eta_1) e^{im(\varphi-\varphi_0)} \times$$

$$\times \begin{cases} R_{|m|n}^{(1)}(c, \xi) R_{|m|n}^{(3)}(c, \xi_0), & \xi_0 > \xi, \\ R_{|m|n}^{(1)}(c, \xi_0) R_{|m|n}^{(3)}(c, \xi), & \xi_0 < \xi. \end{cases} \tag{5.4.2}$$

The expansion formulas for a spherical wave (2.5.1) according to the oblate spheroidal wave functions are obtained from (5.4.1), (5.4.2) by substitution of $i\xi$ for ξ and $-ic$ for c in them.

If we remove the source to infinity and use the asymptotic formula for the functions $R_{mn}^{(3)}(c, \xi_0)$ (5.3.5), from (5.4.1) in the usual manner we find the expansion of the plane wave (2.3.1) according to the prolate spheroidal wave functions:

$$e^{ik(\mathbf{nR})} = 2\sum_{n=0}^{\infty}\sum_{m=0}^{n} i^n \frac{2-\delta_{0m}}{N_{mn}(c)} \times \\ \times S_{mn}(c, \eta_0) S_{mn}(c, \eta) R_{mn}^{(1)}(c, \xi) \cos m(\varphi - \varphi_0), \quad \eta_0 = \cos\theta_0, \tag{5.4.3}$$

where in spheroidal coordinates,

$$(\mathbf{nR}) = f\left[\eta_0\eta\xi + (1-\eta_0^2)^{1/2}(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}\cos(\varphi-\varphi_0)\right] = \\ = r\cos\gamma, \tag{5.4.4}$$

$$\cos\gamma = \cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\varphi-\varphi_0).$$

Formula (5.4.3) may be written in the form,

$$e^{ik(\mathbf{nR})} = 2\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^n \frac{S_{mn}(c, \eta_0)}{N_{mn}(c)} \times \\ \times S_{mn}(c, \eta) R_{|m|n}^{(1)}(c, \xi) e^{im(\varphi-\varphi_0)}. \tag{5.4.5}$$

If we substitute $i\xi$ for ξ and $-ic$ for c in (5.4.3), (5.4.5), we obtain the expansions of the plane wave according to the oblate spheroidal wave functions.

By multiplying both parts of expansion (5.4.3) by

$$S_{pq}(c, \eta_0) \begin{matrix}\cos\\ \sin\end{matrix}(p\varphi_0)\, d\eta_0\, d\varphi_0$$

and integrating them over η_0 from -1 to 1 and over ϕ_0 from 0 to 2π, we find the integral representation of the prolate spheroidal wave functions (5.1.7)

$$R_{mn}^{(1)}(c, \xi) S_{mn}(c, \eta) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) = \\ = \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} e^{ik(\mathbf{nR})} S_{mn}(c, \eta_0) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\, d\eta_0\, d\varphi_0, \tag{5.4.6}$$

or by substituting $\cos\theta_0$ for η_0,

$$R_{mn}^{(1)}(c,\ \xi)\,S_{mn}(c,\ \eta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) =$$

$$= \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ik(nR)}\,S_{mn}(c,\ \cos\theta_0)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\sin\theta_0\,d\theta_0\,d\varphi_0. \qquad (5.4.7)$$

If we proceed from expansion (5.4.5), then

$$R_{|m|n}^{(1)}(c,\ \xi)\,S_{mn}(c,\ \eta)\,e^{im\varphi} =$$

$$= \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_{-1}^{1} e^{ik(nR)}\,S_{mn}(c,\ \eta_0)\,e^{im\varphi_0}\,d\eta_0\,d\varphi_0 \qquad (5.4.8)$$

or

$$R_{|m|n}^{(1)}(c,\ \xi)\,S_{mn}(c,\ \gamma_1)\,e^{im\varphi} =$$

$$= \frac{i^{-n}}{4\pi}\int_0^{2\pi}\int_0^{\pi} e^{ik(nR)}\,S_{mn}(c,\ \cos\theta_0)\,e^{in\varphi_0}\sin\theta_0\,d\theta_0\,d\varphi_0. \qquad (5.4.9)$$

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For the spheroidal wave functions (5.1.8), the formulas for the integral representation have the form:

$$R_{mn}^{(3)}(c,\ \xi)\,S_{mn}(c,\ \eta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) =$$

$$= \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_{i\infty}^{1} e^{ik(nR)}\,S_{mn}(c,\ \gamma_{,0})\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\,d\eta_0\,d\varphi_0,$$

$$R_{|m|n}^{(3)}(c,\ \xi)\,S_{mn}(c,\ \eta)\,e^{im\varphi} = \qquad (5.4.10)$$

$$= \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_{i\infty}^{1} e^{ik(nR)}\,S_{mn}(c,\ \eta_0)\,e^{im\varphi_0}\,d\eta_0\,d\varphi_0,$$

$$R_{mn}^{(4)}(c,\ \xi)\,S_{mn}(c,\ \eta)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi) =$$

$$= \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_{-1}^{i\infty} e^{ik(nR)}\,S_{mn}(c,\ \eta_0)\begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_0)\,d\eta_0\,d\varphi_0,$$

$$R_{|m|n}^{(4)}(c,\ \xi)\,S_{mn}(c,\ \gamma_1)\,e^{im\varphi} = \qquad (5.4.11)$$

$$= \frac{i^{-n}}{2\pi}\int_0^{2\pi}\int_{-1}^{i\infty} e^{ik(nR)}\,S_{mn}(c,\ \eta_0)\,e^{im\varphi_0}\,d\eta_0\,d\varphi_0.$$

After substituting $i\xi$ for ξ and $-ic$ for c in (5.4.6) - (5.4.11), we find the formulas for the integral representation of the oblate spheroidal wave functions [in (5.4.10), (5.4.11), $\xi\eta > 1$, $\xi \neq 1$, $0 < \eta \leq 1$; if $-1 \leq \eta < 0$, $-ik(\mathrm{nR})$ must be substituted for $ik(\mathrm{nR})$].

By using the formulas for the integral representation of the wave functions, we can establish a series of relationships between the spherical and the spheroidal functions which will be useful in application. In particular, we find the relationships:

$$S_{mn}(c, \eta) R_{mn}^{(1)}(c, \xi) = \sum_{r=0,1}^{\infty}{}' d_r^{mn}(c) P_{m+r}^{m}(\cos\theta) j_{m+r}(kr), \tag{5.4.12}$$

$$P_n^m(\cos\theta) j_n(kr) = \frac{2}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} \times \times \sum_{l=m,\, m+1}^{\infty}{}' \frac{i^{l-n} d_{n-m}^{ml}(c)}{N_{ml}(c)} S_{ml}(c, \eta) R_{ml}^{(1)}(c, \xi), \tag{5.4.13}$$

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$$P_n^m(\cos\theta) = \frac{2}{2n+1} \cdot \frac{(n+m)!}{(n-m)!} \times \times \sum_{l=m,\, m+1}^{\infty}{}' \frac{d_{n-m}^{ml}(c)}{N_{ml}(c)} S_{ml}(c, \eta), \tag{5.4.14}$$

$$J_m(k\rho) = 2 \sum_n i^{n-m} \frac{S_{mn}(c, 0)}{N_{mn}(c)} S_{mn}(c, \eta) R_{mn}^{(1)}(c, \xi). \tag{5.4.15}$$

Furthermore, from them we can obtain the integral representations for the radial spheroidal functions. In particular,

$$R_{mn}^{(1)}(c, \xi) = = \begin{cases} \dfrac{i^{m-n}}{2S_{mn}(c, 0)} \displaystyle\int_{-1}^{1} J_m\left[c(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}\right] S_{mn}(c, \eta)\, d\eta, \\ \qquad (n-m) \text{ is even} \\ \dfrac{i^{m-n+1} c\xi}{2S'_{mn}(c, 0)} \displaystyle\int_{-1}^{1} \eta J_m\left[c(1-\eta^2)^{1/2}(\xi^2-1)^{1/2}\right] S_{mn}(c, \eta)\, d\eta, \\ \qquad (n-m) \text{ is odd} \end{cases} \tag{5.4.16}$$

$$R_{mn}^{(3)}(c, \xi) = \frac{i^{m-n} c^m}{\displaystyle\sum_{r=0,1}^{\infty}{}' \frac{(r+2m)!}{r!} d_r^{mn}(c)} (\xi^2-1)^{m/2} \times \times \int_0^{\frac{\pi}{2}-i\infty} e^{ic\xi\cos\theta_0} S_{mn}(c, \cos\theta_0) (\sin\theta_0)^{m+1} d\theta_0, \tag{5.4.17}$$

$$R_{mn}^{(4)}(c,\ \xi) = \frac{i^{m-n}\, c^m}{\sum_{r=0\ \text{или}\ 1}^{\infty}{}' \frac{(r+2m)!}{r!} d_r^{mn}(c)} (\xi^2-1)^{m/2} \times$$

$$\times \int_{\frac{\pi}{2}-i\infty}^{\pi} e^{ic\xi\cos\theta_0} S_{mn}(c,\ \cos\theta_0)(\sin\theta_0)^{m+1} d\theta_0. \qquad (5.4.18)$$

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n analogous type of formula is obtained for the oblate spheroidal unctions by substituting $i\xi$ for ξ and $-ic$ for c in (5.4.12) - 5.4.18).

§5. Addition Theorems for Spheroidal Wave Functions.

Let us give two spatial rectangular Cartesian coordinate systems (x_1, y_1, z_1) and (x_2, y_2, z_2) with parallel and identically irected respective axes $Oj z_j$, $Oj y_j$, $Oj x_j$ $(j = 1, 2)$ and the conersion formulas associated with them,

$$\begin{aligned} x_j &= f_j\left[(\xi_j^2-1)(1-\eta_j^2)\right]^{1/2}\cos\varphi_j, \\ y_j &= f_j\left[(\xi_j^2-1)(1-\eta_j^2)\right]^{1/2}\sin\varphi_j, \\ z_j &= f_j\xi_j\eta_j \end{aligned} \qquad (5.5.1)$$

of the system of local spheroidal coordinates (ξ_j, η_j, ϕ_j). In ach of the local coordinate systems are given systems of spheroidal ave functions,

$$R_{mn}^{(1)}(c_j,\ \xi_j) S_{mn}(c_j,\ \eta_j) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_j) \qquad (5.5.2)$$

nd

$$R_{mn}^{(3),\ (4)}(c_j,\ \xi_j) S_{mn}(c_j,\ \eta_j) \begin{matrix}\cos\\ \sin\end{matrix}(m\varphi_j) \qquad (5.5.3)$$

$n = 0, 1, 2, \ldots, |m| \leq n;\ j = 1, 2$). Let us find formulas which will allow writing each of the wave functions (5.5.2) or (5.5.3) belonging to the j-th system of coordinates through the wave functions written in the s-th system of coordinates ($j = 1$ or $j = 2$; $j = 1, 2$ and $j \neq s$). For this, let us introduce into study the plane wave (2.3.1), after writing it in the j-th local coordinate system in the form

$$e^{ik(\mathbf{n}\mathbf{r}_j)} = e^{ik(\mathbf{n}\mathbf{r}_{js})}\, e^{ik(\mathbf{n}\mathbf{r}_s)} \qquad (5.5.4)$$

($r_j = r_{js} + r_s$, Fig. 19) where the vector n, which determines the direction of propagation of the wave front, forms with the axis $Oj z_j$ the angle α, and its projection onto the plane $Oj x_j y_j$ forms with the positive direction of the axis $Oj x_j$ the angle β. Therefore, in (5.5.4),

$$(\mathbf{n}\mathbf{r}_j) = z_j \cos\alpha + x_j \sin\alpha \cos\beta + y_j \sin\alpha \sin\beta, \tag{5.5.5}$$

where the Cartesian coordinates are coupled with the spheroidal formulas (5.5.1). In the local spherical coordinates r_j, θ_j, ϕ_j,

$$(\mathbf{n}\mathbf{r}_j) = r_j[\cos\theta_j \cos\alpha + \sin\theta_j \sin\alpha \cos(\varphi_j - \beta)]. \tag{5.5.6}$$

On the basis of (5.4.3) the right hand side of (5.5.4) can be written in the form of an expansion according to the spheroidal wave functions,

$$e^{ik(\mathbf{n}\mathbf{r}_j)} = 2 \sum_{n=0}^{\infty} \sum_{m=0}^{n} i^n \frac{2 - \delta_{0m}}{N_{mn}(c_s)} \times \tag{5.5.7}$$

$$\times S_{mn}(c_s, \cos\alpha) S_{mn}(c_s, \eta_s) R^{(1)}_{mn}(c_s, \xi_s) \cos m(\varphi_s - \beta) e^{ik(\mathbf{n}\mathbf{r}_{js})}.$$

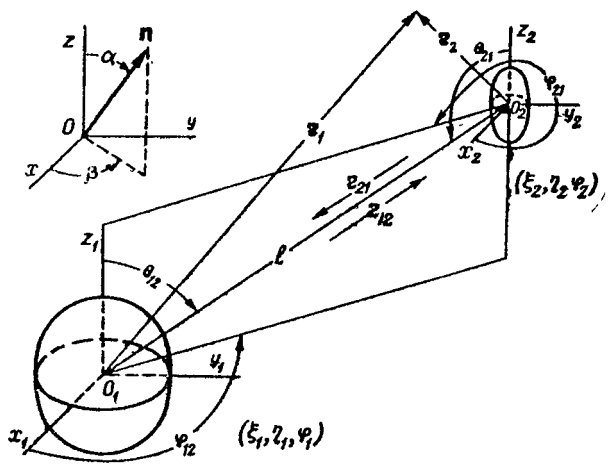

Fig. 19. Addition Theorem for Spheroidal Wave Functions.

After multiplying both parts of (5.5.7) by

$$S_{pq}(c_j, \cos\alpha) \begin{matrix}\cos \\ \sin\end{matrix}(p\beta) \sin\alpha \, d\alpha \, d\beta$$

id integrating them over β from 0 to 2π, and over α along one of ie boundaries c_1, c_3, c_4 (c_1 is the boundary with ends at the oints 0 and π; c_3 is the boundary with ends at the points 0 and $-i\infty$; c_4 is the boundary with ends at the points $\frac{\pi}{4} - i\infty$ and π) c find with the aid of formulas (5.4.7), (5.4.10), (5.4.11), 5.2.2.), (3.6.11), (3.6.14), (3.4.12) and (3.4.13) the desired ormulas, i.e., the addition theorems for the prolate spheroidal ave functions. They may be written in the form, /196

$$R_{pq}^{(1),(3);(4)}(c_j, \xi_j) S_{pq}(c_j, \eta_j) \begin{matrix}\cos\\ \sin\end{matrix} (p\varphi_j) =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{T_{mnpq}^{(1),(3),(4)}(c_j, c_s; l, \theta_{js}) \begin{matrix}\cos\\ \sin\end{matrix} [(p+m)\varphi_{js} - m\varphi_s] +$$

$$+ Q_{mnpq}^{(1),(3),(4)}(c_j, c_s; l, \theta_{js}) \begin{matrix}\cos\\ \sin\end{matrix} [(p-m)\varphi_{js} + m\varphi_s]\} \times$$

$$\times R_{mn}^{(1)}(c_s, \xi_s) S_{mn}(c_s, \eta_s), \qquad (5.5.8)$$

here

$$T_{mnpq}^{(1),(3),(4)} = i^{n-q} \frac{2-\delta_{0m}}{N_{mn}(c_s)} \sum_{r=0,1}^{\infty'} \sum_{t=0,1}^{\infty'} d_r^{pq}(c_j) d_t^{mn}(c_s) \times \qquad (5.5.9)$$

$$\times \sum_{\sigma=|r+p-t-m|}^{r+p+t+m} i^{\sigma} a_{\sigma}^{(t+m, m, r+p, p)} z_{\sigma}^{(1),(3),(4)}(kl) P_{\sigma}^{p+m}(\cos\theta_{js}),$$

$$Q_{mnpq}^{(1),(3),(4)} = i^{n-q} \frac{2-\delta_{0m}}{N_{mn}(c_s)} \sum_{r=0,1}^{\infty'} \sum_{t=0,1}^{\infty'} d_r^{pq}(c_j) d_t^{mn}(c_s) \times \qquad (5.5.10)$$

$$\times \sum_{\sigma=|r+p-t-m|}^{r+p+t+m} i^{\sigma} b_{\sigma}^{(t+m, m, r+p, p)} z_{\sigma}^{(1),(3),(4)}(kl) \times$$

$$\times P_{\sigma}^{m-p}(\cos\theta_{js}).$$

lere through $l = r_{js}$, θ_{js}, ϕ_{js} we denote the spherical coordinates of the point O_s, i.e., the origin of the s-th local system of coordinates, in the j-th system (l is the distance between O_j and O_s; $l = r_{js} = r_{sj}$); through $z_{\sigma}(kl)$ we denote one of the spherical Bessel functions: $z_{\sigma}^{(1)}(kl) = j_{\sigma}(kl)$; $z_{\sigma}^{(3)}(kl) = h_{\sigma}^{(1)}(kl)$, $z_{\sigma}^{(4)}(kl) = h_{\sigma}^{(2)}(kl)$. The values of $a_{\sigma}^{(t+m, m, r+p, p)}$ and $b_{\sigma}^{(t+m, m, r+p, p)}$ are determined just as in (3.4.14), (3.4.15).

In the case of spheroidal wave functions, containing the radial function $R_{pq}^{(1)}(c_j, \xi_j)$ in the left-hand side of (5.5.8) formula (5.5.8) remains valid for values of $\xi_{js} \lesseqgtr \xi_s$. In the case if the wave function in the left-hand side of (5.5.8) contains the radial function $R_{pq}^{(3)}(c_j, \xi_j)$ or $R_{pq}^{(4)}(c_j, \xi_j)$, formula (5.5.8) can be used only under the condition that $l > f_s \xi_s$ (justification of the substitution of the order of summation and integration in deriving these formulas requires that $l \geq f_i \xi_{js} |\eta_{js}| > f_j + f_s$ also).

When $l < f_s \xi_s$ the addition theorems for the prolate spheroidal /1 wave functions assume the form,

$$R_{pq}^{(3),\,(4)}(c_j, \xi_j) S_{pq}(c_j, \xi_j) \begin{matrix}\cos\\ \sin\end{matrix}(p\varphi_j) =$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \{\tilde{T}_{pqmn} R_{m+p,\,n}^{(3),\,(4)}(c_s, \xi_s) S_{m+p,\,n}(c_s, \eta_s) \times \tag{5.5.11}$$

$$\times \begin{matrix}\cos\\ \sin\end{matrix}[(p+m)\varphi_s - m\varphi_{js}] + \tilde{Q}_{pqmn} R_{p-m,\,n}^{(3),\,(4)}(c_s, \xi_s) S_{p-m,\,n}(c_s, \eta_s) \times$$

$$\times \begin{matrix}\cos\\ \sin\end{matrix}[(p-m)\varphi_s + m\varphi_{js}]\},$$

where

$$\tilde{T}_{pqmn} = i^{n-q} \sum_{\sigma=m}^{\infty}{}' i^{\sigma} \frac{2-\delta_{0m}}{N_{m\sigma}(c_j)} \times$$

$$\times A_{pqm\sigma}^{n}(c_j, c_s) R_{m\sigma}^{(1)}(c_j, \xi_{js}) S_{m\sigma}(c_j, \eta_{js}),$$

$$\tilde{Q}_{pqmn} = i^{n-q} \sum_{\sigma=m}^{\infty}{}' i^{\sigma} \frac{2-\delta_{0m}}{N_{m\sigma}(c_j)} \times \tag{5.5.12}$$

$$\times B_{pqm\sigma}^{n}(c_j, c_s) R_{m\sigma}^{(1)}(c_j, \xi_{js}) S_{m\sigma}(c_j, \eta_{js}).$$

Formula (5.5.11) is obtained according to the previous plan after representation of the plane wave in the form of (5.5.4) and expansion according to the spheroidal wave functions of the factor $e^{ik(\mathrm{n}r_{js})}$. Here as the auxiliaries we use the formulas

$$S_{mn}(c_1, \eta) S_{pq}(c_2, \eta) = \sum_{\sigma=m+p}^{\infty}{}' A_{pqmn}^{\sigma}(c_2, c_1) S_{m+p,\sigma}(c_1, \eta), \tag{5.5.13}$$

$$S_{mn}(c_1, \eta) S_{pq}(c_2, \eta) = \sum_{\sigma=p-m}^{\infty}{}' B_{pqmn}^{\sigma}(c_2, c_1) S_{p-m,\,\sigma}(c_1, \eta), \tag{5.5.14}$$

obtained on the basis of (5.2.2), (3.4.12), (3.4.13) and (5.4.14). Here

$$A^{\sigma}_{pqmn} = \sum_{r=0,1}^{\infty\prime} \sum_{t=0,1}^{\infty\prime} d_r^{pq}(c_2)\, d_t^{mn}(c_1) \times \times \sum_{l=|r+p-t-m|}^{r+p+t+m} d^{p+m,\,\sigma}_{l-m-p}(c_1)\, a_l^{(r+p,\,p,\,t+m,\,m)} \frac{N_{p+m,\,l}}{N_{p+m,\,\sigma}(c_1)}, \tag{5.5.15}$$

$$B^{\sigma}_{pqmn} = \sum_{r=0,1}^{\infty\prime} \sum_{t=0,1}^{\infty\prime} d_r^{pq}(c_2)\, d_t^{mn}(c_1) \times \times \sum_{l=|r+p-t-m|}^{r+p+t+m} d^{p-m,\,\sigma}_{l-p+m}(c_1)\, b_l^{(r+p,\,p,\,t+m,\,m)} \frac{N_{p-m,\,l}}{N_{p-m,\,\sigma}(c_1)}, \tag{5.5.16}$$

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but

$$N_{mn} = \frac{2(n+m)!}{(2n+1)(n-m)!}. \tag{5.5.17}$$

With degeneration of the spheroidal prolate coordinates into spherical ones, formulas (5.5.11) and (5.5.8) convert into the formulas in (3.7.20), written respectively for the case $r_{js} > r_s$ and $r_{js} < r_s$.

With degeneration of the j-th system of spheroidal coordinates into a spherical one ($j = 1$ or $j = 2$), from (5.5.8) and (5.5.11) we obtain the addition theorems which give the expansion of the spherical wave functions according to the spheroidal wave functions written in the s-th system of coordinates ($s = 2$ or $s = 1$). On the contrary, with degeneration of the s-th system of coordinates into a spherical one, from formulas (5.5.8) and (5.5.11) we obtain the expansions of the spheroidal prolate wave functions according to the spherical wave functions, written in the s-the system of coordinates.

When the axes $Ojzj$ of both local systems of coordinates coincide (the coordinate spheroids of the systems are coaxial), in formulas (5.5.8) and (5.5.11), $\theta_{js} = +1$ or $\theta_{js} = -1$ and $\phi_j = \phi_s$. As a result, in (5.5.8) we will have $T^{(1),(3),(4)}_{mnpq} \equiv 0$, if $p = m \neq 0$, and formula (5.5.8) is written in the form,

$$R^{(1),(3),(4)}_{pq}(c_j,\ \xi_j)\, S_{pq}(c_j, \eta_j) = = \sum_{n=p}^{\infty} Q^{(1),(3),(4)}_{pnpq}(c_j, c_s; l, \theta_{js})\, R^{(1)}_{pn}(c_s, \xi_s)\, S_{pn}(c_s, \eta_s). \tag{5.5.18}$$

If, furthermore, $p = m = 0$ (the wave functions do not depend on the azimuth angle ϕ), then

$$T_{0n0q} \equiv Q_{0n0q}$$

and the theorem (5.5.8) is written in the form,

$$R_{0q}^{(1),\,(3),\,(4)}(c_j,\ \xi_j)\,S_{0q}(c_j,\ \eta_j) = \\ = 2\sum_{n=0}^{\infty} Q_{0n0q}^{(1),\,(3),\,(4)}(c_j,\ c_s;\ l,\ \theta_{js})\,R_{0n}^{(1)}(c_s,\ \xi_s)\,S_{0n}(c_s,\ \eta_s). \qquad (5.5.19)$$

In analogous cases, from (5.5.11) we obtain the formulas, /19

$$R_{pq}^{(3),\,(4)}(c_j,\ \xi_j)\,S_{pq}(c_j,\ \eta_j) = \\ = 2\sum_{n=p}^{\infty} \tilde{Q}_{pq0n}\,R_{pn}^{(3),\,(4)}(c_s,\ \xi_s)\,S_{pn}(c_s,\ \eta_s), \qquad (5.5.20)$$

$$R_{0q}^{(3),\,(4)}(c_j,\ \xi_j)\,S_{0q}(c_j,\ \eta_j) = \\ = 2\sum_{n=0}^{\infty} \tilde{Q}_{0q0n}\,R_{0n}^{(3),\,(4)}(c_s,\ \xi_s)\,S_{0n}(c_s,\ \eta_s). \qquad (5.5.21)$$

If in deriving the addition theorems for prolate spheroidal wave functions (5.5.2) and (5.5.3) we take instead of (5.4.3) the expansion of the plane wave in the form of (5.4.5) and the respective formulas of the integral representation of the spheroidal and spherical wave functions, the addition theorems may be written in the form,

$$R_{|p|q}^{(1),\,(3),\,(4)}(c_j,\ \xi_j)\,S_{pq}(c_j,\ \eta_j)\,e^{ip\varphi_j} = \\ = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} Q_{mnpq}^{(1),\,(3),\,(4)}(c_j, c_s; l,\ \theta_{js})\,S_{mn}(c_s, \eta_s)\times \\ \times R_{|m|n}^{(1)}(c_s,\ \xi_s)\,e^{im\varphi_s}, \qquad (5.5.22)$$

where

$$Q_{mnpq}^{(1),\,(3),\,(4)} = \frac{2i^{n-q}}{N_{mn}(c_s)}\sum_{r=0,1}^{\infty}{}'\sum_{t=0,1}^{\infty}{}' \overset{\bullet}{d}{}_r^{pq}(c_j)\,d_t^{mn}(c_s)\times \\ \times \sum_{\sigma=|r+p-t-m|}^{r+p+t+m} i^{\sigma}\,b_{\sigma}^{(r+p,\ p,\ t+m,\ m)}\times \\ \times z_{\sigma}^{(1),\,(3),\,(4)}(kl)\,P_{\sigma}^{p-m}(\theta_{js})\,e^{i(p-m)\varphi_{js}}. \qquad (5.5.23)$$

In the case when the coordinate spheroids are coaxial ($p = m \neq 0$, $\phi_j = \phi_s$), we obtain the formula,

$$R^{(1),\,(3),\,(4)}_{|p|q}(c_j,\ \xi_j)\,S_{pq}(c_j,\ \eta_j) = \\ = \sum_{n=p}^{\infty} Q^{(1),\,(3),\,(4)}_{pnpq}(c_j,\ c_s;\ l,\ \theta_{js})\,R^{(1)}_{|p|n}(c_s,\ \xi_s)\,S_{pn}(c_s,\ \eta_s), \qquad (5.5.24)$$

and with axial symmetry ($p = m = 0$),

$$R^{(1),\,(3),\,(4)}_{0q}(c_j,\ \xi_j)\,S_{0q}(c_j,\ \eta_j) = \\ = \sum_{n=0}^{\infty} Q^{(1),\,(3),\,(4)}_{0n0q}(c_j,\ c_s;\ l,\ \theta_{js})\,R^{(1)}_{0n}(c_s,\ \xi_s)\,S_{0n}(c_s,\eta_s). \qquad (5.5.25)$$

PART TWO

CHAPTER 6

DIFFRACTION OF ELECTROMAGNETIC WAVES ON CIRCULAR CYLINDERS

§1. Diffraction of a Plane Wave on One Cylinder.

In an unbounded uniform and isotropic space with electric and magnetic permeabilities ε, μ and a conductivity $\sigma = 0$ $\left(k^2 = \frac{\omega^2}{c^2}\mu\varepsilon\right)$ let there be contained an infinitely long circular cylinder with the permeabilities ε_2, μ_2 and a conductivity σ_2 ($k^2 \blacksquare (\omega^2\mu_2\varepsilon_2 + i4\pi\omega\mu_2\sigma_2)/c^2$), occupying in a cylindrical system of coordinates ρ, ϕ, z an area of space $\rho \leq a$ (the axis Oz of the system is directed along the axis of the cylinder), and let there be incident on the cylinder a plane electromagnetic wave (1.3.4) which propagates in the direction of the unit vector $\mathbf{n}$ ($\mathbf{n}$ | pl. $\phi = 0$ and forms an angle θ with the axis z) and is polarized such that the vector of the electrical intensity E^0 of the wave is directed at an angle α to the plane $\phi = 0$ (Fig. 20). The problem is formulated on determining the secondary fields outside and inside the cylinder, which occur with incidence of the plane wave on the cylinder. As usual, a strict solution to this problem is reduced to solving a uniform system of Maxwell equations (1.1.4) ($\rho = 0$, $J^{(ext)} = 0$) with the given boundary conditions such as (1.1.5) or (1.1.6) (depending on the value of σ_2) on the surface of the cylinder ($\rho = a$) and the condition of radiation at infinity for the components of the vectors of the electromagnetic fields. In the cylindrical system of coordinates the primary field E^0, H^0 of the incident wave permits representation in the form of a superposition of the fields of electrical (TE) and magnetic (TM) types. Therefore, in solving the problem we can look separately at the two cases of polarization of the field of the incident wave relative to the plane $\phi = 0$, when the vector $E \parallel$ pl. $\phi = 0$, the vector $H^0 \parallel$ pl. $\phi = 0$. /200

Solution to the general problem representing the case of an arbitrarily polarized field E^0, H^0 of the incident wave can be obtained then in the form of a linear combination of solutions found for the cases of the TM and TE-polarizations of the incident wave. /201

For example, let the field of the wave (1.3.4) be a field of only one of the types. Then we can show that in the case of normal incidence of the plane wave (1.3.4) on the cylinder with $\sigma_2 \neq \infty$ ($\theta = \pi/2$), and also in the case of an oblique incidence on an ideally conducting cylinder ($\sigma_2 = \infty$), the secondary fields both outside and inside the cylinder (if $\sigma_2 \neq \infty$) will be fields of the same type as the field of the incident wave. However, in the general case of oblique incidence of the plane wave on the cylinder with a

finite conductivity ($\sigma_2 \neq \infty$) it is impossible to satisfy the boundary conditions (1.1.5) on the surface of the cylinder if we seek the secondary fields as fields of the same type as the field of the incident wave. In this case the secondary fields should be sought in the form of a superposition of fields of both types, in spite of the fact that the field of the incident wave is a field of only one type. Therefore if we introduce the auxiliary scalar potential functions u, v, coupled with the vectors of the field by equations (1.6.16), (1.6.18), then depending on the character of the polarization of the field of the incident wave (1.3.4), the direction of its propagation (the angle θ) and the electromagnetic properties of the cylinder, the problem of determining the vectors of the secondary fields will be reduced in mathematical formulation to a problem of seeking the solutions to u, v of the Helmholtz scalar equation, which satisfies one of the boundary conditions (1.7.1) - (1.7.4) on the surface of the cylinder and the supplementary condition of radiation at infinity, applied to the functions u^1, v^1 which determine the secondary field outside the cylinder. For solution to the problem we must know the explicit form of the potential functions u^0, v^0 which represent the field of the incident wave.

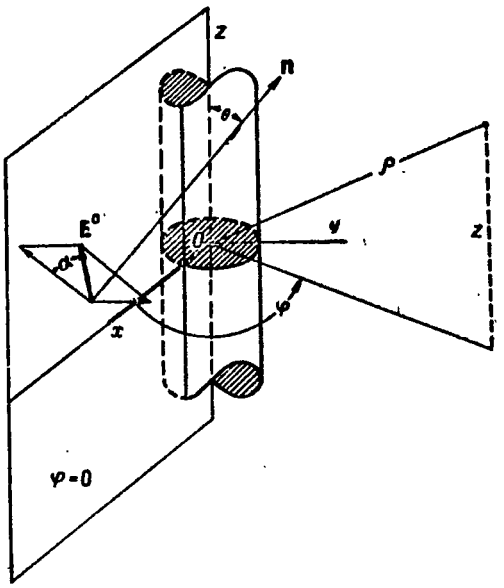

Fig. 20. Excitation of Circular Cylinder of the Plane Wave.

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If the plane wave (1.3.4) which is obliquely incident on the cylinder is a wave of TM-type, the component H_z^0 of the vector H^0 is equal to zero, and the component along the axis z of the vector E^0 is equal in cylindrical coordinates to the expression,

$$E_z^0 = E \sin\theta \, e^{ikz\cos\theta - ik\rho\sin\theta\cos\varphi}, \tag{6.1.1}$$

and then from (1.6.16) we find that $v^0 \equiv 0$, but

$$u^0 = \frac{E}{k\lambda} e^{ikz\cos\theta - i\lambda\rho\cos\varphi}, \tag{6.1.2}$$

where $\lambda = k \sin\theta$.

If the wave (1.3.4) which is obliquely incident on the cylinder is a wave of TE-type, then the component E_z^0 of the vector E^0 is equal to zero and in cylindrical coordinates the component along the axis z of the vector H^0 is equal to the expression,

$$H_z^0 = E \sin\theta\, e^{ikz\cos\theta - ik\rho\sin\theta\cos\varphi} \tag{6.1.3}$$

and from (1.6.18) we find that $u^0 \equiv 0$, but

$$v^0 = \frac{E}{k\lambda}\, e^{ikz\cos\theta - i\lambda\rho\cos\varphi}. \tag{6.1.4}$$

On the basis of (2.3.8) the functions (6.1.2), (6.1.4) may be represented in the form of expansions according to the cylindrical wave functions:

$$u^0 = \frac{E}{k\lambda}\, e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n J_n(\lambda\rho)\, e^{in\varphi} \tag{6.1.5}$$

for a field of TM-type and

$$v^0 = \frac{E}{k\lambda}\, e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n J_n(\lambda\rho)\, e^{in\varphi} \tag{6.1.6}$$

for a field of TE-type.

Since the cylinder is infinitely long, then the dependence of the complete field on the space coordinate z must be periodic and must be expressed by the factor $\exp(-ikz\cos\theta)$. Therefore, the potential functions u^j, v^j $(j = 1, 2)$ of the secondary fields in the general case of oblique incidence of the wave (1.3.4) and $\sigma_2 \neq \infty$ in the region outside the cylinder should be sought in the form of an expansion according to the wave functions (2.1.7) with the constant $h = k\cos\theta$: /203

$$u^1 = e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} a_n^{(1)} H_n^{(1)}(\lambda\rho)\, e^{in\varphi},$$

$$v^1 = e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} b_n^{(1)} H_n^{(1)}(\lambda\rho)\, e^{in\varphi}, \tag{6.1.7}$$

and in the region inside the cylinder, in the form of an expansion according to the wave functions (2.1.6) with the same value of $h = z \cos \theta$:

$$u^2 = e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} a_n^{(2)} J_n(\lambda_2 \rho) e^{in\varphi},$$

$$v^2 = e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} b_n^{(2)} J_n(\lambda_2 \rho) e^{in\varphi}, \qquad (6.1.8)$$

where $\lambda_2^2 = k_2^2 - k^2 \cos^2 \theta$.

The expansion coefficients $a_n^{(j)}$, $b_n^{(j)}$ ($j = 1, 2$) are subject to determination from the boundary conditions. By determination of them, and through them the potential functions u^j, v^j also, the strict solution to the problem is generally finished. The expressions for the components of the vectors E, H of the sought fields may be found through u, v from (1.6.16), (1.6.18).

Such then in principle is the scheme for the strict solution to the problem of diffraction of an obliquely incident plane wave of TM or TE-type on an infinitely long circular cylinder. This problem pertains to the number of simplest problems of the diffraction theory solvable by the classical method.

An Ideally Conducting Cylinder. The field inside an ideally conducting cylinder is equal to zero and the problem is reduced to determining the secondary field E^1, H^1, scattered on the cylinder. For example, let the field of the incident wave be a TM-type field. Then $v^0, v^{1,2} \equiv 0$, $u^2 \equiv 0$. The function u^1 is sought in the form of the series (6.1.7). On the surface of the cylinder when $\rho = a$ added to the function u^0 it must satisfy the boundary condition (1.7.3). On the basis of (6.1.5) and (6.1.7), from (1.7.3) we find /204

$$a_n^1 = -\frac{E}{k\lambda} (-i)^n \frac{J_n(\lambda a)}{H_n^{(1)}(\lambda a)}, \qquad (6.1.9)$$

such that

$$u^1 = -\frac{E}{k\lambda} e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n \frac{J_n(\lambda a)}{H_n^{(1)}(\lambda a)} H_n^{(1)}(\lambda \rho) e^{in\varphi}. \qquad (6.1.10)$$

The expressions for the components of the secondary field E^1, H^1 are found from equations (1.6.16). For the complete field,

$$u = \frac{E}{k\lambda} e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n \times$$

$$\left\{ J_n(\lambda\rho) - \frac{J_n(\lambda a)}{H_n^{(1)}(\lambda a)} H_n^{(1)}(\lambda\rho) \right\} e^{in\varphi}. \qquad (6.1.11)$$

If we know the expressions for the components of the vector H of the magnetic intensity of the complete field on the surface of the cylinder, we can establish the distribution density law for the surface currents j, excited on the surface of the cylinder by the field of the incident wave. Thus, from (1.1.7), we find that in cylindrical coordinates $j = \{0, 0, j_z\}$, where $j_z = \frac{c}{4\pi} H_\phi$ $(\rho = a)$. After determining H_ϕ through (6.1.11), from (1.6.16) and using the value of the Wronskian determinant for the Bessel functions (2.2.52), we find

$$j_z = \frac{Ece^{ikz\cos\theta}}{2\pi^2 a\lambda} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=-\infty}^{\infty} \frac{(-i)^n e^{in\varphi}}{H_n^{(1)}(\lambda a)}. \qquad (6.1.12)$$

Let us note here that if the rule governing distribution of the current on the surface of the cylinder were known *a priori*, then the solution to the respective problem of diffraction could be found by a simpler method, for example, through the electrodynamic vector potential A (§2, Ch. 1), associated with the surface currents by the relationship,

$$\mathbf{A} = \frac{\mu}{c} \int_{(S)} \mathbf{j} \frac{e^{ikR}}{R} ds. \qquad (6.1.13)$$

However, in the majority of cases the imaginary distribution of the current on the surface of the body may be found only as a result of a strict solution to the problem (for example, as was done above for the cylinder). Knowledge of the rules in the distribution of currents on the surface of a body, established on the basis of a strict solution to the problem, permits using them then for an approximate solution to the diffraction problems. /205

In the wave band, for which, when $\theta \neq 0$, $k\rho\sin\theta \to \infty$, on the basis of the asymptotic formula (2.2.24) for the first Hankel function, the function u^1 will be approximately equal to

$$u^1 \doteq -\frac{E}{k\lambda} e^{ikz\cos\theta} \sqrt{\frac{2}{\pi\lambda\rho}} e^{i\left(\lambda\rho - \frac{\pi}{4}\right)} \times$$

$$\times \sum_{n=-\infty}^{\infty} (-1)^n \frac{J_n(\lambda a)}{H_n^{(1)}(\lambda a)} e^{in\varphi}. \qquad (6.1.14)$$

In the case of normal incidence of the wave (1.3.4) when $\theta = \pi/2$, the fields do not depend on the spatial coordinate z, by virtue of which in the examined approximation, from (1.6.16) there follows,

$$E_\rho^1 = E_\varphi^1 = H_\rho^1 = H_z^1 = 0, \tag{6.1.15}$$

$$E_z^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = \frac{e^{ik\rho}}{\sqrt{\rho}}\, A(\varphi), \tag{6.1.16}$$

where through $A(\phi)$ we denote the amplitude function of the stray field in the wave band determined by the formula,

$$A(\varphi) = -E\sqrt{\frac{2}{\pi k}}\, e^{-i\pi/4} \times$$
$$\times \sum_{n=-\infty}^{\infty} (-1)^n \frac{J_n(ka)}{H_n^{(1)}(ka)}\, e^{in\varphi}. \tag{6.1.17}$$

On the basis of determining the cross section of back scattering (in this case of the radar scattering cross section) (1.4.11) and formulas (6.1.16), (6.1.17) we find that in the case of a field of TM type and $\theta = \pi/2$,

$$\sigma_B = \frac{4}{k}\left|\sum_{n=-\infty}^{\infty} (-1)^n \frac{J_n(ka)}{H_n^{(1)}(ka)}\right|^2 \tag{6.1.18}$$

or /206

$$\sigma_B = \frac{4}{k}\left|\sum_{n=0}^{\infty} \varepsilon_n (-1)^n \frac{J_n(ka)}{H_n^{(1)}(ka)}\right|^2, \tag{6.1.19}$$

where $\varepsilon_0 = 1$, $\varepsilon_n = 2$, if $n \geq 1$.

Proceeding from the overall determination of the complete cross section of scattering (§4, Ch. 1), per unit of length of the cylinder, the value of σ_S is found to equal

$$\sigma_s = \frac{4}{k}\,\mathrm{Re}\sum_{n=-\infty}^{\infty} \frac{J_n(ka)}{H_n^{(1)}(ka)} \tag{6.1.20}$$

or

$$\sigma_s = \frac{4}{k}\,\mathrm{Re}\sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(ka)}{H_n^{(1)}(ka)}. \tag{6.1.21}$$

In the case when the incident wave is a wave of TE-type, we will have $u^{0,1,2} \equiv 0$, $v^2 \equiv 0$, and the potential v^1 is sought in the form of series (6.1.7). On the surface of the cylinder ($\rho = a$) the function v^1 added to v^0 must satisfy the boundary condition (1.7.4). On the basis of (6.1.6), (6.1.7), from (1.7.4) we find that[1]

$$b_n^1 = -\frac{E}{k\lambda}(-i)^n \frac{J_n'(\lambda a)}{H_n^{(1)'}(\lambda a)}, \tag{6.1.22}$$

and then

$$v^1 = -\frac{E}{k\lambda} e^{ikz\cos\theta} \times \times \sum_{n=-\infty}^{\infty} (-i)^n \frac{J_n'(\lambda a)}{H_n^{(1)'}(\lambda a)} H_n^{(1)}(\lambda\rho) e^{in\varphi}. \tag{6.1.23}$$

The potential of the complete field will be equal to

$$v = \frac{E}{k\lambda} e^{ikz\cos\theta} \times \times \sum_{n=-\infty}^{\infty} (-i)^n \left\{ J_n(\lambda\rho) - \frac{J_n'(\lambda a)}{H_n^{(1)'}(\lambda a)} H_n^{(1)}(\lambda\rho) \right\} e^{in\varphi}. \tag{6.1.24}$$

The expressions for the components of the vectors E, H are found from (1.6.18). From (1.1.7) we find that in cylindrical coordinates $\mathbf{j} = \{0, j_\phi, j_z\}$, where $j_\phi = -\frac{c}{4\pi} H_z$, $j_z = \frac{c}{4\pi} H_\phi$ ($\rho = a$; H_ϕ, H_z are components of the vector H of the complete field). After determining H_ϕ, H_z through (6.1.24), from (1.6.18) on the basis of (2.2.52), we find: /2

$$j_\varphi = -\frac{ciEk}{2\pi^2 ka} e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n \frac{e^{in\varphi}}{H_n^{(1)'}(\lambda a)}, \tag{6.1.25}$$

$$j_z = -\frac{icE\cos\theta}{2\pi^2 a^2\lambda^2} e^{ikz\cos\theta} \sum_{n=-\infty}^{\infty} (-i)^n n \frac{e^{in\varphi}}{H_n^{(1)'}(\lambda a)}. \tag{6.1.26}$$

Thus, unlike the case of a TM-type wave a TE-type incident wave, on the surface of the cylinder when $\theta \neq \pi/2$, excites a current in which generally speaking both the axial j_z and the azimuthal j_ϕ

[1] The prime in the Bessel function everywhere denotes differentiation over the argument.

components of the density vector of the surface current j are nonzero. However the stray field here is created only by the azimuthal component j_ϕ ($u^1 \equiv 0$).

In the wave band, for which when $\theta \neq 0$, $k\rho \sin\theta \to \infty$, approximately,

$$v^1 \quad -\frac{Ee^{ikz\cos\theta}}{k\lambda}\sqrt{\frac{2}{\pi\lambda\rho}}\,e^{i\left(\lambda\rho-\frac{\pi}{4}\right)} \times$$

$$\times \sum_{n=-\infty}^{\infty}(-1)^n \frac{J_n'(\lambda a)}{H_n^{(1)'}(\lambda a)}\,e^{in\varphi}. \qquad (6.1.27)$$

In the case of normal incidence of the plane wave, when $\theta = \pi/2$, from (1.6.18) it follows that in the examined approximation in the wave band

$$E_\rho^1 = E_z^1 = H_\rho^1 = H_\varphi^1 = 0,$$

but

$$H_z^1 = \sqrt{\frac{\varepsilon}{\mu}}\,E_\varphi^1 = \frac{e^{ik\rho}}{\sqrt{\rho}}\,A^*(\varphi), \qquad (6.1.28)$$

where $A^*(\phi)$ is the amplitude function of the TE-type stray field, determined by the expression

$$A^*(\varphi) = E\sqrt{\frac{2}{\pi k}}\,e^{-\frac{i\pi}{4}}\sum_{n=-\infty}^{\infty}(-1)^n\frac{J_n'(ka)}{H_n^{(1)'}(ka)}\,e^{in\varphi}. \qquad (6.1.29)$$

In this case, for the scattering cross section we obtain the formulas, /208

$$\sigma_B = \frac{4}{k}\left|\sum_{n=0}^{\infty}\varepsilon_n(-1)^n\frac{J_n'(ka)}{H_n^{(1)'}(ka)}\right|^2, \qquad (6.1.30)$$

$$\sigma_s = \frac{4}{k}\operatorname{Re}\sum_{n=0}^{\infty}\varepsilon_n\frac{J_n'(ka)}{H_n^{(1)'}(ka)}, \qquad (6.1.31)$$

$$\varepsilon_0 = 1, \quad \varepsilon_n = 2, \quad n \geqslant 1.$$

As we can see, the strict expressions for the components of the vectors E, H of the TM and TE-type fields obtained from (1.6.16) and (1.6.18) as well as the expressions for the components of the densities of the surface currents and the scattering cross sections

have the form of infinite series. Using formulas (2.2.26) - (2.2.28), which are asymptotic with respect to the index n for the Bessel functions, it is easy to establish the absolute and uniform convergence of all series encountered here at each point of space outside the cylinder and on its surface (k and θ are fixed). However, it should be noted that in the numerical calculations the rapidity of the convergence of the series depends substantially (for each θ) on the value of the parameter λa. The series converge rather rapidly, if λa is relatively low (if the radius of the cylinder a does not substantially exceed the wave length λ_0), and with increase in λa, the convergence of the series is significantly worsened, and with large λa, the series become only slightly feasible for numerical computation since in such case the number of terms in the series which must be taken into account during numerical computation have the same order of magnitude as λa.

With large values of λa we use the approximate methods of computing the series by converting them to asymptotic expressions.

We can discuss two approaches to obtaining the asymptotic expressions. The first of these is based on the ideas of Watson and is usually called the Watson transform (method). The Watson transform consists of the fact that the diffraction series to be studied, which gives the solution to the problem, is transformed into a contour integral according to the complex variable (in the complex plane of the index) which, after deformation by integration, is computed using the theory of residues, i.e., it is represented in the form of a sum of residues in the poles of the subintegral expression. For example, by applying the Watson method to the series,

$$S = \sum_{n=-\infty}^{\infty} a_n e^{in\varphi},$$

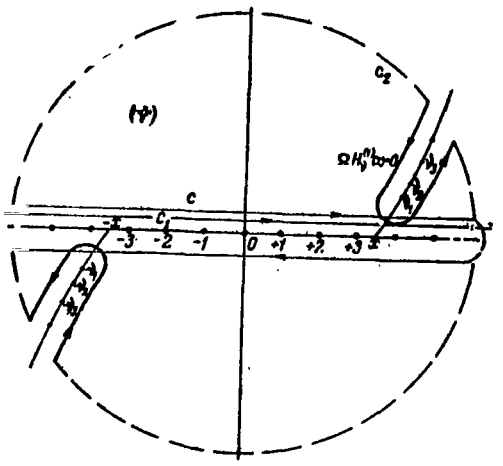
we find after transformation to a contour integral, /20

$$S_1 = \frac{i}{2} \int_{(c)} \frac{e^{i\nu(\varphi-\pi)}}{\sin \pi\nu} a_\nu \, d\nu, \qquad (6.1.32)$$

where the integration boundary (c) on the complex ν-plane includes the real axis as shown on Figure 21 (on the assumption that a_ν has no poles on the real axis). In this case when a_ν is an even function of the index ν, from (6.1.32) we find

$$S = i \int_{(c_1)} \frac{\cos \nu(\varphi - \pi)}{\sin \pi\nu} a_\nu d\nu,$$

:ere as the integration boundary we take the straight line that is rallel to the real axis and is located in the upper half-plane.



g. 2⊥. Watson Transform: is the Integration Bound-y in Formula (6.1.32); c_1 id c_2 are the Deformed oundaries.

By deforming the boundary (c_1) and the loop (c_2) (Fig. 21) which encompasses the poles of the function a_ν and applying the theory of residues to the integral over the boundary (c_2), we find the representation of the series S in the form of a sum of residues, that is suitable for computation for large values of λa (in the problems of diffraction on cylinders the simple poles of the function a_ν are the kernels of the equation $\Omega H_\nu^{(1)}(\lambda a) = 0$, which lie in the first quadrant of the ν-plane, $\Omega = 1$ or $\Omega = \frac{\partial}{\partial(\lambda a)}$). /210

The Watson method, first used by him in a problem of the dipole field, located above the surface of the uniform sphere [143], was then applied to the problem of diffraction of a plane wave on a circular cylinder [125, 144]. Here as ıe original series subject to summation with large values of $\varepsilon = \lambda a$, : take the series,

$$V = \sum_{n=-\infty}^{\infty} (-i)^n \left\{ J_n(\lambda\rho) - \frac{\Omega J_n(\varepsilon)}{\Omega H_n^{(1)}(\varepsilon)} H_n^{(1)}(\lambda\rho) \right\} e^{in\varphi}, \qquad (6.1.33)$$

$$J = \sum_{n=-\infty}^{\infty} (-i)^n \frac{e^{in\varphi}}{\Omega H_n^{(1)}(\varepsilon)} \qquad (6.1.34)$$

$\Omega = 1$ for a TM-type field and $\Omega = \frac{\partial}{\partial\varepsilon}$ for a TE-type field), which iffer from the series (6.1.11), (6.1.24), (6.1.12), (6.1.25) only y the coefficients.

The Watson transform applied to series (6.1.33), set up according to the non-damping wave functions $H_n^{(1)}(\lambda\rho)e^{in\phi}$, leads to the quivalent series

$$V = \pi \sum_s \left\{ \frac{\Omega H_\nu^{(2)*}(\varepsilon)}{\frac{\partial}{\partial\nu}\Omega H_\nu^{(1)}(\varepsilon)} \right\}_{\nu_s} \times$$

$$\times \frac{\cos\nu_s(\pi-\varphi)}{\sin\pi\nu_s} H_{\nu_s}^{(1)}(\lambda\rho)\, e^{\frac{-i\pi\nu_s}{2}} \qquad (6.1.35)$$

for the damping wave functions $H_{\nu_s}^{(1)}(\lambda\rho)e^{i\nu_s\phi}$. Here ν_s represents the kernels of the equation $\Omega H_{\nu_s}^{(1)}(\varepsilon) = 0$, whose approximate values may be found from formulas (2.2.41), (2.2.42). On the basis of the asymptotic formulas in §2, Chapter 2 when $\lambda\rho > \nu_s(\lambda\rho - \lambda a >> (\lambda a)^{1/3})$, from (6.1.35) which appears as the precise expression for the function V, we obtain its asymptotic expression,

$$V = (\lambda a)^{1/3}\frac{e^{i\lambda\sqrt{\rho^2-a^2}+i\pi/12}}{\sqrt{2\lambda}(\rho^2-a^2)^{1/4}}\sum_s\frac{C_s}{1-e^{2\pi i\nu_s}}\times$$

$$\times\left\{e^{i\nu_s\left[\varphi-\arccos\frac{a}{\rho}-\frac{\pi}{2}\right]}+e^{i\nu_s\left[\frac{3\pi}{2}-\varphi-\arccos\frac{a}{\rho}\right]}\right\}, \qquad (6.1.36)$$

/2

where in the case of a TM-type field approximately,

$$C_s = \frac{\pi^{3/2}}{3\cdot 6^{1/3}\{A'(q_s)\}^2}, \qquad (6.1.37)$$

and for a TE-type field, approximately,

$$C_s = \frac{\pi^{3/2}}{6^{1/3}q_s\{A(q_s)\}^2} \qquad (6.1.38)$$

(in (6.1.37) q_s represents the kernels of the equation $A(q_s) = 0$, but in (6.1.38) q_s represents the kernels of the equation $A'(q_s) = 0$; $A(q)$ is the Airy function).

Since the kernels ν_s have a positive imaginary part, which increases with growth in the number s, then each of the waves of the residues (creeping waves [48, 126, 127]) in (6.1.36) rapidly attenuates with growth in s, by virtue of which the series (6.1.36) rapidly converges; if only the values $a\left(\varphi-\arccos\frac{a}{\rho}-\frac{\pi}{2}\right)$, $a\left(\frac{3\pi}{2}-\varphi-\arccos\frac{a}{\rho}\right)$, included in the phase coefficients in (6.1.36), remain positive. This latter is always valid in the region of the geometric shadow, when $\pi/2 < \phi < 3\pi/2$. The convergence will be good when $\lambda a >> 1$ and the values of ϕ are not too near $\pi/2$ and $3\pi/2$. It is easy to see that expression (6.1.36) becomes unsuitable for numerical computations of the field in the wave band of the region of the geometric shadow, when $\rho \to \infty$.

The asymptotic expressions for the function V, which are suitable for numerical computation in the illuminated region in the case of large ε, are obtained on the basis of a slight modification of the Watson method [125] and have in the wave band the form ($\rho\to\infty$)

$$V = V_1 + V_2, \tag{6.1.39}$$

ere asymptotically

$$V_1 = \varepsilon^{1/3} \frac{e^{i\left(\lambda\rho + \frac{\pi}{12}\right)}}{\sqrt{2\lambda\rho}} \sum_s \frac{C_s}{1 - e^{2\pi i \nu_s}} \times$$
$$\times \left\{ e^{\left[i\varepsilon - \frac{\sqrt{3}-i}{2}\left(\frac{\varepsilon}{6}\right)^{1/3} q_s \right](\pi - \varphi)} + \right. \tag{6.1.40}$$
$$\left. + e^{\left[i\varepsilon - \frac{\sqrt{3}-i}{2}\left(\frac{\varepsilon}{6}\right)^{1/3} q_s \right](\pi + \varphi)} \right\}$$

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$$V_2 = e^{-i\lambda\rho\cos\varphi} \mp \sqrt{\frac{a}{2\rho}\cos\frac{\varphi}{2}}\, e^{i\lambda\left[\rho - 2a\cos\frac{\varphi}{2}\right]}. \tag{6.1.41}$$

ıe series (6.1.40) rapidly converges with growth in the number s, nce in the illuminated region the expressions $a\left(\frac{3\pi}{2} + \varphi - \arccos\frac{a}{\rho}\right)$, $a\left(\frac{3\pi}{2} - \varphi - \arccos\frac{a}{\rho}\right)$, that enter into the phase coefficients in 3.1.40), remain positive. For the function J, which is determined / the series (6.1.34), we obtain on the basis of the Watson trans- >rm the asymptotic expressions: in the region of the geometric ıadow ($|\phi - \pi| < \pi/2$),

$$J = \left\{ \begin{array}{c} \frac{\pi\varepsilon^{\frac{2}{3}} e^{\frac{i\pi}{6}}}{2} \\ \frac{\pi\varepsilon}{2i} \end{array} \right\} \sum_s \frac{D_s}{1 - e^{2\pi i \nu_s}} \times \tag{6.1.42}$$
$$\times \left\{ e^{i\nu_s\left(\varphi - \frac{\pi}{2}\right)} + e^{i\nu_s\left(\frac{3\pi}{2} - \varphi\right)} \right\},$$

nd in the illuminated region ($|\phi| < \pi/2$),

$$J = \left\{ \begin{array}{c} \pi\varepsilon\cos\theta \\ -\pi\varepsilon i \end{array} \right\} e^{-i\varepsilon\cos\varphi} + \left\{ \begin{array}{c} \frac{1}{2}\pi\varepsilon^{\frac{2}{3}} e^{\frac{i\pi}{6}} \\ -\frac{i}{2}\pi\varepsilon \end{array} \right\} \times \tag{6.1.43}$$
$$\times \sum_s \frac{D_s}{1 - e^{2\pi i \nu_s}} \left\{ e^{i\nu_s\left(\frac{3\pi}{2} + \varphi\right)} + e^{i\nu_s\left(\frac{3\pi}{2} - \varphi\right)} \right\},$$

where the upper line is taken in the case of a TM-type field and the lower in the case of a TE-type field. Here

$$D_s = -\frac{\pi}{3}\cdot\frac{6^{1/3}}{A'(q_s)}, \quad A(q_s) = 0, \tag{6.1.44}$$

for a TM-type field and

$$D_s = -\frac{\pi}{q_s A(q_s)}, \quad A'(q_s) = 0, \tag{6.1.45}$$

for a TE-type field.

In the vicinity of the geometric shadow ($\phi \simeq 3\pi/2$, $\phi \approx \pi/2$), /2 representations (6.1.42), (6.1.43) become unsuitable for numerical computation since in this region (the so-called "band of half-shadow") the sign of one of the exponents in each of the terms of the sums $\nu_s(\phi - \pi/2)$ or $\nu_s(3\pi/2 - \phi)$, which guarantees convergence of the series, tends to zero. Usually in the band of half-shadow, other methods are used for obtaining the asymptotic approximations.

For the first five terms of the series (6.1.36), (6.1.40), (6.1.42) and (6.1.43), the numerical values of C_s and D_s are given in Table 3 [48].

TABLE 3. NUMERICAL VALUES OF C_s AND D_s.

s	C_s (TM)	C_s (TE)	D_s (TM)	D_s (TE)
1	0,91072	1,53187	—1,79678	1,83243
2	0,69427	0,78520	1,56880	+0,73473
3	0,59820	0,64199	—1,45621	0,54538
4	0,53974	0,56719	1,38324	—0,45333
5	0,49897	0,51840	—1,32996	0,39627

For the cross section of back scattering in the case of normal incidence of the plane wave, when $\theta = \pi/2$, we have the asymptotic expressions

$$\sigma_B = \pi a \left| 1 - 2(\varepsilon)^{-1/6} \times \right.$$
$$\left. \times \sum \frac{C_s \exp\left\{ i\left(2\varepsilon - \nu_s\pi + \frac{\pi}{12}\right)\right\}}{1 - e^{2\pi i \nu_s}} \right|^2 \tag{6.1.46}$$

for a TM-type field and

$$\sigma_B = \pi a \left| 1 + 2\varepsilon^{-1/6} \times \right.$$

$$\left. \times \sum_s \frac{C_s \exp\left\{ i \left(2\varepsilon - \nu_s \pi + \frac{\pi}{12} \right) \right\}}{1 - e^{2\pi i \nu_s}} \right|^2 \qquad (6.1.47)$$

for a TE-type field.

The asymptotic expressions for the series (6.1.11), (6.1.24), (6.1.12) and (6.1.25) are obtained from (6.1.36), (6.1.40), (6.1.42) and (6.1.43) by adding to them the respective coefficients. Series (6.1.11), (6.1.24) refer to the complete diffraction fields. In order to obtain the asymptotic representations for the field, scattered on the cylinder, from the asymptotic expressions of the functions u, v it follows to subtract the functions u^0 and v^0, respectively. /214

On the basis of another approach to obtaining the asymptotic expressions for the fields and currents we can cite the ideas of Fok [49]. Transformation of the investigated series here consists of the fact that the series is first also transformed to a contour integral according to the complex variable, from which then, after analyzing it along the undeformed integration path, the principal term is separated and an evaluation is given of the remainder (here the method of separating the principal term is not predetermined). In application to the problem of diffraction of the plane electromagnetic wave on a circular cylinder it is discussed in [75] (see also [52]). Here as the original series subject to the investigation, we have taken the series,

$$V = \sum_{n=0}^{\infty} \varepsilon_n (-i)^n \frac{\Omega J_n(\varepsilon)}{\Omega H_n^{(1)}(\varepsilon)} H_n^{(1)}(\lambda\rho) \cos n\varphi, \qquad (6.1.48)$$

$$J = \sum_{n=0}^{\infty} \varepsilon_n (-i)^n \frac{\cos n\varphi}{\Omega H_n^{(1)}(\varepsilon)}, \qquad (6.1.49)$$

where Ω has the previous meaning. Series (6.1.48) and (6.1.49) differ from series (6.1.10), (6.1.23) and (6.1.12), (6.1.25) only by their coefficients. In the case of a TM-type field, for J we obtain the asymptotic expression

$$J = -\frac{k\pi\varepsilon}{2M} \left[\sum_{n=0}^{\infty} e^{i\varepsilon \left(\frac{4n+3}{2} \pi - \varphi \right)} f(\xi_{2n+1}) + \right.$$

$$\left. + \sum_{n=0}^{\infty} e^{i\varepsilon \left(\frac{4n-1}{2} \pi + \varphi \right)} f(\xi_{2n+2}) \right], \qquad (6.1.50)$$

where $M = \left(\frac{\varepsilon}{2}\right)^{1/3}$, $\xi_{2n+1} = M\left(\frac{4n+3}{2}\pi - \varphi\right)$, $\xi_{2n+2} = M\left(\frac{4n-1}{2}\pi + \varphi\right)$, $0 \leqslant \varphi \leqslant \pi$, , but through $f(\xi)$ we denote the Fok function,

$$f(\xi) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} \frac{e^{i\xi t}}{w_1(t)}\, dt, \qquad (6.1.51)$$

which allows, when $\xi > 0$, expansion into a series of residues, /215

$$f(\xi) = 2\sqrt{\pi i} \sum_{s=1}^{\infty} \frac{e^{i\xi t_s}}{w_1'(t_s)} \qquad (6.1.52)$$

(t is the s-th kernel of the equation $w_1(t) = 0$; $w_1(t)$ is the Fok-Airy function; the boundary Γ is shown on Figure 22). The function $f(\xi)$, which enters into representation (6.1.50), decreases monotonically in absolute value with growth in ξ. With large values of ξ it is convenient to use the function,

$$F(\xi) = e^{i\xi^3/3} f(\xi), \qquad (6.1.53)$$

which, when $\xi \to \pm\infty$, has the asymptotic representations

$$\begin{aligned} F(\xi) &= 2i\sqrt{\xi}, \quad \xi \to -\infty, \\ F(\xi) &= 0, \quad \xi \to +\infty. \end{aligned} \qquad (6.1.54)$$

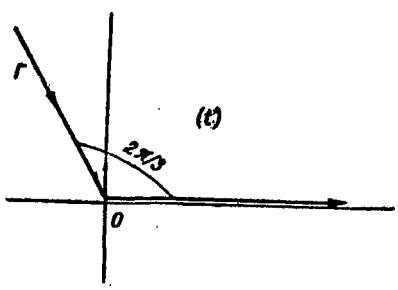

Fig. 22. Integration Boundary in Formula (6.1.51).

Therefore, in (6.1.50) the major role is played by the terms with negative and small positive arguments. In the region $0 \leq \phi \leq \pi$, the basic contribution to solving (6.1.50) is made by the first two terms.

For the case of TE-polarization, we have

$$J = -\frac{i}{2}\pi\varepsilon\left[\sum_{n=0}^{\infty} e^{i\varepsilon\left(\frac{4n+3}{2}\pi - \varphi\right)} g(\xi_{2n+1}) + \sum_{n=0}^{\infty} e^{i\varepsilon\left(\frac{4n-1}{2}\pi + \varphi\right)}\right], \qquad (6.1.55)$$

where through $g(\xi)$ we denote the Fok function,

$$g(\xi) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} \frac{e^{i\xi t}}{w_1'(t)}\, dt, \qquad (6.1.56)$$

which allows, when $\xi > 0$, expansion into a series of residues, /216

$$g(\xi) = 2\sqrt{\pi}\, i \sum_{s=1}^{\infty} \frac{e^{i\xi t_s}}{t_s' w_1(t_s')} \tag{6.1.57}$$

(t_s' represents the kernels of the equation $w_1'(t) = 0$). With large values of ξ it is convenient to use the function $G(\xi)$ introduced by the relationships,

$$\begin{aligned} G(\xi) &= e^{i\xi^3/3} g(\xi), \\ G(\xi) &= 2, \quad \xi \to -\infty, \\ G(\xi) &= 0, \quad \xi \to +\infty. \end{aligned} \tag{6.1.58}$$

The functions $g(\xi)$ decrease monotonically in absolute value with increase in ξ; when $\xi \to \pm\infty$, representations (6.1.58) are valid. Therefore, in series (6.1.55) the major role is played by terms with negative and small positive arguments. In the region $0 \leq \phi \leq \pi$, the basic contribution to solving (6.1.55) is made only by the first two terms.

The functions $f(\xi)$, $g(\xi)$ were studied in [50, 169, 170].

Near $\phi = \pi/2$ and for $\phi \lesssim \pi/2$, approximately,

$$J_{TM} = \frac{\pi e}{2iM}\left[e^{-ie\left(\varphi-\frac{3\pi}{2}\right)} f(\xi_1) + e^{-ie\cos\varphi} F(\xi_2')\right], \tag{6.1.59}$$

$$J_{TE} = \frac{i}{2}\pi e\left[e^{ie\left(\frac{3\pi}{2}-\varphi\right)} g(\xi_1) + e^{-ie\cos\varphi} G(\xi_2)\right], \tag{6.1.60}$$

where $\xi_2' = -M\cos\phi < 0$. The asymptotic representation of the function V, in the case of a TM-type field, has the form,

$$V = M^{3/2}\frac{e^{i\lambda\rho}}{\sqrt{\lambda\rho}}\, e(\varphi, \varepsilon), \tag{6.1.61}$$

where through $e(\phi, \varepsilon)$ we denote the so-called complex directional diagram. For all values of ϕ, other than a small region where $M(\pi - \phi) < 0.2$,

$$\begin{aligned} e(\varphi, \varepsilon) = \sqrt{\frac{2}{M}}\Bigg[& e^{ie(\pi-\varphi)}\hat{f}(M(\pi-\varphi)) + \\ & + e^{-i2e\sin\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)}\hat{F}\left(-2M\sin\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)\right)\Bigg]. \end{aligned} \tag{6.1.62}$$

In the region where $M(\pi - \phi) > 0.2$, /21

$$e(\varphi, \varepsilon) = -\frac{2e^{\frac{i3\pi}{4}}}{\sqrt{\pi\varepsilon}} \cdot \frac{\sin\varepsilon(\pi-\varphi)}{\pi-\varphi} + \\ + \sqrt{\frac{2}{M}} \left[e^{i\varepsilon(\pi-\varphi)} \tilde{f}\left(M(\pi-\varphi)\right) + \right. \\ \left. + e^{-i\varepsilon(\pi-\varphi)} \tilde{f}\left(-M(\pi-\varphi)\right)\right]. \tag{6.1.63}$$

Here

$$\hat{f}(\xi) = \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}} \int_\Gamma e^{i\xi t} \frac{v(t)}{w_1(t)}\, dt = \\ = \frac{e^{-\frac{i3\pi}{4}}}{2\sqrt{\pi}\xi} + \tilde{f}(\xi), \tag{6.1.64}$$

where

$$\tilde{f}(\xi) = \frac{e^{-\frac{i\pi}{12}}}{\sqrt{\pi}} \int_0^\infty e^{-\frac{\xi t}{2}(\sqrt{3}+i)} \frac{v(t)}{w_2(t)}\, dt + \\ + \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}} \int_0^\infty e^{i\xi t} \frac{v(t)}{w_1(t)}\, dt.$$

With large negative arguments, the function $\hat{f}(\xi)$ is represented in the form,

$$\hat{f}(\xi) = \frac{\sqrt{-\xi}}{2} e^{-i\xi^3/12}. \tag{6.1.65}$$

In the case of a TE-type field we have

$$V = M^{3/2} \frac{e^{i\lambda\rho}}{\sqrt{\lambda\rho}} h(\varphi, \varepsilon), \tag{6.1.66}$$

where the complex directional diagram $h(\phi, \varepsilon)$ is determined in the region in which $M(\pi - \phi) > 0.2$, by the expression

$$h(\varphi, \varepsilon) = \sqrt{\frac{2}{M}} \left[e^{i\varepsilon(\pi-\varphi)} \hat{g}\left(M(\pi-\varphi)\right) + \right.$$
$$\left. + e^{-i2\varepsilon \sin\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)} \hat{G}\left(-2M \sin\left(\frac{\pi}{2}-\frac{\varphi}{2}\right)\right)\right] \qquad (6.1.67)$$

and in the region where $M(\pi - \phi) > 0.2$, [sic] in the form /218

$$h(\varphi, \varepsilon) = -\frac{2e^{i3\pi/4}}{\sqrt{\pi\varepsilon}} \cdot \frac{\sin\varepsilon(\pi-\varphi)}{\pi-\varphi} +$$
$$+ \sqrt{\frac{2}{M}} \left[e^{i\varepsilon(\pi-\varphi)} \tilde{g}\left(M(\pi-\varphi)\right) + \right. \qquad (6.1.68)$$
$$\left. + e^{-i\varepsilon(\pi-\varphi)} \tilde{g}\left(M(\varphi-\pi)\right)\right].$$

Here

$$\hat{g}(\xi) = \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_\Gamma e^{i\xi t} \frac{v'(t)}{w_1(t)}\, dt = \frac{e^{-i3\pi/4}}{2\sqrt{\pi}\,\xi} + \tilde{g}(\xi), \qquad (6.1.69)$$

where

$$\tilde{g}(\xi) = \frac{e^{-\frac{i\pi}{12}}}{\sqrt{\pi}} \int_0^\infty e^{\frac{-\xi t}{2}\sqrt{3}+t)} \frac{v'(t)}{w_2(t)}\, dt +$$
$$+ \frac{e^{i\pi/4}}{\sqrt{\pi}} \int_0^\infty e^{i\xi t} \cdot \frac{v'(t)}{w_1(t)}\, dt. \qquad (6.1.70)$$

In (6.1.62) and (6.1.67) we have set

$$\hat{F}(\xi) = e^{i\xi^3/12} \hat{f}(\xi), \quad \hat{G}(\xi) = e^{i\xi^3/12} \hat{g}(\xi). \qquad (6.1.71)$$

The functions $\tilde{f}(\xi)$ and $\tilde{g}(\xi)$ were investigated in [168]. The graphs of the functions $\tilde{f}(\xi)$, $\tilde{g}(\xi)$, $\hat{f}(\xi)$ and $\hat{g}(\xi)$ in the intervals required for numerical computations are shown on Figures 23, 24 taken from [75].

With large negative arguments, the function $\hat{g}(\xi)$ is represented in the form,

$$\hat{g}(\xi) = -\frac{\sqrt{-\xi}}{2} e^{-i\xi^3/12}. \qquad (6.1.72)$$

As noted in [75] comparison of the results obtained from the formulas for the strict theory and from the approximate ones shows that even when $\varepsilon \geq 5$ we obtain a satisfactory agreement which rapidly becomes better with increase in ε.

Thus, we have

$$2e(\varphi, \varepsilon)\sqrt{\pi\varepsilon}\, e^{\frac{i\pi}{4}} = \sum_{n=0}^{\infty} \varepsilon_n \frac{J_n(\varepsilon)}{H_n^{(1)}(\varepsilon)} \cos n(\pi - \varphi), \qquad (6.1.73)$$

$$2h(\varphi, \varepsilon)\sqrt{\pi\varepsilon}\, e^{\frac{i\pi}{4}} = \sum_{n=0}^{\infty} \varepsilon_n (-1)^n \frac{J_n'(\varepsilon)}{H_n^{(1)'}(\varepsilon)} \cos n\varphi \qquad (6.1.74)$$

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(the functions $e(\phi, \varepsilon)$, $h(\phi, \varepsilon)$ are symmetrical relative to ϕ by virtue of the symmetry of the problem). Relationships (6.1.73), (6.1.74) easily permit writing the respective asymptotic expressions for the values of the cross sections.

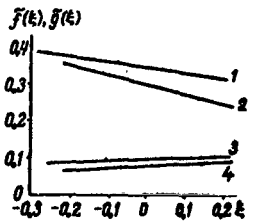

Fig. 23. The Functions $\tilde{f}(\xi)$ [1. Re $\tilde{f}(\xi)$; 3. Im $\tilde{f}(\xi)$] and $\tilde{g}(\xi)$ [(2. Re $\tilde{g}(\xi)$; 4. Im $\tilde{g}(\xi)$].

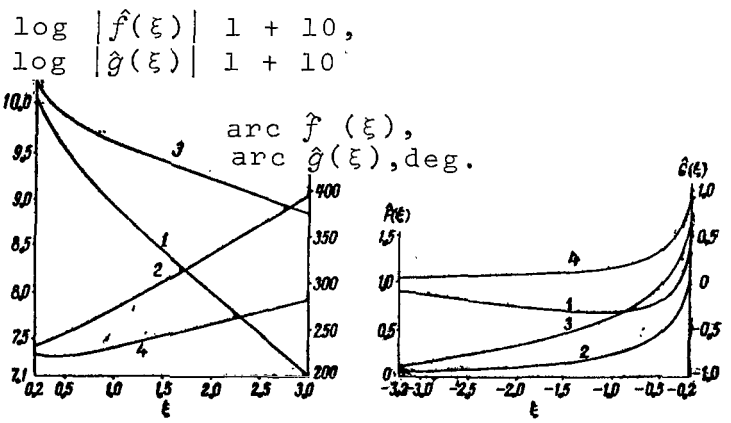

Fig. 24. The Functions $\hat{f}(\xi)$ [1. log $|\hat{f}(\xi)| + 10$; 2. arc $\hat{f}(\xi)$], $\hat{g}(\xi)$ [3. log $|\hat{g}(\xi)| + 10$; 4. arc $\hat{g}(\xi)$] and $\hat{F}(\xi)$. [1. Re $\hat{F}(\xi)$; 2. Im $\hat{F}(\xi)$], $\hat{G}(\xi)$. [3. Re $\hat{G}(\xi)$; 4. Im $\hat{G}(\xi)$].

A Cylinder with Finite Conductivity. In the case of a cylinder with finite conductivity ($\sigma_2 \neq \infty$) the diffraction problem consists of determining the secondary field E^1, H^1, scattered on the cylinder and the secondary field E^2, H^2 excited by the incident wave inside the cylinder.

For example, if the field of the obliquely incident wave is a TM-type field, then for it $v^0 \equiv 0$, the potential u^0 is represented in the form of (6.1.5), and the potentials $u^{1,2}$, $v^{1,2}$ of the secondary fields are sought in the form of series (6.1.7) and (6.1.8), /220. whose coefficients must be found from the boundary conditions (1.7.2) [with normal incidence of the plane wave we will have $v^{1,2} \equiv 0$, but the expansion coefficients of the potentials $u^{1,2}$ are found from the boundary conditions (1.7.1)]. Their computation leads to the expressions

$$a_n^{(2)} = \left(\frac{Eai^n}{kv} J_n(v) + a_n^{(1)} H_n^{(1)}(v) \right) \frac{v^2}{u^2 J_n(u)}, \qquad (6.1.75)$$

$$b_n^{(2)} = \frac{v^2}{u^2} \cdot \frac{H_n^{(1)}(v)}{J_n(u)} b_n^{(1)}, \qquad (6.1.76)$$

where

$$a_n^{(1)} = -\frac{Eai^n k}{v} \left\{ \frac{J_n(v)}{H_n^{(1)}(v)} + \right.$$
$$\left. + \frac{2i \left[\frac{H_n^{(1)'}(v)}{vH_n^{(1)}(v)} - \frac{KJ_n'(u)}{uJ_n(u)} \right]}{\pi v^2 H_n^{(1)2}(v) D_n} \right\},$$
$$b_n^{(1)} = \frac{Ean \cos\theta \, i^n}{k_0 \mu v} \left[\frac{1}{v^2} - \frac{1}{u^2} \right] \frac{2}{\pi v^2 H_n^{(1)2}(v) D_n}, \qquad (6.1.77)$$
$$D_n = n^2 \cos^2\theta \left[\frac{1}{v^2} - \frac{1}{u^2} \right] - k^2 \left[\frac{H_n^{(1)'}(v)}{vH_n^{(1)}(v)} - \frac{KJ_n'(u)}{uJ_n(u)} \right] \times$$
$$\times \left[\frac{H_n^{(1)'}(v)}{vH_n^{(1)}(v)} - \frac{N^2 J_n'(u)}{KuJ_n(u)} \right],$$
$$N^2 = (k_2/k)^2, \quad K = \mu_2/\mu, \quad v = \lambda a, \quad u = \lambda_2 a,$$
$$\lambda = k \sin\theta, \quad \lambda_2^2 = k_2^2 - k^2 \cos^2\theta.$$

By determining the coefficients $a_n^{1,2}$, $b_n^{1,2}$ the problem is fully solved. The components of the electromagnetic field outside and inside the cylinder are found from equations (1.6.15), written in a cylindrical coordinate system.

The expressions in (6.1.77) for the coefficients $a_n^{(1)}$, $b_n^{(1)}$, which have a rather awkward form, may in several instances be simplified to a significant degree. For example, with normal incidence of the plane wave, when $\theta = \pi/2$, we have $b_n^{(1)} = b_n^{(1)} = 0$, but

$$a_n^{(2)} = \left(\frac{Ei^n}{k^2} J_n(ka) + a_n^{(1)} H_n^{(1)}(ka) \right) \frac{k^2}{k_2^2 J_n(k_2 a)}, \qquad (6.1.78)$$

where now

$$a_n^{(1)} = -\frac{Ei^n}{k^2} \times$$

$$\times \frac{\left(\frac{J_n'(ka)}{J_n(ka)} - \frac{J_n'(k_2a)N}{J_n(k_2a)K}\right)}{\left(\frac{H_n^{(1)'}(ka)}{H_n^{(1)}(ka)} - \frac{J_n'(k_2a)N}{J_n(k_2a)K}\right)} \cdot \frac{J_n(ka)}{H_n^{(1)}(ka)} \qquad (6.1.79)$$

($N = k_2/k$, $K = \mu_2/\mu$). In the wave band ($\theta = \pi/2$),

$$E_z^{(1)} = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = \frac{e^{ikp}}{\sqrt{\rho}} A(\varphi), \qquad (6.1.80)$$

where the amplitude function

$$A(\varphi) = \sqrt{\frac{2}{\pi k}}\, e^{-\frac{i\pi}{4}} \sum_{n=-\infty}^{\infty} (-i)^n a_n^{(1)} e^{in\varphi}. \qquad (6.1.81)$$

For the scattering cross sections we obtain the expressions:

$$\sigma_B = \frac{4}{E^2 k} \left| \sum_{n=-\infty}^{\infty} i^n a_n^{(1)} \right|^2 \qquad (6.1.82)$$

$$\sigma_s = \frac{4}{E^2 k} \sum_{n=-\infty}^{\infty} |a_n^{(1)}|^2 \qquad (6.1.83)$$

A number of other cases (along with $\theta = \pi/2$), leading to simplification of the expressions for the coefficients of the series, were studied in [52], where a similar problem was solved for a dielectric cylinder.

In the case, when the wave which is obliquely incident on the cylinder is a TE-type wave, the problem is solved in a similar manner. The difference will be only that now $u = u^{(1)}$, but $v = v^0 + v^{(1)}$; $u^0 \equiv 0$, $v^0 \neq 0$.

In the same way the problem of the diffraction of a plane electromagnetic wave of TM type or TE type can be solved on a composite cylinder (a cylinder with a different type of covering, Fig. 25), if we seek the potentials of the secondary fields in Region II in the form of series of linear combinations of wave functions (2.1.6) and (2.1.7), and in Region III in the form of series of wave functions (2.1.6) with the proper choice of the parameter k. With oblique incidence of the plane wave, the coefficients of the series

must be found from boundary conditions such as (1.7.2), given on the cylindrical surfaces $\rho = a$ and $\rho = b$. In the case of normal incidence of the plane wave the secondary fields should be sought in the form of fields whose polarization coincides with the polarization of the field of the incident wave. Here the coefficients of the series may be found from boundary conditions such as (1.7.1), given when $\rho = a$ and $\rho = b$. Solution to this problem is given in [153] (for a TM-type incident wave). The problem on the diffraction of an obliquely incident wave of TM type on a composite cylinder was studied in [128], where expressions are found for the coefficients of the series. /222

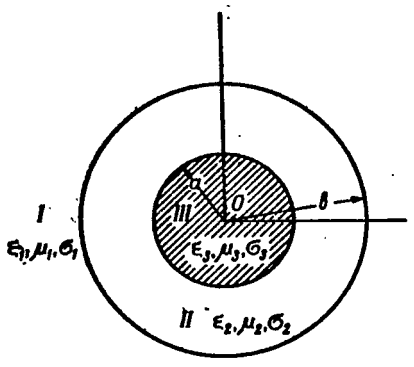

Fig. 25. Composite Cylinder (Plane z = const).

§2. Field of a Longitudinal Dipole Emitter in the Presence of an Ideally Conducting Circular Cylinder.

In an unbounded uniform and isotropic space with electrical and magnetic permeabilities ε, μ and a conductivity $\sigma = 0$ ($k = \omega/c\sqrt{\mu\varepsilon}$) let there be contained an infinitely long ideally conducting circular cylinder, whose surface in cylindrical coordinates is given by the equation $\rho = a$ (the axis Oz of the coordinate system is directed along the axis of the cylinder), and at the point ρ^0, ϕ^0, z^0 inside the cylinder let there be found a unit oscillational dipole, whose dipole moment is oriented parallel to the generatrix of the cylinder in the positive direction of the axis Oz (Fig. 26,a). The problem is posed of determining the diffracted field, caused by the presence of the cylinder in space [52]. Here we can differentiate two cases: the case of an electric dipole and the case of a magnetic dipole with moments $\mathbf{p}$ and $\mathbf{m}$, respectively. It is obvious that in each of the cases the components of the vectors $\mathbf{E}^0$, $\mathbf{H}^0$ of the dipole field in free space may be expressed as a function of the case through the components of the vectors $\mathbf{p}$, $\mathbf{m}$ from formulas (1.2.13), (1.2.17) using formulas (1.3.9), (1.3.10), which connect the Hertz vectors $\vec{\Pi}^0$, $\vec{\Pi}^{0*}$ with $\mathbf{p}$, $\mathbf{m}$, respectively. /223

From comparison of (1.2.13), (1.2.17) with (1.6.16), (1.6.18) it is easy to see that for the field of the electric dipole in free space, on the basis of (2.5.9), $v^0 \equiv 0$, but

$$u^0 = \Pi_z^0 = p\,\frac{e^{ikR}}{R} =$$

$$= \frac{ip}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi^0)} \int_{-\infty}^{\infty} \begin{cases} J_n(v\rho) H_n^{(1)}(v\rho^0), \\ J_n(v\rho^0) H_n^{(1)}(v\rho), \end{cases} \quad \begin{matrix} \rho < \rho^0 \\ \rho > \rho^0 \end{matrix} \Bigg\} e^{ih(z-z^0)}\, dh, \tag{6.2.1}$$

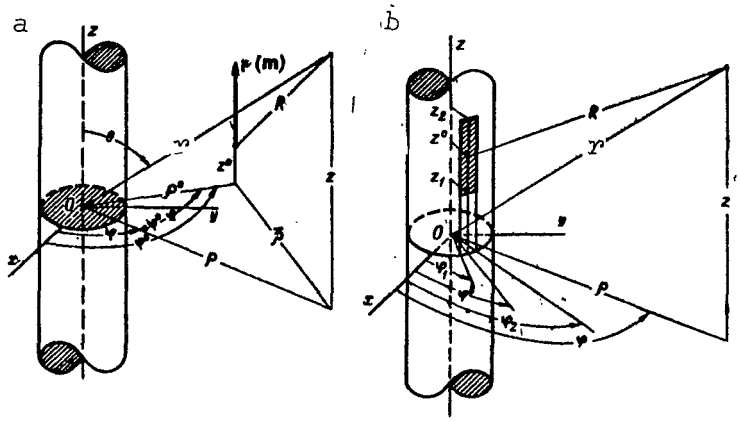

Fig. 26. Excitation of a Circular Cylinder by a Longitudinal Dipole (a) and Generalization of the Problem of the Case of a Longitudinal Slot (b).

and in the case of a magnetic dipole $u^0 = 0$, but

$$v^0 = \Pi_z^{0*} = m \frac{e^{ikR}}{R} =$$

$$= \frac{im}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi^0)} \int_{-\infty}^{\infty} \begin{cases} J_n(v\rho) H_n^{(1)}(v\rho^0), \\ J_n(v\rho^0) H_n^{(1)}(v\rho), \end{cases} \quad \begin{matrix} \rho < \rho^0 \\ \rho > \rho^0 \end{matrix} \Bigg\} e^{ih(z-z^0)}\, dh, \tag{6.2.2}$$

since $\mathrm{p} = \{0, 0, p\}$, $\mathrm{m} = \{0, 0, m\}$. From (1.2.13), (1.2.17) and (1.3.9), (1.3.10) it also follows that the primary field of the dipole is a field of only one of the polarizations, either a TM type field in the case of an electric dipole or a TE type in the case of a magnetic dipole. Here we find that the boundary conditions on the surface of an ideally conducting cylinder will be necessarily satisfied if the secondary field, scattered on the cylinder, is also sought in the form of a field of only one of the TM and TE types as a function of the type of the dipole field in free space. As a result the problem of determining the vectors $\mathrm{E} = \mathrm{E}^0 + \mathrm{E}^1$, $\mathrm{H} + = \mathrm{H}^0 + \mathrm{H}^1$ of the diffracted field is reduced to seeking the auxiliary scalar potential functions u^1 or v^1, i.e., the solutions to the Helmholtz equation which satisfy, when added /22

to u^0 or respectively v^0, the boundary condition (1.7.3) in the case of an electric dipole or condition (1.7.4) in the case of a magnetic dipole on the surface of the cylinder ($\rho = a$) and the condition of radiation at infinity.

Electric Dipole. If we seek the potential u^1 of the stray field in the form of a series of Fourier integrals such as (2.1.9),

$$u^1 = \frac{p}{2i} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi^0)} \int_{-\infty}^{\infty} x_n(h) H_n^{(1)}(v\rho) e^{ih(z-z^0)} dh, \qquad (6.2.3)$$

then from (1.7.3) and (6.2.1) we find that

$$x_n(h) = \frac{J_n(va)}{H_n^{(1)}(va)} H_n^{(1)}(v\rho^0), \quad v = (k^2 - h^2)^{1/2},$$

and then for the complete field we have

$$u = \frac{ip}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi^0)} \int_{-\infty}^{\infty} \left\{ \begin{array}{l} \left[J_n(v\rho^0) - \right. \\ \left[J_n(v\rho) - \right. \end{array} \right. \qquad (6.2.4)$$

$$\left. \begin{array}{l} - \frac{J_n(va)}{H_n^{(1)}(va)} H_n^{(1)}(v\rho^0) \Big] H_n^{(1)}(v\rho), \quad \rho > \rho^0 \\ - \frac{J_n(va)}{H_n^{(1)}(va)} H_n^{(1)}(v\rho) \Big] H_n^{(1)}(v\rho^0), \quad \rho < \rho^0 \end{array} \right\} e^{ih(z-z^0)} dh.$$

It is quite difficult to carry out the integration in (6.2.4) in the general case. However, in the particular case of the wave band when as usual $k\rho >> 1$ ($\rho \to \infty$) for the Fourier integrals entering into (6.2.4),

$$I_n = \frac{i}{2} \int_{-\infty}^{\infty} f_n(h) H_n^{(1)}(v\rho) e^{ih(z-z^0)} dh, \qquad (6.2.5)$$

where $f_n(h) = J_n(v\rho^0) - \frac{J_n(va)}{H_n^{(1)}(va)} H_n^{(1)}(v\rho^0)$, on the basis of the method of steepest descents, we obtain the asymptotic expressions, /225

$$I_n = (-i)^n f_n(k\cos\theta) \frac{e^{ikR'}}{R}, \qquad (6.2.6)$$

if first in (6.2.5) we take the first term of its asymptotic expansion (2.2.24) instead of the Hankel function $H_n^{(1)}(v\rho)$. Here θ is

the spherical coordinate. As a result, in the wave band, approximately,

$$u = p\frac{e^{ikr}}{r}\exp(-ikz^0\cos\theta)\,M(\theta,\ \varphi) \tag{6.2.7}$$

(r, θ are spherical coordinates with the origin at the point $z = 0$), where through M we denote the expression,

$$M(\theta,\ \varphi) = \sum_{n=-\infty}^{\infty}(-i)^n\left\{J_n(k\rho^0\sin\theta) - \frac{J_n(ka\sin\theta)}{H_n^{(1)}(ka\sin\theta)}H_n^{(1)}(k\rho^0\sin\theta)\right\}e^{in(\varphi-\varphi^0)}. \tag{6.2.8}$$

The function $M(\theta, \phi)$, which corresponds to the approximation in which we can ignore the terms which are proportional to the powers r^{-2}, r^{-3}, etc., is usually assumed to be a correction factor to the characteristics of the directionality of the dipole in free space.

From (1.6.16) we find that in this approximation in the wave band

$$\sqrt{\frac{\mu}{\varepsilon}}H_\varphi = E_\theta = -k^2\sin\theta\,u. \tag{6.2.9}$$

Since for a dipole with a linear current $I(z^0)$ in it,

$$p = \frac{cI(z^0)\,dz^0}{-4\pi i\omega\varepsilon},$$

then by integrating (6.2.9) over z^0 from z_1 to z_2 we find expressions which generalize the solution of the problem to the case of a linear antenna with a given electrical current $I(z^0)$:

$$E_\theta = \frac{k\sin\theta}{4i\pi}\sqrt{\frac{\mu}{\varepsilon}}\frac{e^{ikr}}{r}\left\{\int_{z_1}^{z_2}I(z^0)\,e^{-ikz^0\cos\theta}dz^0\right\}M(\theta,\varphi). \tag{6.2.10}$$

By expanding the integration over z^0 from $-\infty$ to $+\infty$ in (6.2.4) we find on the basis of (2.5.8) the expression for the potential of the complete field of the linear infinitely long source in the presence of the cylinder when $\rho > \rho^0$, /22

$$u = -\frac{I\sqrt{\frac{\mu}{\varepsilon}}}{4k} \sum_{n=-\infty}^{\infty} \left\{ J_n(k\rho^0) - \right.$$

$$\left. - \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(k\rho^0) \right\} H_n^{(1)}(k\rho) e^{in(\varphi-\varphi^0)} \qquad (6.2.11)$$

hen $\rho < \rho^0$ we must exchange places for ρ and ρ^0). Here the only n-zero component of the electrical field is

$$E_z = k^2 u \qquad (6.2.12)$$

ormulas (6.2.11), (6.2.12) give a strict solution to the problem the diffraction of a cylindrical wave $\frac{-I}{4k}\sqrt{\frac{\mu}{\varepsilon}} H_0^{(1)}(k\bar{\rho})$ on an eally conducting cylinder of infinite length).

Magnetic Dipole. If we seek the potential v^1 of the field attered on the cylinder in the form of series (6.2.3), then from .7.4) and (6.2.2) for the expansion coefficients we find the ex- ession

$$x_n(h) = \frac{J_n(va)}{H_n^{(1)}(va)} H_n^{(1)}(v\rho^0),$$

n (6.2.2) p is substituted by m) and then for the complete field,

$$v = \frac{im}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi-\varphi^0)} \int_{-\infty}^{\infty} \left\{ \left[J_n(v\rho) - \right. \right.$$

$$\left. \left[J_n(v\rho^0) - \frac{J_n'(va)}{H_n^{(1)'}(va)} H_n^{(1)}(v\rho) \right] H_n^{(1)}(v\rho^0), \quad \rho^0 > \rho \right.$$

$$\left. - \frac{J_n'(va)}{H_n^{(1)'}(va)} H_n^{(1)}(v\rho^0) \right] H_n^{(1)}(v\rho), \quad \rho > \rho^0 \left. \right\} e^{ih(z-z^0)}\, dh. \qquad (6.2.13)$$

he prime denotes differentiation of the functions over the argu- nt). In the wave band, approximately,

$$v = m\frac{e^{ikr}}{r} \exp(-ikz^0\cos\theta) M^*(\theta, \varphi), \qquad (6.2.14)$$

ıere the correction factor is /227

$$M^*(\theta, \varphi) = \sum_{n=-\infty}^{\infty} (-i)^n \left[J_n(k\rho^0 \sin\theta) - \right.$$

$$\left. - \frac{J_n'(ka\sin\theta)}{H_n^{(1)'}(ka\sin\theta)} H_n^{(1)}(k\rho^0\sin\theta) \right] e^{in(\varphi-\varphi^0)}. \qquad (6.2.15)$$

From (1.6.18) we find that in this approximation in the wave band

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\theta = k^2 \sin\theta \sqrt{\frac{\mu}{\varepsilon}}\, v. \qquad (6.2.16)$$

Since for a dipole with linear magnetic current $I^*(z^0)$

$$m = \frac{cI^*(z^0)\,dz^0}{-4\pi i\,\omega\mu},$$

then by integrating (6.2.14) over z^0 from z_1 to z_2 we find from (6.2.16), for the component of the field of the longitudinal linear magnetic current in the presence of a cylinder in the wave band, the expression

$$E_\varphi = \frac{ik\sin\theta}{4\pi}\,\frac{e^{ikr}}{r}\left\{\int_{z_1}^{z_2} I^*(z^0)\,e^{-ikz\cos\theta}\,dz^0\right\} M^*(\theta,\varphi). \qquad (6.2.17)$$

At the limit when $\rho^0 \bullet a$ this problem in the electrodynamic sense will be equivalent to the problem on the field of the unit longitudinal emitting slot, located on the surface of an ideally conducting cylinder. On the basis of (2.2.52) at the limit,

$$M^*(\theta,\varphi) = \frac{2i}{\pi ka\sin\theta}\sum_{n=0}^{\infty}(-i)^n\varepsilon_n\,\frac{\cos n(\varphi-\varphi^0)}{H_n^{(1)'}(ka\sin\theta)}, \qquad (6.2.18)$$

and then formula (6.2.17) will give a generalization of the solution of the problem to the case of an infinitely narrow longitudinal slot see Fig. 26, b), expanded to a length from z_1 to z_2. If in (6.2.17) we substitute the current distribution $I^*(z^0)$ in the slot by determining the intensity $E_\phi\ (a,\ \phi^0,\ z^0)ad\phi^0$, then after integration over ϕ^0 from ϕ_1 to ϕ_2 in the right-hand part of (6.2.17) we find a further generalization of the solution of the problem to the case of a rectangular emitting slot of finite width $a(\phi_2 - \phi_1)$ with a given current distribution $E_\phi\ (a,\ \phi^0,\ z^0)$ in the opening of the slot.

The distribution of the field in the slot, which is near the actual, has the form, /22

$$E_\varphi(a,\ \varphi^0,\ z^0) = \frac{V(z^0)}{a\pi\left[(\Delta/2)^2 - \varphi^{02}\right]^{1/2}} \qquad (6.2.19)$$

($\Delta \square \phi_2 - \phi_1$). In this case (the wave band)

$$E_\varphi = \frac{e^{ikr}}{r}\,S(\theta)\,M^*(a,\ \varphi-\varphi^0), \qquad (6.2.20)$$

here

$$S(\theta) = \frac{ik \sin\theta}{4\pi} \int_{z_1}^{z_2} V(z^0) \exp(-ikz^0 \cos\theta)\, dz^0, \qquad (6.2.21)$$

$$M^*(\alpha, \varphi) = \frac{2i}{\pi ka \sin\theta} \times$$

$$\times \sum_{n=0}^{\infty} \frac{(-i)^n \varepsilon_n \cos n\varphi}{H_n^{(1)'}(ka \sin\theta)} G_n, \quad \alpha = ka \sin\theta, \qquad (6.2.22)$$

$$G_n = \frac{1}{\pi \cos n\varphi} \int_{-\Delta/2}^{\Delta/2} \frac{\cos n(\varphi - \varphi^0)}{\sqrt{\left(\frac{\Delta}{2}\right)^2 - \varphi^{02}}}\, d\varphi^0.$$

fter substituting $\phi^0 = -(\Delta/2)\cos\theta$, we can write

$$G_n = \frac{1}{\pi} \int_0^{\pi} e^{i(\Delta/2)n\cos\theta}\, d\theta = J_0\left(\frac{\Delta}{2} n\right), \qquad (6.2.23)$$

here $J_0(x)$ is the Bessel function of zero-th order, $\phi^0 = (\phi_1 + \phi_2)$ 2, and $V(z^0) = E_\phi(a, \phi^0, z^0)a(\phi_2 - \phi_1)$ is the voltage at the slot t is obvious that when $\left|\frac{\Delta}{2} n\right| \ll 1$, we will have $J_0 \approx 1$.

The function $S(\theta)$, which enters into (6.2.20), represents the haracteristics of the directionality of the slot located on an nfinite conducting plane. Therefore the function $M^*(\alpha, \phi)$ may be ssumed as a correction factor which takes into account the finiteess of the radius of curvature of the cylinder.

Reference [52], where this problem is solved, gives graphs of $M^*|$ as a function of the angle ϕ for different values of the paraeter α in the range from 1.5 to 21.

Let $z_1 = -l$, and $z_2 = l$. Then in the case of a sufficiently /229
arrow slot the voltage distribution in the opening of the slot is lmost sinusoidal and can be written in the form,

$$V(z) = V_0 \frac{\sin k(l - |z|)}{\sin kl}, \qquad (6.2.24)$$

here V_0 is the voltage in the middle of the slot. From (6.2.21) e obtain

$$S(\theta) = \frac{iV_0[\cos(kl\cos\theta) - \cos kl]}{2\pi \sin kl \sin\theta}, \qquad (6.2.25)$$

and when $2l = \lambda_0/2$ (length of the slot is equal to half the wave length),

$$S(\theta) = \frac{iV_0}{2\pi} \cdot \frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta}, \qquad (6.2.26)$$

and then instead of (6.2.20) we obtain an expression for the component of the electric field E_ϕ:

$$E_\varphi = -\frac{e^{ikr}}{r} \cdot \frac{V_0}{\pi^2 a} \cdot \frac{\cos\left[\frac{\pi}{2}\cos\theta\right]}{\sin\theta} \times \times \sum_{n=0}^{\infty} \frac{(-i)^n \varepsilon_n \cos n(\varphi - \varphi_0)}{H_n^{(1)'}(x)} J_0\left(\frac{\Delta}{2} n\right). \qquad (6.2.27)$$

The coefficient of the directed effect of the emitting slot, located on the surface of the cylinder, may be found from formula (1.4.18) in which it follows to set

$$\Psi(\theta, \varphi) \approx r^2 |E_\varphi|^2, \qquad (6.2.28)$$

where E_ϕ is determined by formula (6.2.27). The quantity $g(\theta, \phi)$ may be found by numerical integration in (1.4.18). By numerical integration we can also find the values of the resistance of the emission from the dipole field in the presence of a cylinder and the conductivity of the emission from the slot on the cylinder, if we use formulas (1.4.14) and (1.4.17), respectively, after substituting there

$$2\overline{P} = \frac{c}{4\pi} \sqrt{\frac{\varepsilon}{\mu}} \int_0^{2\pi} \int_0^{\pi} \Psi(\theta, \varphi) \sin\theta \, d\theta \, d\varphi$$

(under the condition that in (1.4.14) the integration is carried out over a sphere of infinitely large radius with the center at the origin). Then in (1.4.14) the component E_ϕ is determined by formula (6.1.16), and in (1.4.19) by formula (6.2.27).

§3. Diffraction of a Plane Wave on Two Cylinders. /230

In an unbounded uniform and isotropic space with permeabilities ε, μ and a conductivity $\sigma = 0$, let there be contained two parallel infinitely long ideally conducting cylinders, occupying in local cylindrical coordinate systems ρ_s, ϕ_s, z_s, $s = \pm 1$, the regions of space $\rho_{-1} \leq a_{-1}$, $\rho_{+1} \leq a_{+1}$ (the axes $Q_s z_s$ of the systems are directed along the respective axes of the cylinders) and let there be

incident on the cylinders the plane wave (1.3.4) propagating in the direction of the unit vector n, which lies on the plane of the normal cross section of the cylinders and forms a certain angle ϕ_0 with the lines of the centers $\overline{O_{-1}O_{+1}}$ of the cross sections of the cylinders. In cylindrical coordinates ρ, ϕ, z with the origin at point

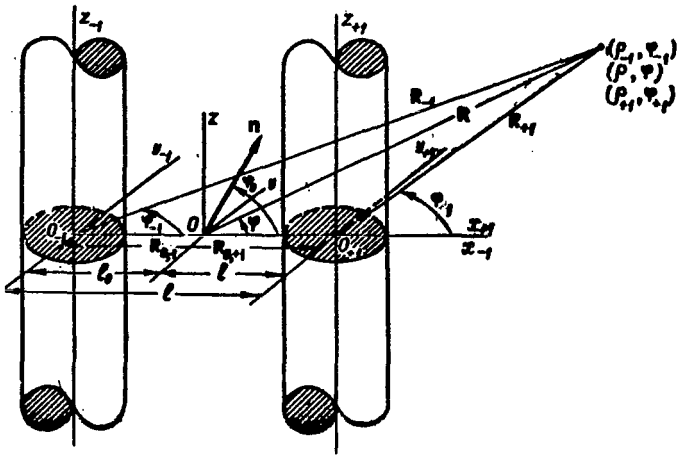

Fig. 27. Diffraction of a Plane Wave on Two Circular Cylinders.

O, lying on the line of the centers $O_{-1}O_{+1}$ midway between them (the axis Oz Os_s, the axis Ox is directed along the line of the centers, Fig. 27), the wave (1.3.4) may be written then in the form,

$$\mathbf{E}^0 = Ee^{ikp\cos(\varphi-\varphi_0)} \qquad (6.3.1)$$

($\theta = \pi/2$; the case of oblique incidence of the wave may be assumed to be analogous). We pose the problem of determining the secondary field E^1, H^1, scattered on the cylinders. The strict solution to this problems consists, just as in the case of one cylinder, of seeking the solution to a homogeneous system of Maxwell equations (1.1.4) ($\rho = 0$, $J^{(ext)} = 0$) under boundary conditions (1.1.6) on the surface of each of the cylinders for the components of the complete field E, H and the condition of radiation at infinity for the components of the stray field E^1, H^1. If the vector E^0 of the field of the incident wave is directed at a certain angle to the plane zOy, then the solution to the problem, after representing the primary field in the form of a superposition of particular fields of TM and TE types, can be obtained as a linear combination of the solutions to the two independent problems, corresponding to the cases of TM and TE-polarizations of the field of the incident wave. Here the field E^1, H^1, scattered on the cylinders, may be found as a function of the character of the polarization of the primary field through the potential functions u^1 or v^1 from (1.6.16) or (1.6.18), respectively. /231

- If the incident wave is a TM type field, then its potential u^0, written in cylindrical coordinates ρ, ϕ, z on the basis of (6.3.1), will be the function.

$$u^0 = \frac{E}{k^2} e^{ik\rho\cos(\varphi-\varphi_0)} \tag{6.3.2}$$

($\theta = \pi/2$, $(nR) = x\cos\phi_0 + y\sin\phi_0$). If this wave (6.3.1) is a TE type wave, the potential of its field will be the function,

$$v^0 = \frac{E}{k^2} e^{ik\rho\cos(\varphi-\varphi_0)}. \tag{6.3.3}$$

Since $R = R_s + R_{0s}$ (see Fig. 27) and $(nR) = (nR_s) + (nR_{0s})$, then it is obvious that the function (6.3.2) and the function (6.3.3) on the basis of (2.3.8) can be written in local cylindrical coordinates of the s-th cylinder in the form of the expansions:

$$u^0 = \frac{Ee^{ik(n\,R_{0s})}}{k^2} \sum_{n=-\infty}^{\infty} i^n J_n(k\rho_s) e^{in(\varphi_s-\varphi_0)} \tag{6.3.4}$$

$$v^0 = \frac{Ee^{ik(n\,R_{0s})}}{k^2} \sum_{n=-\infty}^{\infty} i^n J_n(k\rho_s) e^{in(\varphi_s-\varphi_0)}. \tag{6.3.5}$$

It is also obvious that on the strength of the constancy of the electromagnetic properties of the medium surrounding the cylinder, and the properties of the cylinders themselves along the axis Oz and the independence of the wave (6.3.1) from the spatial coordinate z, the complete field E, H will not depend on z. Therefore the potentials u^1 and v^1 of the secondary fields must be solutions to the two-dimensional Helmholtz equation which, when added to u^0 or respectively to v^0, will satisfy the boundary conditions (1.7.3) or (1.7.4) as a function of the type of incident wave and the conditions of radiation at infinity.

The problem is fully solved by seeking the functions u^1 and v^1. /232

If we use the method of perturbations (§9, Ch. 1) to seek u^1, we then obtain

$$u^1 = \sum_{s=\pm 1} \sum_{m=1}^{\infty} {}^s u^m, \tag{6.3.6}$$

where through ${}^s u^m$ we denote the potential of the second field of m-th order of scattering, produced by the presence of the s-th

ylinder in space. The potential of the resulting field is repre-
ented by the expression,

$$u = u^0 + u^1. \tag{6.3.7}$$

f we seek ${}^s u^m$ in the form of a series

$${}^s u^m = \sum_{n=-\infty}^{\infty} {}^s A_n^m H_n^{(1)}(k\rho_s)\, e^{in\varphi_s} \tag{6.3.8}$$

ccording to the cylindrical wave functions which satisfy the con-
ition of radiation at infinity, then

$$u^1 = \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} \left(\sum_{m=1}^{\infty} {}^s A_n^m \right) H_n^{(1)}(k\rho_s)\, e^{in\varphi_s}. \tag{6.3.9}$$

he coefficients ${}^s A_n^m$ of the series (6.3.8) (we denote them as coef-
icients of scattering of m-th order) are found from the condition,

$${}^s u^m + {}^{-s} u^{m-1} = 0, \quad \rho_s = a_s; \quad s = \pm 1, \tag{6.3.10}$$

hich is satisfied simultaneously on the surface of both cylinders
then $u = 0$ when $\rho_s = a_s$, $s = \pm 1$). After representing the poten-
ial u_0 of the field of the incident wave in the form of the series
6.3.4) and using the addition theorem (2.4.5) for the cylindrical
ave functions, from (6.3.10) we find

$$\begin{aligned} {}^s A_n^m = &- \frac{J_n(ka_s)}{H_n^{(1)}(ka_s)} \times \\ &\times \sum_{p=-\infty}^{\infty} {}^{-s} A_p^{m-1} H_{p-n}^{(1)}(kl)\, e^{i(p-n)\varphi_{-s,s}}, \quad s = \pm 1. \end{aligned} \tag{6.3.11}$$

y substituting (6.3.11) into (6.3.9) we find the formal solution
) the problem for the potential u^1.

From (6.3.11) it is obvious that the scattering coefficients /233
^m_n of arbitrary order are expressed through all the preceding,
id in the end may be expressed through the scattering coefficients
first order,

$${}^s A_n^1 = - \frac{E J_n(ka_s)\, i^n}{k^2 H_n^{(1)}(ka_s)}\, e^{ik(n\,\mathbf{R}_{0s}) - in\varphi^0} \tag{6.3.12}$$

Similarly we can solve the problem for the potential v^1 under the boundary conditions of Neumann on the surface of the cylinders.

In the presence of a field inside the s-th cylinder its potential is represented by the sum,

$$u_s^2 = \sum_{n=-\infty}^{\infty} \left(\sum_{m=1}^{\infty} {}^sB_n^m \right) J_n(k_2 \rho_s) e^{in\varphi_s}, \tag{6.3.13}$$

if we seek the potential ${}^su_2^m$ of the field of "m-th order of refraction" in the form of a series,

$$ {}^su_2^m = \sum_{n=-\infty}^{\infty} {}^sB_n^m J_n(k_2 \rho_s) e^{in\varphi_s}. \tag{6.3.14}$$

In this case the coefficients ${}^sA_n^m$ and ${}^sB_n^m$ are found from the conjugate conditions on the surface of each cylinder.

The potential v_s^2 of a refracted field of TE type inside the s-th cylinder may be found analogously.

However, it follows to note that solution to the problem of the interaction of two diffracting cylinders, obtained by the method of perturbations, is found to be only slightly feasible in practice for the numerical computations in the case when the distance between cylinders is not great and their transverse dimensions are comparable or considerably higher than the wave length of excitation. As already noted, this method is suitable only in the case of strongly different bodies with transverse dimensions considerably smaller than the wave length. Here we have shown that for future analysis of the solution to the problem found by the classical method of separation of variables we can make several conclusions based on comparison of the results.

In application to the problem of the interaction of two cylinders, the classical method of separation of variables permits effectively finding the numerical results for a sufficiently wide range of variation in the parameters. Using this method the problem is solved according to the following scheme.

TM Type Field. The potential u^1 of the secondary field scat- /23
tered on the cylinders is sought in the form of a sum,

$$u^1 = \sum_{s=\pm 1} u_s \tag{6.3.15}$$

of two terms, each of which represents a diffraction series,

$$u_s = \sum_{n=-\infty}^{\infty} x_n^s H_n^{(1)}(k\rho_s)\, e^{in\varphi_s}, \quad s = \pm 1, \tag{6.3.16}$$

with the coefficients x_n^s, subject to determination from the boundary conditions (1.7.3) which must be satisfied simultaneously on the surface of each of the cylinders. Using expansion (6.3.4) and formula (2.4.5) we arrive at the conclusion that conditions (1.7.3) will be satisfied when $\rho_s = a_s$, $s = \pm 1$, if x_n^s is the solution to the infinite system of linear equations,

$$x_n^s + \sum_{m=-\infty}^{\infty} \alpha_{nm}^{(-s,s)} x_m^{-s} = f_n^s \quad (n = 0, \pm 1, \ldots; \quad s = \pm 1), \tag{6.3.17}$$

where

$$\alpha_{nm}^{(-s,s)} = \frac{J_n(ka_s)}{H_n^{(1)}(ka_s)} H_{m-n}^{(1)}(kl)\, e^{i(m-n)\varphi_{-s,s}}, \quad \varphi_{-1,+1} = 0, \quad \varphi_{+1,-1} = \pi;$$
$$f_n^s = -\frac{i^n E J_n(ka_s)}{k^2 H_n^{(1)}(ka_s)}\, e^{ik(n\,\mathbf{R}_{0s}) - in\varphi_s}. \tag{6.3.18}$$

Proceeding from the behavior of the Bessel functions when $m \to \infty$, we can establish that the coefficients $\alpha_{nm}^{(-s,s)}$ of the system do not satisfy the condition of regularity of the infinite system because of the convergence of the series $\sum_{m=-\infty}^{\infty} |\alpha_{nm}^{(-s,s)}|$ for each n. System (6.3.17) will not be quasi-regular since it does not satisfy the conditions of quasi-regularity of infinite systems. Moreover the coefficients of the system (6.3.17) do not satisfy all other known characteristics of the solvability of infinite systems (for example, the conditions $\sum_{n,m} |\alpha_{nm}^{(-s,s)}| < \infty$ or $\sum_{n,m} |\alpha_{n,m}^{(-s,s)}|^2 < \infty$, are not satisfied; each of these conditions is a characteristic of the solvability of an infinite system much as (6.3.17) [20]). Therefore, without making additional studies of system (6.3.17) it is impossible to justifiably use any of the known methods of solving the infinite systems. /235

To carry out the investigation of system (6.3.17) let us make in it a substitution of the unknowns x_n^s with new unknowns X_n^s, after setting

$$x_n^s = J_n'(ka_s) X_n^s, \quad s = \pm 1. \tag{6.3.19}$$

On the basis of (6.3.19), from (6.3.17) for X_n^s we obtain an infinite system of linear equations,

$$X_n^s + \sum_{m=-\infty}^{\infty} c_{nm}^{(-s,s)} X_m^{-s} = F_n^s \qquad (n = 0, \pm 1, \ldots; \; s = \pm 1) \tag{6.3.20}$$

with complex coefficients

$$c_{nm}^{(-s,s)} = \frac{J_m(ka_{-s})}{H_n^{(1)}(ka_s)} H_{m-n}^{(1)}(kl) e^{i(m-n)\varphi_{-s,s}} \tag{6.3.21}$$

and with complex right-hand parts

$$F_n^s = -\frac{i^n E e^{iskl_0 \cos\varphi_0}}{k^2 H_n^{(1)}(ka_s)} e^{-in\varphi_0}, \quad l_0 = \frac{l}{2}. \tag{6.3.22}$$

Its determinant Δ of infinite order has the form ($\alpha_{nm}^{-} = c_{nm}^{(-1,\,+1)}$, $\alpha_{nm}^{+} = c_{nm}^{(+1,\,-1)}$)

$$\Delta = \begin{vmatrix}
1 & \alpha_{00}^{-} & 0 & \alpha_{0,-1}^{-} & 0 & \alpha_{0,+1}^{-} \\
\alpha_{00}^{+} & 1 & \alpha_{0,-1}^{+} & 0 & \alpha_{0,+1}^{+} & 0 \\
0 & \alpha_{-1,0}^{-} & 1 & \alpha_{-1,-1}^{-} & 0 & \alpha_{-1,+1}^{-} \\
\alpha_{-1,0}^{+} & 0 & \alpha_{-1,-1}^{+} & 1 & \alpha_{-1,+1}^{+} & 0 \\
0 & \alpha_{+1,0}^{-} & 0 & \alpha_{+1,-1}^{-} & 1 & \alpha_{+1,+1}^{-} \\
\alpha_{+1,0}^{+} & 0 & \alpha_{+1,-1}^{+} & 0 & \alpha_{+1,+1}^{+} & 1 \\
0 & \alpha_{-2,0}^{-} & 0 & \alpha_{-2,-1}^{-} & 0 & \alpha_{-2,+1}^{-} \\
\alpha_{-2,0}^{+} & 0 & \alpha_{-2,-1}^{+} & 0 & \alpha_{-2,+1}^{+} & 0 \\
0 & \alpha_{+2,0}^{-} & 0 & \alpha_{+2,-1}^{-} & 0 & \alpha_{+2,+1}^{-} \\
\alpha_{+2,0}^{+} & 0 & \alpha_{+2,-1}^{+} & 0 & \alpha_{+2,+1}^{+} & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots
\end{vmatrix}$$

$$\begin{vmatrix}
0 & \alpha_{0,-2}^{-} & 0 & \alpha_{0,+2}^{-} & \cdots \\
\alpha_{0,-2}^{+} & 0 & \alpha_{0,+2}^{+} & 0 & \cdots \\
0 & \alpha_{-1,-2}^{-} & 0 & \alpha_{-1,+2}^{-} & \cdots \\
\alpha_{-1,-2}^{+} & 0 & \alpha_{-1,+2}^{+} & 0 & \cdots \\
0 & \alpha_{+1,-2}^{-} & 0 & \alpha_{+1,+2}^{-} & \cdots \\
\alpha_{+1,-2}^{+} & 0 & \alpha_{+1,+2}^{+} & 0 & \cdots \\
1 & \alpha_{-2,-2}^{-} & 0 & \alpha_{-2,+2}^{-} & \cdots \\
\alpha_{-2,-2}^{+} & 1 & \alpha_{-2,+2}^{+} & 0 & \cdots \\
0 & \alpha_{+2,-2}^{-} & 1 & \alpha_{+2,+2}^{-} & \cdots \\
\alpha_{+2,-2}^{+} & 0 & \alpha_{+2,+2}^{+} & 1 & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{vmatrix} \tag{6.3.23}$$

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Using formulas (2.2.26), (2.2.28) which are asymptotic relative to the index n for the Bessel functions, the Stirling asymptotic formula,

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \tag{6.3.24}$$

and the inequalities [28]

$$|J_n(z)| \leqslant \frac{\left|\left(\frac{z}{2}\right)^n\right|}{\Gamma(n+1)}\, e^{|\mathrm{Im}z|} \tag{6.3.25}$$

we can show that the matrix elements $c_{nm}^{(-s,s)}$ of system (6.3.20) for all n and m satisfy the inequalities

$$|c_{nm}^{(-s,s)}| < \mathrm{const}_1 \frac{(|n|+|m|)!}{|n|!\,|m|!} \left(\frac{a_{-s}}{l}\right)^{|m|} \left(\frac{a_s}{l}\right)^{|n|}, \tag{6.3.26}$$

$$|F_n^s| < \mathrm{const}_2 q^{|n|}, \quad 0 < q < 1 \tag{6.3.27}$$

(see, for example, [85]). Now if we construct an infinite system of linear equations in which, as the matrix elements (coefficients), we take the expressions from the right-hand part of (6.3.26), and as the right-hand parts the expressions from the right-hand part of (6.3.27), then such a system obviously will be a majorant with respect to the system (6.3.20). Its matrix coefficients will satisfy (1.9.19), since

$$\sum_{n,m=-\infty}^{\infty} \left[\frac{(|n|+|m|)!}{|n|!\,|m|!} \left(\frac{a_{-s}}{l}\right)^{|m|} \left(\frac{a_s}{l}\right)^{|n|}\right]^2 < \infty, \tag{6.3.28}$$

if

$$l > a_{-1} + a_{+1} \tag{6.3.29}$$

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(condition (6.3.29) means that the cylinders do not intersect one another), and the right-hand parts will satisfy condition (1.9.20), since

$$\sum_{n=-\infty}^{\infty} q^{|n|} < \infty.$$

Consequently, the system of equations which is a majorant to (6.3.20) will possess a completely continuous form. On the basis of (6.3.26), the majorizing system (6.3.20) will also possess the same form. Therefore, the Gilbert alternative on the solvability of finite

systems is applicable to system (6.3.20): either system (6.3.20) has a single determined solution $\{X_n^s\}$, which satisfies the condition,

$$\sum_{n=-\infty}^{\infty} |X_n^s|^2 < \infty, \tag{6.3.30}$$

or the homogeneous system which corresponds to the heterogeneous system (6.3.20) has the solution $\{X_n^s\}$ which satisfies condition (6.3.30) and is non-zero. Since the assumptions on the possibility of the existence of a non-trivial solution to the homogeneous system leads to a contradiction with the theorem of uniqueness in solving the boundary problem for the potential u^1, then in such case only the first part of the alternative will be valid. Using (6.3.26), (6.3.27) we can establish that system (6.3.20) is a quasi-regular system in the general case, and converts with the proper choice of the parameter kl with given ka_s, $s \pm 1$) into a regular one (when $kl \to \infty$ all $c_{nm}^{(\bar{s},s)} \to 0$, as a result of which the determinant (6.3.23) of the system tends to unity). The unique solution to system (6.3.20) may be found by the method of truncation (reduction). The boundedness of the unknown X_n^s, according to the absolute values of all n, following from (6.3.30) along with (6.3.19) permits easily showing the absolute and uniform convergence of the series (6.3.16) for all ϕ and $\rho_s \geq a_s$, $s = \pm 1$, and also the possibility of a term-by-term differentiation of these series over ϕ and ρ the necessary number of times at each point of the given region outside the cylinders[2]. In their construction, series (6.3.15), (6.3.16) satisfy the Helmholtz equation and the condition of radiation at infinity. Therefore by determining the unknowns X_n^s, and through them the potential u^1 as well, the mathematical problem for u^1 is fully solved.

By the introduction of new designations, system (6.3.20) may be written in normal form, /238

$$z_i + \sum_{l=1}^{\infty} a_{il} z_l = b_i, \quad i = 1, 2, \ldots, \tag{6.3.31}$$

if we set

$$\begin{aligned} z_{|4l-1|} &= X_{-l}^{+1}, \quad z_{|4l-1|+1} = X_{-l}^{-1}, \quad l = 0, 1, \ldots, \\ z_{4l+1} &= X_l^{+1} \qquad z_{4l+2} = X_l^{-1}, \quad l = 1, 2, \ldots, \end{aligned} \tag{6.3.32}$$

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[2] On the basis of (6.3.26), conditions $|X_n^s| < \text{const}$ for all n after computation of the absolute value of (6.3.22) may be established from (6.3.20) which in fact is $|X_n^s| < [a_s/(l - a_{-s})]^{|n|}$. Then $\sum_{n=-\infty}^{\infty} |X_n^s| < \infty$.

$$b_{|4l-1|} = f_{-l}^{+1}, \quad b_{|4l-1|+1} = f_{-l}^{-1}, \quad l = 0, 1, \ldots,$$
$$b_{4l+1} = f_l^{+1}, \qquad b_{4l+2} = f_l^{-1}, \quad l = 1, 2, \ldots, \tag{6.3.33}$$

$$a_{|4l-1|,\,|4j-1|+1} = c_{-l,-j}^{(-1,+1)}, \quad a_{|4l-1|+1;|4j-1|} = c_{-l,-j}^{(+1,-1)},$$
$$a_{|4l-1|,\,4j+2} = c_{-l,j}^{(-1,+1)}, \quad a_{4l+2,\,|4j-1|} = c_{l,-j}^{(+1,-1)},$$
$$a_{|4l-1|+1,\,4j+1} = c_{-l,j}^{(+1,-1)}, \quad a_{4l+1,\,|4j-1|+1} = c_{l,-j}^{(-1,+1)}, \tag{6.3.34}$$
$$a_{4l+1,4j+2} = c_{l,j}^{(-1,+1)}, \quad a_{4l+2,4j+1} = c_{l,j}^{(+1,-1)},$$

$$a_{|4l-1|,4j+1} = 0, \quad a_{4l+1,|4j-1|} = 0,$$
$$a_{|4l-1|+1,4j+2} \neq 0, \quad a_{4l+2,|4j-1|+1} = 0, \tag{6.3.35}$$

$$a_{ll} = 0. \tag{6.3.36}$$

Let us find (formally) the main solution to system (6.3.17) after applying the method of successive approximations to it with the original values,

$$x_{n,0}^s = 0, \quad s = \pm 1. \tag{6.3.37}$$

We find:

$$x_{n,1}^s = f_n^s = -\frac{Ei^n J_n(ka_s)}{k^2 H_n^{(1)}(ka_s)}\, e^{ik(\mathbf{n}\mathbf{R}_{0s}) - in\varphi^s},$$
$$x_{n,2}^s = f_n^s - \sum_{m=-\infty}^{\infty} \alpha_{nm}^{(-s,s)}\, x_{m,1}^{-s}, \tag{6.3.38}$$

$$x_{n,p}^s = f_n^s - \sum_{m=-\infty}^{\infty} \alpha_{nm}^{(-s,s)}\, x_{m,p-1}^{-s}, \quad p \geqslant 2, \tag{6.3.39}$$

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Taking (6.3.18) into account and comparing (6.3.38), (6.3.39) with expressions (6.3.11), (6.3.12), we find

$$x_{n,1}^s = {}^sA_n^1,$$
$$x_{n,2}^s = {}^sA_n^1 + {}^sA_n^2,$$
$$x_{n,3}^s = {}^sA_n^1 + {}^sA_n^2 + {}^sA_n^3,$$

$$x^s_{n,p} = \sum_{m=1}^{p} {}^sA^m_n,$$

On the assumption that a limit exists $\lim_{p\to\infty} x^s_{n,p} = x^s_n$, we find

$$x^s_n = \sum_{m=1}^{\infty} {}^sA^m_n. \qquad (6.3.40)$$

Consequently, the formal application of the method of successive approximations to system (6.3.17) permits obtaining expressions for the scattering coefficients of any orders [in particular, scattering coefficients of m-th order will be the m-th term of the sum (6.3.40)] and in the same manner the secondary waves of any order of scattering can be defined. The solution to system (6.3.17) itself [coefficients of series (6.3.16)] represents simply the sums of the scattering coefficients of the diffraction series for the potentials of the secondary fields of all sequential orders of scattering, producing the interaction of the cylinders.

Through the solution $\{X^s_n\}$ of system (6.3.20) the potential u^1 of the secondary field on the basis of (6.3.19) is written in the form,

$$u^1 = \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} X^s_n J_n(ka_s) H^{(1)}_n(k\rho_s) e^{in\varphi_s}, \qquad (6.3.41)$$

and the potential of the complete field,

$$u = u^0 + \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} X^s_n J_n(ka_s) H^{(1)}_n(k\rho_s) e^{in\varphi_s}, \qquad (6.3.42)$$

where in coordinates with the origin at point O the potential u^0 of the primary field has the form of (6.3.2) and in local coordinates of the s-th cylinder has the form of (6.3.4). /21

Using the addition theorem (2.4.5) and formulas (6.3.20), (6.3.42), from (1.1.7) we find that the law of the density distribution j of the surface currents induced on the surface of the s-th cylinder by the incident wave and the secondary waves of all orders of scattering, will be given by the formula,

$$j^s_z = -\frac{\omega\varepsilon}{2\pi^2 a_s} \sum_{n=-\infty}^{\infty} X^s_n e^{in\varphi_s}, \qquad (6.3.43)$$

where j_z is the only non-zero longitudinal component of the vector j.

In the wave band for which, in coordinates with the origin at point O, approximately,

$$\rho_s = \rho - sl_0 \cos\varphi, \quad \varphi_s = \varphi, \quad \rho_s = \rho \quad (l \ll \rho; \quad l_0 = l/2), \tag{6.3.44}$$

after using the asymptotic formula (2.2.24) for the first Hankel function, approximately,

$$u^1 = \sqrt{\frac{2}{\pi k \rho}}\, e^{i\left(k\rho - \frac{\pi}{4}\right)} \sum_{s=\pm 1} e^{-iskl_0\cos\varphi} \times \\ \times \sum_{n=-\infty}^{\infty} (-i)^n X_n^s J_n(ka_s) e^{in\varphi}, \tag{6.3.45}$$

and from (1.6.16) we find that in this approximation,

$$E_\rho^1 = E_\varphi^1 = H_\rho^1 = H_z^1 = 0,$$

but

$$E_z^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = \frac{e^{ik\rho}}{\sqrt{\rho}} A(\varphi), \tag{6.3.46}$$

where the amplitude function $A(\phi)$ is determined by the expression,

$$A(\varphi) = k^2 \sqrt{\frac{2}{\pi k}}\, e^{-i\frac{\pi}{4}} \sum_{s=\pm 1} e^{-iskl_0\cos\varphi} \times \\ \times \sum_{n=-\infty}^{\infty} (-i)^n X_n^s J_n(ka_s) e^{in\varphi}. \tag{6.3.47}$$

From (1.4.11) and (6.3.46), (6.3.47), (6.3.1) we find that the linear cross section of back scattering (radar transverse scattering cross section) for two ideally conducting parallel infinitely long circular cylinders will be /241

$$\sigma_B = \frac{4k^3}{E^2} \left| \sum_{s=\pm 1} e^{iskl_0\cos\varphi_0} \times \right. \\ \left. \times \sum_{n=-\infty}^{\infty} i^n X_n^s J_n(ka_s) e^{in\varphi_0} \right|^2 \tag{6.3.48}$$

and from (6.3.46), (6.3.47), that the complete transverse scattering cross section for two parallel circular cylinders will be

$$\sigma_s = \frac{4k^3}{E^2} \operatorname{Re}\left\{ \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} |X_n^s J_n(ka_s)|^2 + \right.$$

$$+ \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_n^{-1} X_k^{+1*} J_n(ka_{-1}) J_k(ka_{+1}) J_{n-k}(kl) + \qquad (6.3.49)$$

$$\left. + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_n^{+1} X_k^{-1*} J_n(ka_{+1}) J_k(ka_{-1}) J_{k-n}(kl) \right\}.$$

In order to obtain numerical results using the formulas cited above, we must first of all compute the values of the coefficients X_n^s of the expansion of the potential u^1 into the series (6.3.15). The approximate values of several of the first coefficients X_n^s may be found with *a priori* given accuracy of computing by comparing the successive solutions of the finite systems obtained from (6.3.20) with the truncation for different increasing values of N, i.e., the order of truncation of system (6.3.20). The system of N-th order of truncation has the form,

$$\left.\begin{aligned} X_n^{+1} + \sum_{m=-N}^{N} c_{nm}^{(-1,1)} X_m^{-1} &= F_n^{+1}, \\ X_n^{-1} + \sum_{m=-N}^{N} c_{nm}^{(+1,-1)} X_m^{+1} &= F_n^{-1}, \end{aligned}\right\} \qquad (6.3.50)$$

$$n = 0, \pm 1, \ldots, \pm N,$$

or on the basis of (6.3.31) /24

$$z_i + \sum_{j=1}^{4N+2} a_{ij} z_j = b_i,$$

where $i = 1, 2, \ldots, 4N + 2$, but N assumes one of the values 0, 1, ... and where z_i, a_{ij}, b_i are given by formulas (6.3.32) - (6.3.36). As the original value of N is seeking approximate values of the coefficients X_n^s with the required degree of accuracy we can take $N = [2ka]$, where as usual the brackets indicate the integral part of the number ka ($a = \max_{s=\pm 1} a_s$), since in practice the series which enter into (6.3.45), converge with the same rapidity as does series (6.1.10), which determines the potential of the secondary field scattered on one cylinder.

As noted above, in the case when kl is sufficiently high, the alues of the matrix elements $c_{nm}^{(-s,s)}$ in (6.3.50) can be neglected nd then we will have approximately $X_n^s = F_n^s$ (in this case the amp- itudes of the seconadary waves of all orders of scattering, begin- ing from the second, and the amplitudes of their sum are small in omparison with the amplitude of the secondary waves of the first rder of scattering). System (6.3.50) is a system of $4N + 2$ equa- ions with $4N + 2$ unknowns X_n^{+1} and X_n^{-1}. If we exclude X_n^{-1} or X_n^{+1} rom (6.3.50), instead of system (6.3.50), we can solve two indepen- ent systems

$$X_n^s + \sum_{k=-N}^{N} \bar{c}_{nk} X_k^s = \bar{F}_n^s, \quad s = \pm 1, \tag{6.3.51}$$

ch of which contains $2N + 1$ equations with $2N + 1$ unknowns X_n^{-1} only X_n^{+1}. Here

$$\bar{c}_{nk} = -\sum_{m=-N}^{N} c_{nm}^{(-s,s)} c_{mk}^{(s,-s)},$$
$$\bar{F}_n^s = F_n^s - \sum_{m=-N}^{N} c_{nm}^{(-s,s)} F_m^{-s}. \tag{6.3.52}$$

owever, conversion to system (6.3.51) leads to a complication of ie expressions for the coefficients of the system. In the partic- lar case of two cylinders of different radii we can avoid the com- ications of the coefficients of the system and instead of sys- em (6.3.50) solve two independent systems such as (6.3.51) with ie same coefficients as in (6.3.50) or take into account that hen $a_{-1} = a_{+1}$, /243

$$c_{nm} = c_{nm}^{(-1,+1)} = (-1)^{n+m} c_{nm}^{(+1,-1)} \tag{6.3.53}$$

$_{-1,+1} = 0$, $\phi_{+1,-1} = \pi$, $ka_{-1} = ka_{+1} = ka$). On the basis of (6.3.53), ystem (6.3.50) is written in the form,

$$\left.\begin{aligned} X_n^{+1} + \sum_{m=-N}^{N} c_{nm} X_m^{-1} &= F_n^{+1}, \\ (-1)^n X_n^{-1} + \sum_{m=-N}^{N} c_{nm} (-1)^m X_m^{+1} &= (-1)^n F_n^{-1}, \\ n = 0, \pm 1, \ldots, \pm N. \end{aligned}\right\} \tag{6.3.54}$$

ter adding term-by-term both equations in (6.3.54), we find

$$z_n + \sum_{m=-N}^{N} c_{nm} z_m = f_n, \tag{6.3.55}$$

where we have denoted

$$z_n = X_n^{+1} + (-1)^n X_n^{-1}, \quad f_n = F_n^{+1} + (-1)^n F_n^{-1}. \tag{6.3.56}$$

After term-by-term subtraction of the second term from the first, we find

$$u_n - \sum_{m=-N}^{N} c_{nm}(-1)^m u_m = \varphi_n, \tag{6.3.57}$$

where

$$u_n = X_n^{+1} - (-1)^n X_n^{-1}, \quad \varphi_n = F_n^{+1} - (-1)^n F_n^{-1}.$$

In the same manner, for computation of X_n^{+1} and X_n^{-1}, we must first solve systems (6.3.55), (6.2.57), each of which is a system of $(2N + 1)$-th order, after which we use formulas

$$X_n^{+1} = \frac{z_n + u_n}{2}, \quad X_n^{-1} = \frac{z_n - u_n}{2}(-1)^n. \tag{6.3.58}$$

When $\phi_0 = \pi/2$ (direction of propagation of the wave is normal to the line of the centers of the cross section of the cylinders with the plane Oxy) and $ka_{-1} = ka_{+1} = ka$, we can see, for example from (6.3.51) and relationships (6.3.53) and /2

$$c_{-n,-m}^{(-s,s)} = c_{nm}^{(-s,s)}, \tag{6.3.59}$$

that

$$X_n^s = (-1)^n X_{-n}^{-s}, \quad n = 0, \pm 1, \ldots; \quad s = \pm 1, \tag{6.3.60}$$

$$X_{-n}^s = (-1)^n X_n^{-s}, \quad n = 0, \pm 1, \ldots; \quad s = \pm 1. \tag{6.3.61}$$

By virtue of the uniform convergence of series (6.3.41), which determines the potential of the field scattered on the cylinders, and the convergence of the method of reduction applied to system (6.3.20), the approximate values of the coefficients of these series are computed with the required degree of accuracy after a finite number of operations. Therefore, in each specific case (for each ka_s and kl) in (6.3.41) and in (6.3.20) we assume in fact a finiteness of the values of the indices n and m ($n, m = 0, \pm 1, \ldots, \pm N$). On the basis of such an assumption it is possible to obtain analytical expressions for the approximate solution to the system (6.3.20), having a "closed" form under the condition that

he cylinders are located at a great distance l from one another, hen $kl >> 1$ (the wave length is much less than l). For this pur- ose let us apply to system (6.3.20) (formally) the method of iter- tions, after setting as the zero approximation, $X^s_{n,0} = 0$, $s = \pm 1$. hen $X^s_{n,1} = F^s_n$, $X^s_{n,2} = F^s_n - \sum_{n'=-\infty}^{\infty} c^{(-s,s)}_{nn'} \times F^{(-s)}_{n'}$ etc. The expression for the + 1-st approximation is obtained in the form,

$$X^s_{n,p+1} = F^s_n + \sum_{t=1}^{p} (-1)^t \sum_{n'=-\infty}^{\infty} \sum_{n''=-\infty}^{\infty} \dots$$

$$\dots \sum_{n^{(t)}=-\infty}^{\infty} c^{(-s,s)}_{nn'} c^{(s,-s)}_{n'n''} \dots c^{(-1)^t s,(-1)^{t-1}s}_{n^{(t-1)}n^{(t)}} F^{(-1)^t s}_{n^{(t)}},$$

hen on the assumption that a limit exists $\lim_{p\to\infty} X^s_{n,p} = X^s_n$ we obtain

$$X^s_n = F^s_n + \sum_{t=1}^{\infty} (-1)^t \sum_{n',n'',\dots,n^{(t)}} c^{(-s,s)}_{nn'} \times \tag{6.3.62}$$

$$\times c^{(s,-s)}_{n'n''} \dots c^{(-1)^t s,(-1)^{t-1}s}_{n^{(t-1)}n^{(t)}} F^{(-1)^t s}_{n^{(t)}}$$

$c^{(-s,\ s)}_{nm}$ and F^s_n are determined by formulas (6.3.21), (6.3.22)). /245
ow we shall assume that in (6.3.21), $kl >> 1$ and $kl >> |n - m|$ by virtue of the above we assume here a finiteness of the values f n and m). Then (6.3.21) is approximately written as the expres- ion,

$$c^{(-s,s)}_{nm} = L a^{(-s)}_m b^{(s)}_n, \tag{6.3.63}$$

here we have set

$$L = \sqrt{\frac{2}{\pi kl}}\, e^{i(kl-\pi/4)}, \quad a^s_n = J_n(ka_s)\, e^{-in(\pi/2-\varphi_{s,-s})},$$

$$b^s_n = \frac{e^{in(\pi/2-\varphi_{-s,s})}}{H^{(1)}_n(ka_s)}, \tag{6.3.64}$$

nd from (6.3.62) we find the approximate value,

$$X^s_n = F^s_n + b^s_n \sum_{t=1}^{\infty} (-1)^t L^t \left[\sum_{n'=-\infty}^{\infty} a^{-s}_{n'} b^{-s}_{n'}\right] \times$$

$$\times \left[\sum_{n''=-\infty}^{\infty} a^s_{n''} b^s_{n''}\right] \dots \left[\sum_{n^{(t-1)}=-\infty}^{\infty} a^{(-1)^{t-1}s}_{n^{(t-1)}} b^{(-1)^{t-1}s}_{n^{(t-1)}}\right] \times$$

$$\times \sum_{n^{(t)}=-\infty}^{\infty} a_{n^{(t)}}^{(-1)^t s} F_{n^{(t)}}^{(-1)^t s_t},$$

which, after introduction of the designations,

$$S(x, \varphi) = \sum_{n=-\infty}^{\infty} \frac{J_n(x)}{H_n^{(1)}(x)} e^{in\varphi}, \tag{6.3.65}$$

$$Q^s = S(x_s, \pi),$$

$$F^s = \frac{-E}{k^2} S(x_s, -\varphi_0 + \varphi_{s,-s}) e^{iskl_s \cos\varphi_s}, \; x_s = ka_s \tag{6.3.66}$$

and separate summation over even and odd numbers we can write the formula

$$X_n^s = F_n^s + b_n^s L \frac{LQ^{-s}F^s - F^{-s}}{1 - [L^2 Q^{-s} Q^s]}, \tag{6.3.67}$$

if we set

$$|L^2 Q^{-s} Q^s| < 1 \tag{6.3.68}$$

(inequality (6.3.68) always may be satisfied by virtue of the obvi- /21
ous convergence of series (6.3.65) with the proper choice of kl, since $L \to 0$, if $kl \to \infty$).

On the basis of (6.3.37) the potential u^1 from (6.3.41) is represented by the sum

$$u^1 = \psi_1 + \psi_2, \tag{6.3.69}$$

where

$$\psi_1 = \frac{-E}{k^2} \sum_{s=\pm 1} e^{iskl_s \cos\varphi_s} \times$$

$$\times \sum_{n=-\infty}^{\infty} i^n \frac{J_n(ka_s)}{H_n^{(1)}(ka_s)} H_n^{(1)}(k\rho_s) e^{in(\varphi_s - \varphi_s)}, \tag{6.3.70}$$

$$\psi_2 = L \sum_{s=\pm 1} \frac{LQ^{-s}F^s - F^{-s}}{1 - [L^2 Q^{-s} Q^s]} \times$$

$$\times \sum_{n=-\infty}^{\infty} i^n \frac{J_n(ka_s)}{H_n^{(1)}(ka_s)} H_n^{(1)}(k\rho_s) e^{in(\varphi_s - \varphi_{-s,s})}. \tag{6.3.71}$$

The expressions of the components of the stray electromagnetic field will be found from (1.6.16). In the wave band they will have the form of (6.3.46), where the amplitude function is now equal to

$$A(\varphi) = -E\sqrt{\frac{2}{\pi k^2}}\, e^{-\frac{i\pi}{4}} \sum_{s=\pm 1} \left\{ S(x_s,\ \varphi - \varphi_0) e^{is\delta} + \right.$$
$$\left. + L \frac{LS(x_{-s},\ \pi) S(x_s,\ -\varphi_0 + \varphi_{s,-s}) e^{is\delta} - S(x_{-s},\ -\varphi_0 + \varphi_{-s;s}) e^{-is\delta_1}}{1 - [L^2 S(x_{-s},\ \pi) S(x_s,\ \pi)]} \right\}, \tag{6.3.72}$$

where $\delta = kl_0(\cos\varphi_0 - \cos\phi)$, $\delta_1 = kl_0(\cos\phi_0 + \cos\phi)$.

In the case of cylinders of equal radii, when $S(x_{\pm 1}, \phi) = S(x, \phi)$, $x = ka$, in the wave band

$$\psi_1 = -\frac{2E}{k^2}\sqrt{\frac{2}{\pi k\rho}}\, e^{i(k\rho - \pi/4)} S(x, \varphi - \varphi_0)\cos\delta,$$
$$\psi_2 = -\frac{E}{k^2}\sqrt{\frac{2}{\pi k\rho}}\, L \frac{e^{i(k\rho - \pi/4)}}{1 - [LS(x,\ \pi)]^2} \times \tag{6.3.73}$$
$$\times \sum_{s=\pm 1} [LS(x,\ \pi) S(x,\ -\varphi_0 + \varphi_{s,-s})\, e^{is\delta} -$$
$$- S(x,\ -\varphi_0 + \varphi_{-s,s}) e^{-is\delta_1}]\, S(x,\ \varphi - \varphi_{-s,s}),$$

and the amplitude function /247

$$A(\varphi) = -E\sqrt{\frac{2}{\pi k}}\, e^{-\frac{i\pi}{4}} \left\{ 2S(x,\ \varphi - \varphi_0)\cos\delta + \right.$$
$$+ \frac{L}{1 - [LS(x,\ \pi)]^2} \sum_{s=\pm 1} [LS(x,\ \pi) S(x,\ -\varphi_0 + \varphi_{s,-s})\, e^{is\delta} - \tag{6.3.74}$$
$$\left. - S(x,\ -\varphi_0 + \varphi_{-s,s}) e^{-is\delta_1}]\, S(x,\ \varphi - \varphi_{-s,s}) \right\}.$$

The physical sense of the functions in (6.3.73) is obvious. The expression of the function ψ_1 coincides with the ordinary expression for the function, which transfers the distribution of the amplitude of the stray field as a function of the angle of observation in the Fraunhofer diffraction problem on the grid, composed of identical elements; here cos ϕ expresses the result of the interaction of the grid (of two cylinders), and the coefficient for this function characterizes the effect of one element of the grid (one cylinder). The function ψ_2 approximately takes into account the effect of all mutual successive diffractions of the secondary waves on the cylinders.

Thus, the substitution of the first Hankel function in (6.3.21) by the dominant term (2.2.24) of the asymptotic expansion when $kl >> 1$ leads to a geometric progression (6.3.64'), by which the

scattering coefficients of the secondary waves of all orders of scattering are summarized. As a result we obtain an approximate expression (6.3.67) for the coefficient X_n^s of the expansion of the total (complete) secondary wave, scattered only by the s-th cylinder taking into account the interaction of the cylinders, according to the wave functions of the s-th cylinder. Here, if we proceed from the meaning of the terms in (6.3.64') (relative to the summation index t), the primary diffraction has an order of $(kl)^{-1/2}$, the secondary, an order of $(kl)^{-1}$, the tertiary, an order of $(kl)^{-3/2}$, etc. At the same time in using formula (2.2.24) for the first Hankel function we discarded terms of the order of $(kl)^{-3/2}$, $(kl)^{-5/2}$, etc., contained in the asymptotic expansion of the Hankel function. In this respect here there arises the unique question of the error involved in this method and the criteria of its applicability. Since in using (2.2.24) we assume a finiteness of the values of the indices n and m ($n, m = 0, \pm 1, \ldots, \pm N$) in (6.3.21), then as one of these we can take the condition,

$$kl \gg \max(m+n) \doteq 2N \approx [4ka], \quad ka = \max_{s=\pm 1} ka_s. \tag{6.3.75}$$

The characteristic feature of the functions (6.3.72) - (6.3.74) is that all of them are finally expressed through series (6.3.65), for which we know the asymptotic representations corresponding to large values of the parameter $x = ka$ ($ka \geq 5$; see §1, Ch. 6). Therefore, they may be used (under condition (6.3.75)) for an approximate solution to the problem of diffraction of a plane electromagnetic wave on two cylinders also in the case of high frequencies, when $ka >> 1$ (it is obvious also that $kl > 2ka >> 1$) and when in the process of using the formulas of the strict theory it is necessary to solve systems of high orders of truncation. /248

TE Type Field. In this case the problem is solved analogously. If we seek the potential v^1 of the stray field in the form of the sum,

$$v^1 = \sum_{s=\pm 1} v_s, \tag{6.3.76}$$

where

$$v_s = \sum_{n=-\infty}^{\infty} x_n^s H_n^{(1)}(k\rho_s) e^{in\varphi_s}, \tag{6.3.77}$$

then on the basis of (6.3.5), (2.4.5), from boundary conditions (1.7.4) on the surface of each of the cylinders after substitution of the unknowns x_n^s by the new X_n^s according to the formula,

$$x_n^s = J_n'(ka_s) X_n^s, \quad n = 0, \pm 1, \ldots; \; s = \pm 1 \tag{6.3.78}$$

we find an infinite system of linear equations relative to X_n^s of the same type as (6.3.20), where now

$$c_{nm}^{-s,s} = \frac{J_m'(ka_{-s})}{H_n^{(1)'}(ka_s)} H_{m-n}^{(1)}(kl)\, e^{i(m-n)\varphi_{-s,s}}, \qquad (6.3.79)$$

$$F_n^s = -\frac{i^n E e^{iskl_s\cos\varphi_s - in\varphi_s}}{k^2 H_n^{(1)'}(ka_s)}. \qquad (6.3.80)$$

This system, just as in the previous case, is uniquely solvable by the method of reduction and its solution will satisfy the condition $\sum_{n=-\infty}^{\infty} |X_n^s|^2 < \infty$ (in fact, the condition given in the footnote on page 224). By seeking X_n^s the mathematical problem for the potential v^1 is fully solved.

On the basis of (6.3.78) the potential v^1 of the stray field is written in the form,

$$v^1 = \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} X_n^s J_n'(ka_s) H_n^{(1)}(k\rho_s)\, e^{in\varphi_s}, \qquad (6.3.81)$$

and for the complete field

$$v = v^0 + \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} X_n^s J_n'(ka_s) H_n^{(1)}(k\rho_s) e^{in\varphi_s}, \qquad (6.3.82)$$

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where in coordinates with the origin at point O the potential of the primary field of the incident wave may be written in the form of (6.3.3) and in local coordinates of the s-th cylinder in the form of series (6.3.5).

On the surface of the s-th cylinder the azimuth component of the density of the surface current is equal to

$$j_\varphi^s = -\frac{c}{4\pi} H_z; \quad \rho_s = a_s, \; s = +1 \text{ or } s = -1,$$

where H_z is the component of the vector H of the complete field. On the basis of the addition theorem (2.4.5), the systems (6.3.20) with (6.3.79), (6.3.80)] and (2.2.47), we find that the law governing the density distribution of the surface current on the s-th cylinder will be determined by the formula,

$$j_\varphi^s = \frac{cik}{2\pi^2 a_s} \sum_{n=-\infty}^{\infty} X_n^s e^{in\varphi_s}. \qquad (6.3.83)$$

In the wave band in coordinates with the origin at point O we have approximately,

$$v^1 = \sqrt{\frac{2}{\pi k \rho}}\, e^{i\left(k\rho - \frac{\pi}{4}\right)} \sum_{s=\pm 1} e^{-iskl_0\cos\varphi} \times \\ \times \sum_{n=-\infty}^{\infty} (-i)^n X_n^s J_n'(ka_s)\, e^{in\varphi}, \tag{6.3.84}$$

and then in this approximation, from (1.6.18) we have

$$E_\rho^1 = E_z^1 = H_\rho^1 = H_\varphi^1 = 0,$$

but

$$H_z^1 = \sqrt{\frac{\varepsilon}{\mu}}\, E_\varphi^1 = \frac{e^{ik\rho}}{\sqrt{\rho}} A^*(\varphi), \tag{6.3.85}$$

where the amplitude function $A^*(\phi)$ is given by the expression

$$A^*(\varphi) = k^2 \sqrt{\frac{2}{\pi k}}\, e^{-\frac{i\pi}{4}} \sum_{s=\pm 1} e^{-iskl_0\cos\varphi} \times \\ \times \sum_{n=-\infty}^{\infty} (-i)^n X_n^s J_n'(ka_s)\, e^{in\varphi}. \tag{6.3.86}$$

For the cross section of back scattering from (1.4.11) we obtain /2

$$\sigma_B = \frac{4k^3}{E^2} \left| \sum_{s=\pm 1} e^{iskl_0\cos\varphi_0} \sum_{n=-\infty}^{\infty} i^n X_n^s J_n'(ka_s)\, e^{in\varphi_0} \right|^2 \tag{6.3.87}$$

(radar scattering cross section), and for the complete scattering cross section from (1.4.10),

$$\sigma_s = \frac{4k^3}{E^2} \operatorname{Re} \left\{ \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} |X_n^s J_n'(ka_s)|^2 + \right. \\ + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_n^{-1} X_k^{+1*} J_n'(ka_{-1})\, J_k'(ka_{+1})\, J_{n-k}(kl) + \\ \left. + \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} X_n^{+1} X_k^{-1*} J_n'(ka_{+1})\, J_k'(ka_{-1})\, J_{k-n}(kl) \right\}. \tag{6.3.88}$$

All the above in this case of TM-polarization of a plane wave refers to system (6.3.20) with the coefficients $c_{nm}^{(-s,s)}$ and the right-hand parts F_n^s determined by formulas (6.3.79) and (6.3.80), respectively. In particular when $kl >> 1$, $kl >> |n - m|$ ($n, m = 0, \pm 1, \ldots, \pm N$) we obtain the approximate formula (6.3.67) in which F_n^s will now be given by expression (6.3.80), but

$$Q^s = S^*(x_s, \pi), \quad F^s = \frac{-E}{k^2} S^*(x_s, -\varphi_0 + \varphi_{-s,s}) e^{isk l_s \cos\varphi_0}, \qquad x_s = ka_s, \tag{6.3.89}$$

where

$$S^*(x, \varphi) = \sum_{n=-\infty}^{\infty} \frac{J_n'(x)}{H_n^{(1)'}(x)} e^{in\varphi}, \tag{6.3.90}$$

$$b_n^s = \frac{e^{in\left(\frac{\pi}{2} - \varphi_{-s,s}\right)}}{H_n^{(1)'}(ka_s)} \tag{6.3.91}$$

(on the assumption that $|L^2 Q^s Q^{\bar{s}}| < 1$). On the basis of (6.3.67) the potential v^1 is written in the form of the sum $v^1 = \psi^1 + \psi^2$, in which the functions ψ^1 and ψ^2 are determined by expressions (6.3.70), (6.3.71), respectively, with $S^*(x, \phi)$ instead of $S(x, \phi)$ and with $J_n'(x)/H_n^{(1)'}(x)$ instead of $J_n(x)/H_n^{(1)}(x)$. /251

The function $A^*(\phi)$ may be written in this case in the form of (6.3.72), if in the expression for the amplitude function $A(\phi)$ (6.3.72) we substitute $S(x, \phi)$ by $S^*(x, \phi)$. In the case of cylinders of equal radii, the functions ψ_1 and ψ_2 will be given by formula (6.3.73), but the amplitude $A^*(\phi)$ will be given by formula (6.3.74, in which $S(x, \phi)$ is substituted by $S^*(x, \phi)$.

Now let us note the following. In the past the potentials u, v of both the primary and the secondary fields were represented by series (6.3.4), (6.3.5) and (6.3.15), (6.3.76), respectively, in which the summation was carried out over positive and negative values of the index of summation n. As a result, in seeking approximate values of the coefficients of the diffraction series (6.3.15) and (6.3.76), it is necessary to solve finite (truncated) systems of equations such as (6.3.50) or (6.3.55), (6.3.57), which contain for a given N (order of truncation) $4N + 2$ or $2N + 1$ equations, respectively. It is obvious that with a large value of the parameter ka ($N \approx 2ka$, $a = \max\limits_{s=\pm 1}\{a_s\}$; $a_{-1}, a_{+1} \neq 0$), the solution to such systems will involve tremendous difficulties. However, we can obtain new reductions in the order of truncated systems of equations to be solved without changing N and by retaining the accuracy of

computation, and instead of solving (6.3.50) or (6.3.55), (6.3.57) we can solve the systems $2(N+1)$ or $N+1$ equations, respectively with $2(N+1)$ or $N+1$ unknowns, if instead of (6.3.4), (6.3.5) the potentials of the primary field are represented by the expansions,

$$\left.\begin{matrix} u^0 \\ v^0 \end{matrix}\right\} = \frac{E}{k^2} e^{ik(\mathbf{n}\mathbf{R}_s)} \sum_{n=0}^{\infty} \varepsilon_n i^n J_n(k\rho_s)\cos n(\varphi_s - \varphi^0), \qquad (6.3.92)$$

$$\varepsilon_0 = 1, \quad \varepsilon_n = 2, \quad n \geqslant 1,$$

with summation in them only over non-negative values of the index n and if we seek the potentials of the secondary field in the form of a series such as

$$\left.\begin{matrix} u^1 \\ v^1 \end{matrix}\right\} = \sum_{s=\pm 1} \sum_{n=0}^{\infty} \left[\left\{ \begin{matrix} a_n^s \\ \tilde{a}_n^s \end{matrix} \right\} H_n^{(1)}(k\rho_s)\cos n\varphi_s + \right. \qquad (6.3.93)$$

$$\left. + \left\{ \begin{matrix} b_n^s \\ \tilde{b}_n^s \end{matrix} \right\} H_n^{(1)}(k\rho_s)\sin n\varphi_s \right].$$

In this case (for example, for a TM type field) after substitution of the unknowns a_n^s and b_n^s with new unknowns A_n^s and B_n^s according to the formulas,

$$a_n^s = J_n(ka_s)A_n^s, \quad b_n^s = J_n(ka_s)B_n^s \qquad (6.3.94)$$

using the addition theorems of (2.4.8) [instead of (2.4.5)], from the boundary conditions for the potential $u = u^0 + u^1$ on the surfaces of the cylinders we obtain two mutually independent infinite systems of equations,

$$A_n^s + \sum_{m=0}^{\infty} \alpha_{nm}^{-s,s} A_m^{-s} = F_n^s, \qquad (6.3.95)$$

$$B_n^s + \sum_{m=0}^{\infty} \beta_{nm}^{(-s,s)} B_m^{-s} = \Phi_n^s \qquad (6.3.96)$$

$$(n = 0, 1, 2, \ldots; \quad s = \pm 1)$$

with the matrix elements,

$$\alpha_{nm}^{(-s,s)} = \frac{J_m(ka_{-s})}{H_n^{(1)}(ka_s)} \left\{ H_{m-n}^{(1)}(kl)\cos(m-n)\varphi_{-s,s} + \right. \qquad (6.3.97)$$

$$\left. + (-1)^n \delta_n H_{m+n}^{(1)}(kl)\cos(m+n)\varphi_{-s,s} \right\},$$

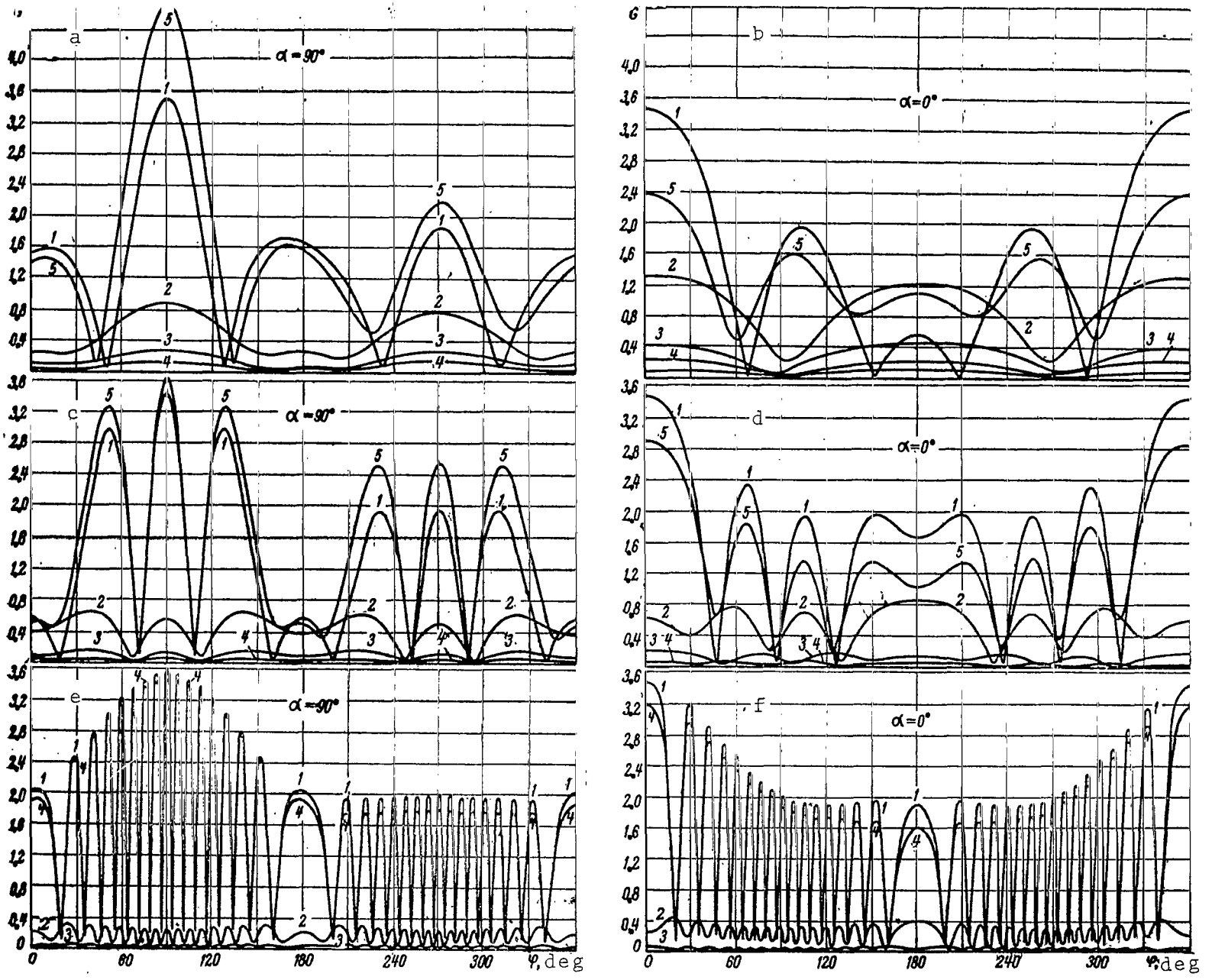

Fig. 28. The Functions G_2^m and G_3 when ka = 1 and kl = 5 (a,b); kl = 10 (c,d), kl = 50 (e,f): a. 1. G_2^1; 2. G_2^2; 3. G_2^3; 4. G_2^4; 5. G_3; b. 1. 5 same; 6. G_2^5; c,d. 1-5. Same; e,f. 1-3. Same; 4. G_3

$$\beta_{nm}^{(-s,s)} = \frac{J_m(ka_{-s})}{H_n^{(1)}(ka_s)} \{H_{m-n}^{(1)}(kl)\cos(m-n)\varphi_{-s,s} - \\ -(-1)^n \delta_n H_{m+n}^{(1)}(kl)\cos(m+n)\varphi_{-s,s}\} \quad (6.3.98)$$

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and with the free terms

$$F_n^s = -\frac{\varepsilon_n i^n E e^{iskl_0\cos\alpha}\cos n\alpha}{k^2 H_n^{(1)}(ka_s)}, \quad (6.3.99)$$

$$\Phi_n^s = -\frac{\varepsilon_n i^n E e^{iskl_0\cos\alpha}\sin n\alpha}{k^2 H_n^{(1)}(ka_s)} \quad (6.3.100)$$

respectively. In both systems the indices n and m now assume only non-negative values. Here $\delta_0 = 0$ and $\delta_n = 1$, if $n \geq 1$.

Just as in the previous cases, the matrix elements of systems (6.3.95), (6.3.96) satisfy the relationships ($\alpha_{-1} = \alpha_{+1}$) /25

$$\alpha_{nm} = (-1)^{n+m}\alpha_{nm}^{(+1,-1)} = \alpha_{nm}^{(-1,+1)}; \\ \beta_{nm} = (-1)^{n+m}\beta_{nm}^{(+1,-1)} = \beta_{nm}^{(-1,+1)}, \quad (6.3.101)$$

by the use of which we can reduce the problem of seeking approximate values of the coefficients of the diffraction series (6.3.93) to solving four monotypic systems of $N + 1$ equations with $N + 1$ unknowns:

$$z_n + \sum_{m=0}^{N}(-1)^m \alpha_{nm} z_m = F_n, \\ u_n - \sum_{m=0}^{N}(-1)^m \alpha_{nm} u_m = P_n, \quad (6.3.102)$$

where

$$z_n = A_n^{+1} + (-1)^n A_n^{-1}, \quad F_n = F_n^{+1} + (-1)^n F_n^{-1}, \\ u_n = A_n^{+1} - (-1)^n A_n^{-1}, \quad P_n = F_n^{+1} - (-1)^n F_n^{-1}, \quad (6.3.103)$$

and

$$v_n + \sum_{m=0}^{N}(-1)^m \beta_{nm} v_m = \Phi_n, \quad (6.3.104)$$

$$w_n - \sum_{m=0}^{N}(-1)^m \beta_{nm} w_m = Q_n,$$

where

$$v_n = B_n^{+1} + (-1)^n B_n^{-1}, \quad \Phi_n = \Phi_n^{+1} + (-1)^n \Phi_n^{-1},$$

$$w_n = B_n^{+1} - (-1)^n B_n^{-1}, \quad Q_n = \Phi_n^{+1} - (-1)^n \Phi_n^{-1}. \qquad (6.3.105)$$

If the radii of the cylinders are different, seeking the approximate values of the diffraction coefficients is reduced to solving two monotypic systems of $2(N + 1)$ equations with $2(N + 1)$ unknowns, which are obtained from (6.3.95), (6.3.96) after their truncation to the N-th order (n, m = 0, 1, ..., N; $s = \pm 1$).

From the uniqueness of the solution to this diffraction problem, let us find after comparison of series (6.3.15), (6.3.76) with the corresponding series such as (6.3.93), that

$$x_n^s + (-1)^n \delta_n x_{-n}^s = a_n^s, \quad b_n^s = i\left[x_n^s - (-1)^n \delta_n x_{-n}^s\right], \qquad (6.3.106)$$

whence

$$x_n^s = \frac{1}{2}(a_n^s - ib_n^s), \quad n \geqslant 0,$$

$$x_{-n}^s = \frac{(-1)^n}{2}(a_n^s + ib_n^s), \quad n \geqslant 1. \qquad (6.3.107)$$

A similar type of relationship is obtained also for the coefficients of series (6.3.76), (6.3.93), which determine the fields of TE type. In this case the corresponding diffraction coefficients will be the solutions to systems (6.3.95), (6.3.96) with the matrix elements and with free terms, which differ from (6.3.97) - (6.3.100) only in the fact that in their expressions the values $J_m(ka_{-s})$, $H_n^{(1)}(ka_s)$ are substituted by the values $J_m'(ka_{-s})$ and $H_n^{(1)}(ka_s)$ [in the formulas in (6.3.94) instead of $J_n(ka_s)$ we take $J_n'(ka_s)$]. Using formulas (6.3.107) in all the series obtained earlier, which give solutions to the problems, we can convert to a summation only over non-negative values of the summation indices by determining the coefficients of the transformed series from systems (6.3.95), (6.3.96).

With numerical solution to the problems on the interaction of two cylinders using computers, the use of formulas (6.3.92) - (6.3.107) may give significant reduction in time required for computing the coefficients of the diffraction series and also will permit with one and the same volume of the computer memory carrying out the computations for a wider range of values of the parameter ka.

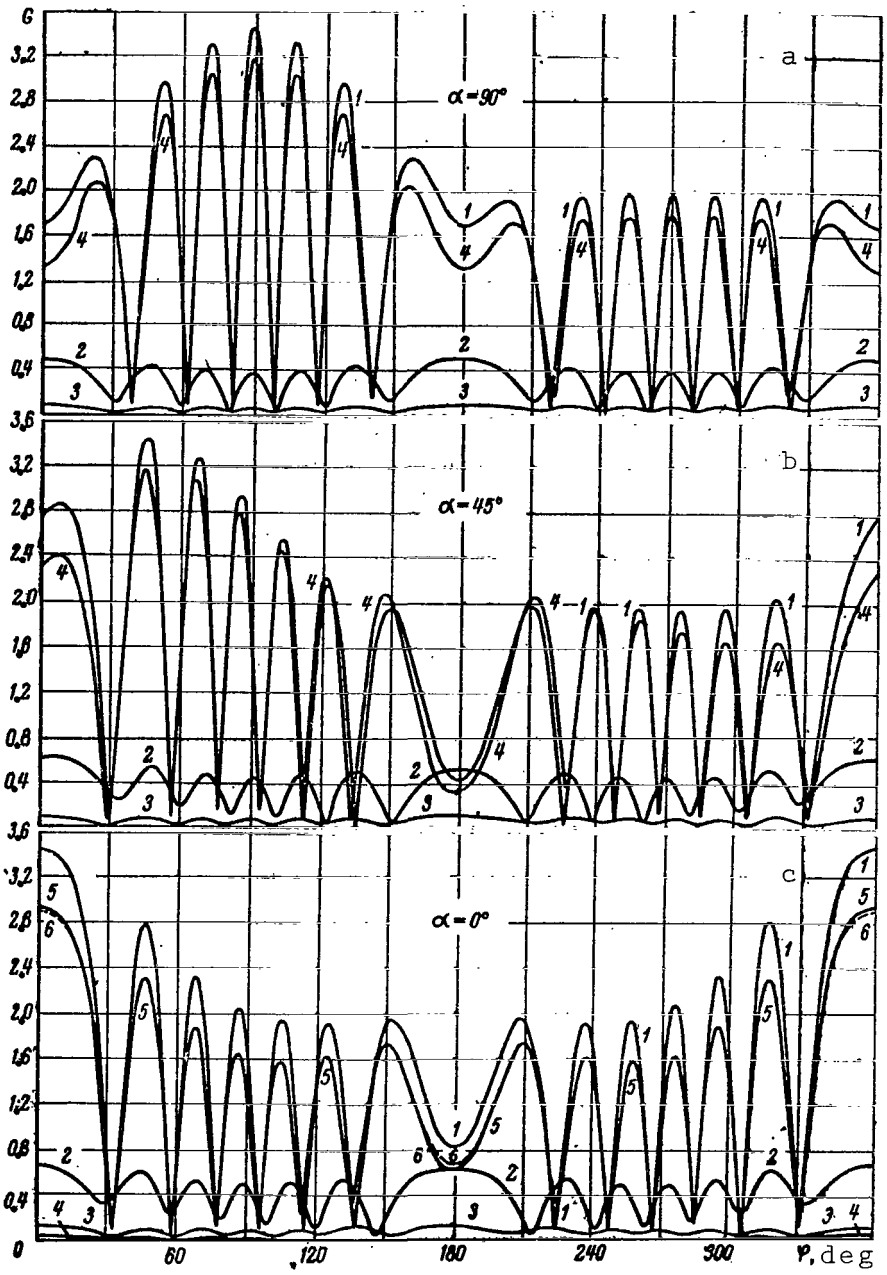

Fig. 29. The Functions G_1, G_2^m and G_3 When $ka = 1$, $kl = 20$: a,b. 1. G_2^1; 2. G_2^2; 3. G_2^3 and 4. G_3; c. 1-3 Same; 4. G_2^4; 5. G_3; 6. G_1.

As examples we have computed[3] the functions $G_1(\phi)$, $G_2^m(\phi)$, $G_3(\phi)$ /257 (Figs. 28 -30), determined by the expressions:

(1) $G_1(\varphi) = \left| \left(\sqrt{\frac{\pi k \rho}{2}} E_z^1 \right) \Big/ E \right|$ where E_z^1 is given by formulas (6.3.46) - (6.3.47);

(2) $G_2^m(\varphi) = \left| \left(\sqrt{\frac{\pi k \rho}{2}} \, {}^m E_z^1 \right) \Big/ E \right|$, where in approximation of the wave band,

$$ {}^m E_z^1 = \frac{e^{ik\rho}}{\sqrt{\rho}} A^m(\varphi), \tag{6.3.108} $$

and the function

$$ A^m(\varphi) = k^2 \sqrt{\frac{2}{\pi k}} \, e^{-i\pi/4} \times \times \sum_{s=\pm 1} e^{-iskl_0 \cos\varphi} \sum_{n=-\infty}^{\infty} (-i)^n {}^s A_n^m e^{in\varphi} \tag{6.3.109} $$

is the amplitude of the stray field of TM type of m-th order of scattering with the scattering coefficients ${}^s A_n^m$ from (6.3.11), (6.3.12);

(3) $G_3(\varphi) = \sum_m G_2^m(\varphi)$. In meaning, the functions $G_1(\phi)$ and $G_3(\phi)$ must be identical; both of them determine one and the same secondary field in the wave band.

The functions G_1, G_2^m, G_3 were assumed as functions of the polar angle α and calculated on the computer for the various values of the parameters ka, kl and $\alpha = \phi_0$ ($a_{-1} = a_{+1} = a$). In the case $ka = 1$, truncation of system (6.3.20) was carried out up to the value $|n| = N = 3$. In the same way the series which determine the secondary fields of the successive orders of scattering were also truncated. In the case $ka = 3$ we took $N = 7$ ($n, m = 0, \pm 1, \ldots, \pm N$). When $ka = 1$ the computations were carried out for cylinders placed apart from one another at a distance l, coupled with their radii by the relationships: $l/a = 5, 10, 20, 50$. When $ka = 3$, we studied the case $l/a = 5$. The direction of propagation of the wave (in the

[3] Computations on the "Minsk-2" computer were made by G.M. Koblova.

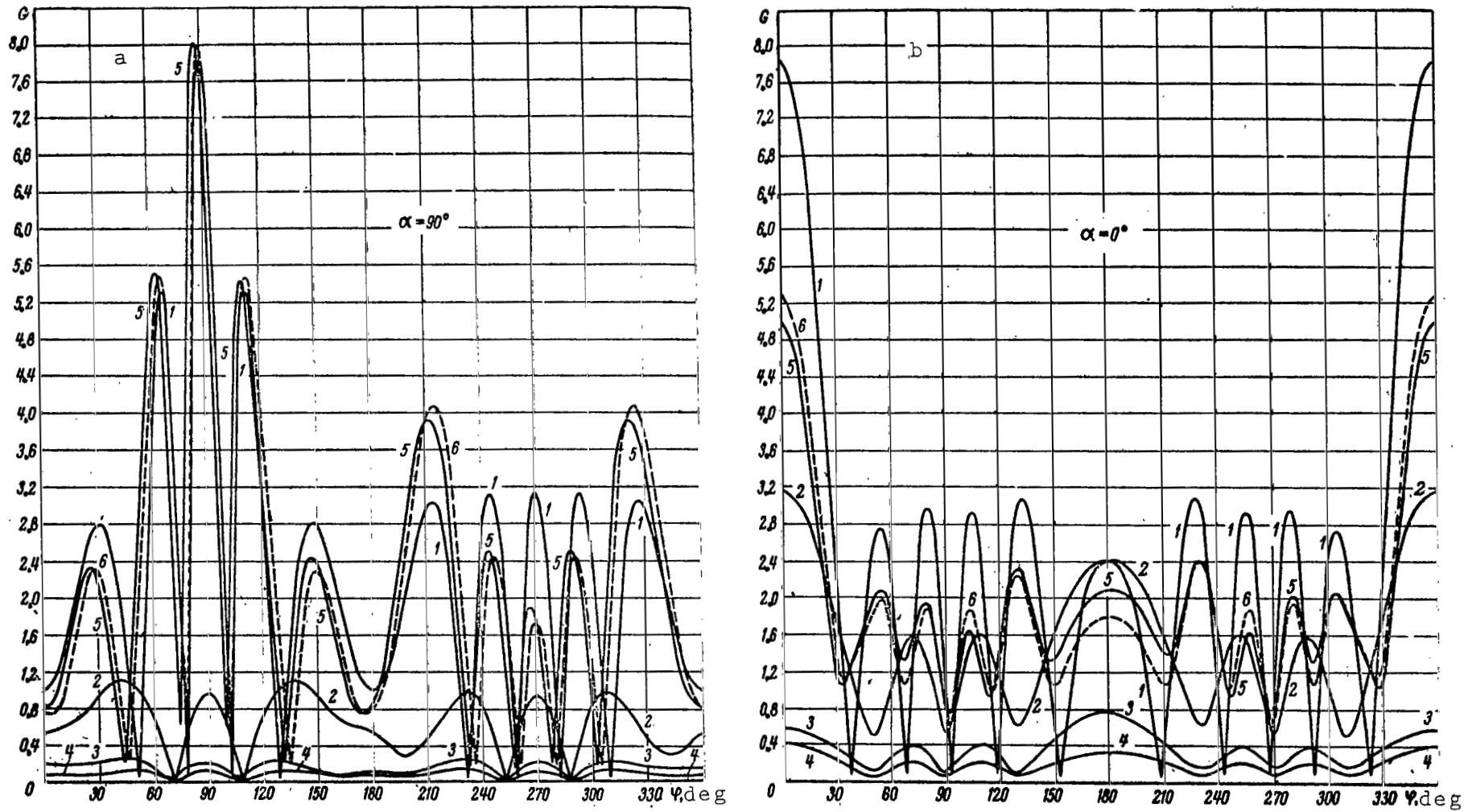

Fig. 30. The Functions G_1, G_2^m and G_3 When $ka = 3$, $kl = 15$: a,b. 1. G_2^1; 2. G_2^2; 3. G_2^3; 4. G_2^4; 5. G_3; and 6. G_1.

TABLE 4* SOLUTIONS TO SYSTEM (6.3.50) FOR VARIOUS N. /260 /261

$ka = 3$, $kl = 15$, $\alpha = 0°$

$X^{+1}_{\pm n}$	n	$N=5$	p	$N=6$	p	$N=7$	p	$N=8$	p	$N=9$	p
$X^{+1}_{n} = X^{+1}_{-n} = X^{N}_{n} \cdot 10^{p}$	9									+0,8877346	−4
										− 2851397	−2
	8							−0,1276242	−1	− 1276245	−1
								+ 1103120	−2	+ 1103009	−2
	7					+0,8015451	−2	+ 8017136	−2	+ 8017179	−2
						+ 4766178	−1	+ 4766130	−1	+ 4766147	−1
	6			+0,1417945	0	+ 1418071	0	+ 1418045	0	+ 1418044	0
				− 3457359	−1	− 3455310	−1	− 3455261	−1	− 3455285	−1
	5	−0,9505103	−1	− 9601088	−1	− 9606621	−1	− 9606306	−1	− 9606295	−1
		− 3016830	0	− 3016594	0	− 3017006	0	− 3017015	0	− 3017013	0
	4	− 4035011	0	− 4023567	0	− 4022835	0	− 4022885	0	− 4022888	0
		+ 1638093	0	+ 1648406	0	+ 1649780	0	+ 1649802	0	+ 1649799	0
	3	+ 1699122	0	+ 1698872	0	+ 1698837	0	+ 1698895	0	+ 1698899	0
		+ 3436247	0	+ 3425272	0	+ 3423528	0	+ 3423529	0	+ 3423533	0
	2	+ 3260161	0	+ 3258199	0	+ 3258263	0	+ 3258259	0	+ 3258256	0
		− 1675667	0	− 1679215	0	− 1678096	0	− 1678114	0	− 1678118	0
	1	− 4407840	0	− 4413912	0	− 4414721	0	− 4414782	0	− 4414781	0
		− 2200976	0	− 2185739	0	− 2186446	0	− 2186453	0	− 2186452	0
	0	+ 4186355	0	+ 4197081	0	+ 4198284	0	+ 4198371	0	+ 4198369	0
		+ 3722323	0	+ 3703537	0	+ 3704178	0	+ 3704203	0	+ 3704203	0
$X^{-1}_{n} = X^{-1}_{-n} = X^{N}_{n} \cdot 10^{p}$	0	− 1433175	+1	− 1433725	+1	− 1433605	+1	− 1433612	+1	− 1433612	+1
		+ 1880906	+1	+ 1882786	+1	+ 1882883	+1	+ 1882880	+1	+ 1882880	+1
	1	+ 1016559	+1	+ 1016393	+1	+ 1016478	+1	+ 1016474	+1	+ 1016473	+1
		− 1666915	+1	− 1665364	+1	− 1665271	+1	− 1665272	+1	− 1665272	+1
	2	− 4313145	−1	− 4264459	−1	− 4263149	−1	− 4262924	−1	− 4262948	−1
		+ 2122902	+1	+ 2123385	+1	+ 2123495	+1	+ 2123498	+1	+ 2123497	+1
	3	− 1077853	+1	− 1077599	+1	− 1077603	+1	− 1077594	+1	− 1077595	+1
		− 1001608	+1	− 1002460	+1	− 1002307	+1	− 1002306	+1	− 1002307	+1
	4	+ 1074303	+1	+ 1073521	+1	+ 1073573	+1	+ 1073580	+1	+ 1073580	+1
		− 4414894	0	− 4423948	0	− 4422702	0	− 4422723	0	− 4422726	0
	5	+ 1898396	0	+ 1890882	0	+ 1891310	0	+ 1891350	0	+ 1891350	0
		+ 4659249	0	+ 4658830	0	+ 4659230	0	+ 4659223	0	+ 4659219	0
	6			− 1796199	0	− 1796128	0	− 1796096	0	− 1796096	0
				+ 3758306	−1	+ 3760190	−1	+ 3760185	−1	+ 3760154	−1
	7					− 1789760	−1	− 1789550	−1	− 1789549	−1
						− 5038497	−1	− 5038515	−1	− 5038536	−1
	8							+ 1008399	−1	+ 1008400	−1
								− 3067882	−2	− 3068021	−2
	9									+ 5110769	−3
										+ 2293796	−2

* In each line the first number is the real part and the second is the imaginary part of the solution.

$ka = 5,\ kl = 25,\ \alpha = \pi/2,\ X_n^{-1} = (-1)^n X_{-n}^{+1} = X_n^N \cdot 10^p$

n	X_n^5	p	X_n^6	p	X_n^7	p	X_n^8	p	X_n^9	p	X_n^{10}	p
10											+0,4412780	−2
											− 3191626	−1
9									−0,1027315	−1	− 1027453	−1
									− 1023344	0	− 1023365	0
8							−0,7640710	−1	− 7640150	−1	− 7639765	−1
							− 3335482	0	− 3335464	0	− 3335548	0
7					−0,1661983	0	− 1672401	0	− 1672044	0	− 1671813	0
					− 8987983	0	− 8982269	0	− 8982572	0	− 8982569	0
6			−0,2207014	0	− 2433329	0	− 2439663	0	− 2438135	0	− 2437988	0
			− 1666590	+1	− 1653823	+1	− 1654948	+1	− 1654902	+1	− 1654866	+1
	−0,6219575	0	− 6239118	0	− 6554659	0	− 6539569	0	− 6538543	0	− 6538841	0
5	− 1965355	+1	− 1945744	+1	− 1963075	+1	− 1963353	+1	− 1963079	+1	− 1963043	+1
	− 1489468	+1	− 1534106	+1	− 1544164	+1	− 1543131	+1	− 1543269	+1	− 1543310	+1
4	− 1090276	+1	− 1101114	+1	− 1139300	+1	− 1136683	+1	− 1136326	+1	− 1136332	+1
	− 1722646	+1	− 1765992	+1	− 1753267	+1	− 1755055	+1	− 1755360	+1	− 1755372	+1
3	+ 1130130	+1	+ 1057060	+1	+ 1023299	+1	+ 1027281	+1	+ 1027529	+1	+ 1027498	+1
	+ 2960190	0	+ 2865289	0	+ 3064103	0	+ 3020132	0	+ 3017111	0	+ 3017345	0
2	+ 3034495	+1	+ 2912273	+1	+ 2893822	+1	+ 2897226	+1	+ 2897341	+1	+ 2897316	+1
	+ 3184258	+1	+ 3210499	+1	+ 3224255	+1	+ 3218615	+1	+ 3218432	+1	+ 3218476	+1
1	+ 1565661	+1	+ 1416070	+1	+ 1411145	+1	+ 1413444	+1	+ 1413504	+1	+ 1413499	+1
0	+ 2114840	+1	+ 2155859	+1	+ 2157704	+1	+ 2152135	+1	+ 2152123	+1	+ 2152173	−1
	− 2065698	+1	− 2230115	+1	− 2225660	+1	− 2223908	+1	− 2223842	+1	− 2223831	+1
−1	− 1733152	+1	− 1708288	+1	− 1717796	+1	− 1722230	+1	− 1722057	+1	− 1722010	+1
	− 9641288	0	− 1135933	0	− 1122904	0	− 1120960	0	− 1120896	+1	− 1120882	+1
−2	+ 4780681	0	+ 4563479	0	+ 4418193	0	+ 4394267	0	+ 4397519	0	+ 4397957	0
	+ 2518304	+1	+ 2355904	+1	+ 2379956	+1	+ 2382218	+1	+ 2382195	+1	+ 2082199	+1
−3	+ 2956190	+1	+ 2876057	+1	+ 2868754	+1	+ 2868773	+1	+ 2869145	+1	+ 2869145	+1
	− 1519971	+1	− 1636400	+1	− 1600971	+1	− 1599337	+1	− 1599550	+1	− 1599556	+1
−4	− 2016542	+1	− 2121603	+1	− 2108664	+1	− 2107396	+1	− 2107238	+1	− 2107239	+1
	− 1800405	+1	− 1829457	+1	− 1793095	+1	− 1793556	+1	− 1793937	+1	− 1793940	+1
−5	+ 4092635	0	+ 3548859	0	+ 3854196	0	+ 3852291	0	+ 3850886	0	+ 3850833	0
	+ 1212521	+1	+ 1260244	+1	+ 1275894	+1	+ 1273619	+1	+ 1273317	+1	+ 1273333	+1
−6			− 7123446	0	− 6911444	0	− 6934684	0	− 6936658	0	− 6936517	0
			− 1082106	+1	− 1093438	+1	− 1094351	+1	− 1094367	+1	− 1094348	+1
−7					+ 1033740	0	+ 1021177	0	+ 1020911	0	+ 1021086	0
					+ 1090233	+1	+ 1091526	+1	+ 1091595	+1	+ 1091593	+1
−8							+ 1413950	0	+ 1414088	0	+ 1414119	0
							− 3511304	0	− 3511293	0	− 3511372	0
−9									+ 1559759	−1	+ 1559631	−1
									+ 7701849	−1	+ 7701638	−1
−10											− 1417801	−1
											− 4956791	−1

coordinate system Oxy) formed an angle $\alpha = 0$ or $\alpha = \pi/2$ with the axis Ox (on Fig. 29, b, the case $\alpha = \pi/4$ is also shown). /264

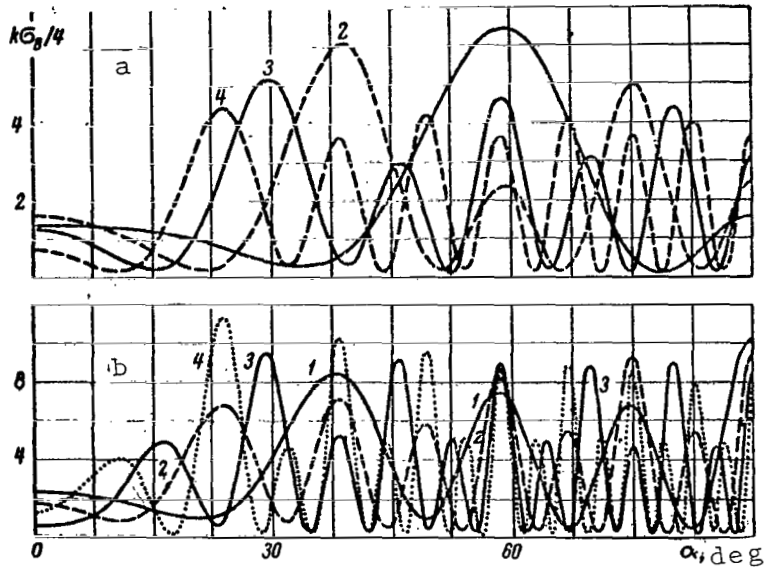

Fig. 31. The Functions $k\sigma_B/4$: a. $ka = 1$: 1. $kl = 6$; 2. 12; 3. 18; and 4. 24: b. $ka = 2$; 1. $kl = 12$; 2. 24; 3. 36; and 4. 48.

As is obvious from the figures, when $ka = 1$, beginning from $l/a = 5$, the basic contribution to the diffraction effect is made by the secondary waves only of the first and second orders of scattering and the contribution of these latter is decreased with increase in l/a (Fig. 28, e.f; Fig. 29). The same picture is observed also in the case $ka = 3$. /265

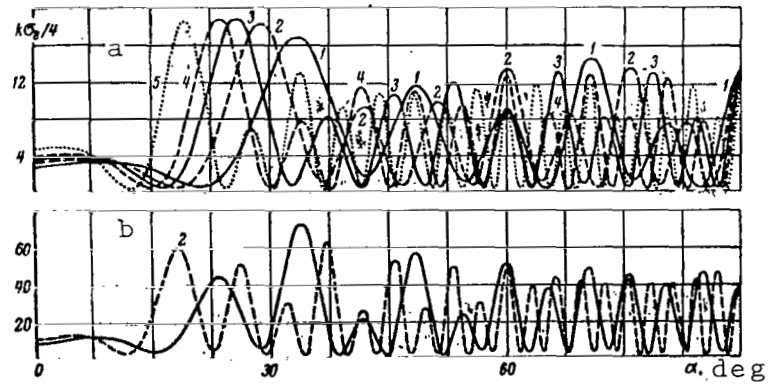

Fig. 32. The Function $k\sigma_B/4$: a. $ka = \pi$: 1. $kl = 6\pi$; 2. 8π; 3. 10π; 4. 12π; 5. 18π; b. $ka = 2\pi$; 1. $kl = 12\pi$; 2. 20π

From the computational point of view, irrefutable advantage is possessed by formulas (6.3.46), (6.3.47), coupled with system (6.3.20), in comparision with formulas (6.3.108), (6.3.109) and

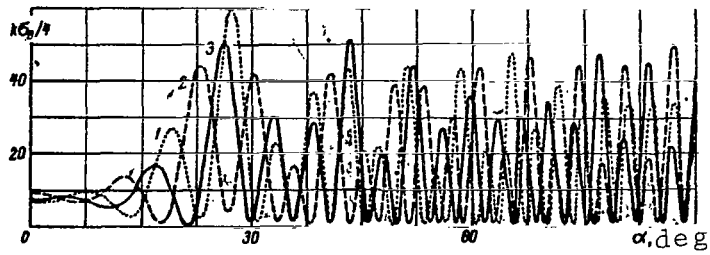

Fig. 33. The Function $k\sigma_B/4$ When $ka = 10$ and $kl = 56$ (1) $kl = 58$ (2) and $kl = 60$ (3).

$G_3(\phi)$: with one and the same $N(n = 0, \pm 1, \ldots, \pm N)$; the use of the first group requires considerably less time for computation of $G_1(\phi)$ that does the use of the second group for computation of $G_3(\phi)$. Moreover, with one and the same N, formulas (6.3.46), (6.3.47), (6.3.20) give a more accurate result.

The radar scattering cross section for two identical circular cylinders (graphs of the functions $k\sigma_B/4$) are computed as a function of the angle of incidence of the wave α and the different values of ka and kl from formula (6.3.48), $a_{-1} = a_{+1} = a$ (Figs. 31 -33). /2

TABLE 5. CHANGE IN THE VALUES OF THE FUNCTION $G_1(\phi)$ WITH CHANGE IN N.

$G_1(\varphi) - A_N(\varphi) \cdot 10^p$; $\alpha = 0^0$, $ka = 3$, $kl = 15$

N	$A_N(0)$	p	$A_N(\pi/4)$	p	$A_N(\pi/2)$	p	$A_N(3\pi/4)$	p	$A_N(\pi)$	p
3	0,4755724	1	0,9541952	0	0,1479034	1	0,1927497	1	0,1190423	1
4	4983550	1	1198494	1	1440365	1	1608175	1	1539182	1
5	5030671	1	1216967	1	1433385	1	1618337	1	1483014	1
6	5035749	1	1216122	1	1433272	1	1618081	1	1488775	1
7	5036171	1	1216072	1	1433478	1	1617952	1	1488313	1
8	5036188	1	1216069	1	1433478	1	1617957	1	1488332	1
9	5036189	1	1216069	1	1433476	1	1617957	1	1488331	1

$\alpha = \pi/2$, $ka = 5$, $kl = 25$

N	$A_N(\pi/2)$	p	$A_N(\pi/4)$	p	$A_N(0)$	p	$A_N(-\pi/4)$	p	$A_N(-\pi/2)$	p
5	0,1086338	2	0,1818911	1	0,2439528	1	0,2898114	1	0,3769828	1
6	1135712	2	1921418	1	2128466	1	3072116	1	3337710	1
7	1149160	2	1879077	1	2121535	1	3134955	1	3383929	1
8	1149309	2	1869377	1	2127585	1	3118716	1	3389068	1
9	1149387	2	1869080	1	2127489	1	3119852	1	3389433	1
10	1149402	2	1869078	1	2127387	1	3119930	1	3389407	1
11	1149402	2	1869075	1	2127386	1	3119926	1	3389396	1

Table 4 characterizes the rate of convergence of the computational process, associated with the solution to the truncated systems (6.3.50) for various increasing values of N, i.e., the order of truncation. Table 5 shows the respective rapidity of change in the values of the function $G_1(\phi)$.

Let us note that use of the classical method of separation of variables may solve a number of other problems of the theory of diffraction of a plane wave on two circular cylinders as well. For example, in [91], using this method the problem on the scattering of a normally incident wave on two composite circular cylinders (cylinders with a covering) was solved. In [105] a solution was given to the problem of scattering of an obliquely incident wave on two parallel circular cylinders of arbitrary conductivity. In [192] the problem was solved on scattering of a plane electromagnetic wave on two parallel infinitely long ferromagnetic cylinders.

§4. Field of a Longitudinal Emitter in the Presence of Two Parallel Circular Cylinders.

Let us look at the problem of a dipole field (electric and magnetic) emitter in a uniform and isotropic space with permeabilities ε, μ and a conductivity $\sigma = 0$, containing within itself two parallel infinitely long ideally conducting circular cylinders on the assumption that the dipole moment (p or m) is directed parallel to the generatrices of the cylinders. In local cylindrical coordinates the surfaces of the cylinders are given by the equations $\rho_s = a_s$ ($s = \pm 1$) (the Oz axis of the systems is directed along the respective axes of the cylinders, Fig. 34, a), and the position of the dipole in space is determined by the values of ρ_s^0, ϕ_s^0, $z^0 = z_s^0$, $s = \pm 1$. In the coordinates ρ, ϕ, z of the system with the origin at point O, lying midway on the line of the centers of the cross section of the cylinders with the plane Oxy (the Ox axes of all systems have an identical direction and lie on the lines of the centers, $Oz \quad O_s z_s$); the position of the dipole is determined by the values of $\rho = \rho^0$, $\phi = \phi^0$, $z = z^0$. The problem consists of seeking the dipole field in the presence of cylinders. Just as in the case of one cylinder, it may be reduced to a mathematical problem for the potential functions u or v depending on the type of dipole. /267

Electric Dipole. In this case, $v^{0,1} \equiv 0$ both for the dipole field in free space and for the field scattered on the cylinders. The potential u^0 of the dipole field in free space in local coordinates of the s-th cylinder may be written in the form of a series of Fourier integrals,

$$u^0 = \frac{ip}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi_s - \varphi_s^0)} \times$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} J_n(v\rho_s^0) H_n^{(1)}(v\rho_s), & \rho_s^0 < \rho_s \\ J_n(v\rho_s) H_n^{(1)}(v\rho_s^0), & \rho_s^0 > \rho_s \end{array} \right\} e^{ih(z-z^0)} dh \qquad (6.4.1)$$

[in coordinates with the origin at point O the expansion has the form of (6.2.1)]. Therefore, if we seek the potential u^1 of the field scattered on the cylinders in the form of analogous series of Fourier integrals,

$$u^1 = -\frac{ip}{2} \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} e^{in(\varphi_s - \varphi_s^0)} \times$$
$$\times \int_{-\infty}^{\infty} x_n^s(h) H_n^{(1)}(v\rho_s) e^{ih(z-z^0)}\, dh, \qquad (6.4.2)$$

then from boundary conditions (1.7.3) on the surface of each of the cylinders, on the basis of the addition theorem (2.4.5), we find, after substituting for $x_n^s(h)$ the unknowns $X_n^s(h)$ according to the formula,

$$x_n^s(h) = J_n(va_s) X_n^s(h), \quad n = 0, \pm 1, \ldots; \quad s = \pm 1, \qquad (6.4.3)$$

an infinite system of linear equations relative to $X_n^s(h)$:

$$X_n^{(s)}(h) + \sum_{m=-\infty}^{\infty} c_{nm}^{(-s,s)}(h) X_m^{-s}(h) = F_n^s(h), \qquad (6.4.4)$$
$$n = 0, \pm 1, \ldots; \quad s = \pm 1,$$

where

$$c_{nm}^{-s,s}(h) = \frac{J_m(va_{-s})}{H_n^{(1)}(va_s)} \times \qquad (6.4.5)$$
$$\times H_{m-n}^{(1)}(vl) e^{i(m-n)\varphi_{-s,s} + in\varphi_s^0 - im\varphi_{-s}^0},$$

$$F_n^s(h) = \frac{H_n^{(1)}(v\rho_s^0)}{H_n^{(1)}(va_s)}, \quad v = \sqrt{k^2 - h^2}. \qquad (6.4.6)$$

The determinant of the system (6.4.4) has the same form as in system (6.3.20) and may be written in the form of (6.3.23). Moreover, system (6.4.4) may be written in the normal form (6.3.31), if we introduce designations which are similar to those used in (6.3.32) - (6.3.36). In the case of a fixed value of the parameter v the difference in system (6.4.4) from (6.3.20) will be expressed only in the expressions of their right-hand parts (6.3.22) and (6.4.6) and in the insignificant difference in the expressions of the matrix elements (6.3.21) and (6.4.5) (if in (6.4.4) we substitute

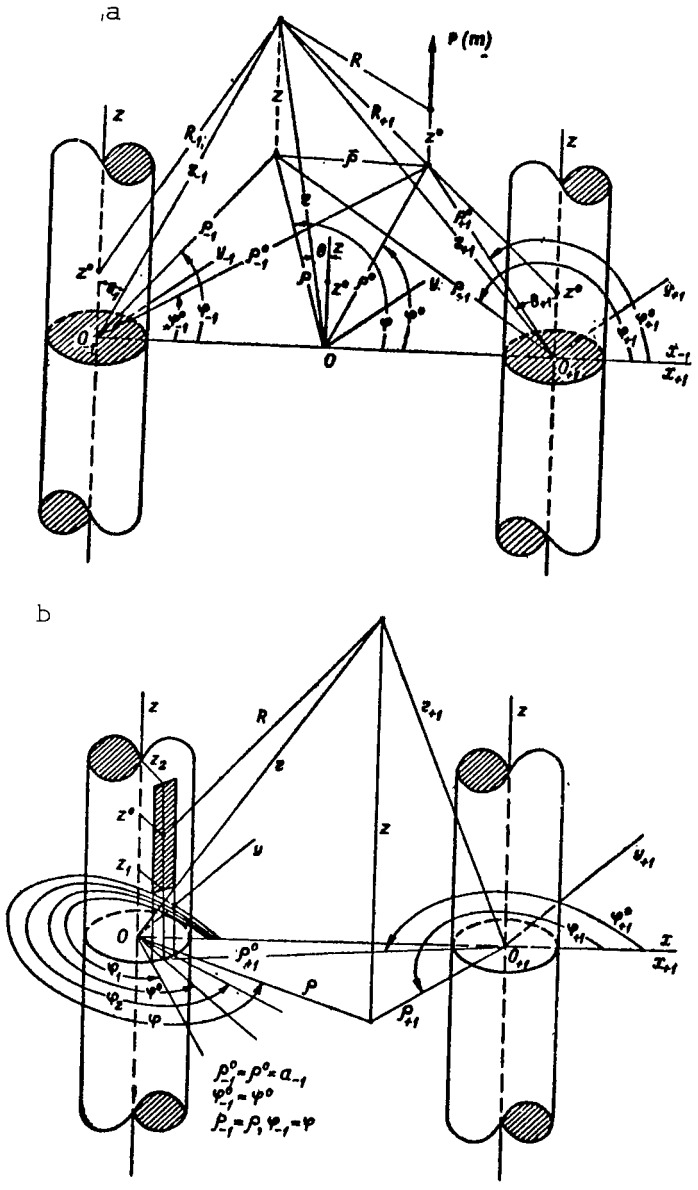

Fig. 34. Excitation of Two Circular Cylinders by a Longitudinal Dipole (a) and Generalization of the Problem to the Case of a Longitudinal Gap on One of the Cylinders (b).

$X_n^s e^{in\phi_s^0}$ for X_n^s, then (6.3.21) and (6.4.5) become identical in form). /26

When $\rho_s^0 > a_s$ the right-hand parts of system (6.4.4) satisfy the condition $\sum_{n=-\infty}^{\infty} |F_n^s(h)|^2 < \infty$. Therefore, just as in the previous case, we can show that with satisfaction of condition (6.3.29) and a fixed v the Gilbert alternative on the solvability of the infinite systems with a completely continuous form are applicable to system (6.4.4) if $\rho_s^0 > a_s$ (the dipole does not lie on the surface of one of the cylinders, $s = +1$ or $s = -1$), on the basis of which and the theorem of uniqueness of the solutions to the boundary problem for the Helmholtz equation, the unique solvability of the heterogeneous system (6.4.4) is established. Here we will have $\sum_{n=-\infty}^{\infty} |X_n^s|^2 < \infty$ (in /27(fact $|X_n^s| < \text{const}_1 \left(\frac{a_s}{\rho_s^0}\right)^{|n|} + \text{const}_2 \left(\frac{a_s}{l - a_{-s}}\right)^{|n|}$). If in (6.4.2) we substitute the new unknowns X_n^s for the unknowns x_n^s, according to the formula,

$$x_n^s(h) = \alpha_n J_n(va_s) X_n^s(h), \tag{6.4.7}$$

where $\alpha_0 = 1$ and $\alpha_n = n$, when $|n| \geq 1$, then the infinite system of equations obtained for $X_n^s(h)$ will differ from (6.4.4) only in the fact that its matrix elements are found to be equal to $\frac{\alpha_m}{\alpha_n} c_{nm}^{(-s,s)}(h)$, where $c_{nm}^{(-s,s)}(h)$ is determined by formula (6.4.5) and the free terms will be equal to $\frac{1}{\alpha_n} F_n^s(h)$, where $F_n^s(h)$ is determined by formula (6.4.6). Such a system with a fixed v, just as before, will have a completely continuous form and its right-hand parts will satisfy condition (1.9.20) as well in the case when the dipole is found to be located on the surface of one of the cylinders, i.e., when $\rho_s^0 = a_s$ ($s = +1$ or $s = -1$; from (6.4.6) it is clear that when $\rho_s^0 = a_s$, $F_n^s = 1$ and then condition (1.9.20) for the right-hand part of (6.4.4) is not satisfied).

Solution to system (6.4.4) for each v may be found by the method of truncation according to the scheme given in the previous problem.

By determining $X_n^s(h)$ the formal solution to the problem of seeking the potential u^1 is ended. Formulas (6.4.2), (6.4.4) - (6.4.6), by which the function u^1 is determined, are precise formulas. However, it is quite difficult to use them in the general case to obtain numerical results, since for satisfaction of the integration in (6.4.2) over h we must have explicit expressions for the coefficients $X_n^s(h)$ as functions of h, whereas from system (6.4.4) in practice we can find the numerical values of the quantities $X_n^s(h)$ only for a finite number of discrete values of the parameter $v^2 = k^2 - h^2$, where h may vary from $-\infty$ to $+\infty$. Therefore, in the general case, formulas (6.4.2), (6.4.6) - (6.4.4) may be used only for approximate computation of the potential u^1 by numerical methods

of integration. If rather than looking at the general case, we investigate the field only in the remote band, then the disadvantages mentioned above are eliminated since for such case we will have $v = k \sin \theta$, by virtue of which from system (6.4.4) the quantities $X_n^s(h)$ with $h = k \cos \theta$ are subject to determination.

As usual, in the remote band it is assumed that $k\rho >> 1$ ($\rho >> l$). The Fourier integrals, which enter into expressions (6.4.2) for the potential u^1, are computed by the method of steepest descents according to the same scheme as in the case of one cylinder. And if /271

$$I_n^s = \frac{i}{2} \int_{-\infty}^{\infty} x_n^s(h) H_n^{(1)}(v\rho_s) e^{ih(z-z^0)} dh, \quad s = \pm 1, \tag{6.4.8}$$

then in local coordinates of the s-th cylinder ($s = \pm 1$) in the wave band, approximately,

$$I_n^s = (-i)^n x_n^s(h) \frac{e^{ikR_s}}{R_s}, \quad h = k \cos \theta, \tag{6.4.9}$$

where (see Fig. 34, a), $\rho_s = R_s \sin \theta$, $z - z^0 = R_s \cos \theta$, but R_s and $\theta_s \approx \theta$ are spherical coordinates with the origin at point O_s. Moreover, there we have $\phi_s \approx \phi$, $\rho_s \approx \rho - sl_0 \cos \phi$ ($\rho >> l_0$), and the cylindrical coordinates ρ, ϕ are connected with the spherical coordinates θ, ϕ, r by the relationships

$$r_s = \sqrt{\rho_s^2 + z^2} = \sqrt{\rho^2 + z^2 - 2\rho s l_0 \cos \varphi + l^2 \cos^2 \varphi} \approx$$

$$\approx r \sqrt{1 - \frac{2sl_0 \sin \theta \cos \varphi}{r}} \approx r - sl_0 \sin \theta \cos \varphi$$

($\rho = z \sin \theta$, $R_s \approx r_s - z^0 \cos \theta$, $l = 2l_0$; $s = \pm 1$).

In spherical coordinates with the origin at point O, from (6.4.2) and (6.4.9) in the given approximation, we find

$$u^1 = -p \frac{e^{ikr - ikz^0 \cos \theta}}{r} \times$$

$$\times \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} (-i)^n X_n^s(h) J_n(x_s) e^{in(\varphi - \varphi_s^0) - iskl_0 \sin \theta \cos \varphi}, \tag{6.4.10}$$

where $x_s = ka_s \sin \theta$, $r \approx R + \rho^0 \sin \theta \cos (\phi - \phi^0) + z^0 \cos \theta$ and where the coefficients $X_n^s(h)$ must be found from (6.4.4) with a value of $h = k \cos \theta$.

Since [see (6.2.1)]

$$u^0 = p\frac{e^{ikR}}{R},$$

then in the same approximation and in the same coordinates for the components of the complete field we find the expressions,

$$H_\varphi = \sqrt{\frac{\varepsilon}{\mu}}\,E_\theta = -k^2 p\frac{e^{ikR}}{R}\sin\theta\sqrt{\frac{\varepsilon}{\mu}}\,A(\theta,\varphi), \tag{6.4.11}$$

in which /27

$$A(\theta,\varphi) = 1 - e^{ikp^0\sin\theta\cos(\varphi-\varphi^0)}\times \times \sum_{s=\pm 1}\sum_{n=-\infty}^{\infty}(-i)^n X_n^s(h)\,J_n(x_s)\,e^{in(\varphi-\varphi^0)-iskl_s\sin\theta\cos\varphi}. \tag{6.4.12}$$

It is obvious that the function $A(\theta, \phi)$ may be assumed as a correction factor (attenuation factor), which takes into account the presence in the field of an electric longitudinal dipole of two ideally conducting parallel circular cylinders of infinite length ($h = k\cos\theta$, $x_s = ka_s\sin\theta$). Let us write (6.4.11) in the form,

$$H_\varphi = \sqrt{\frac{\varepsilon}{\mu}}\,E_\theta = -k^2 p\sqrt{\frac{\varepsilon}{\mu}}\,\frac{e^{ikr-ikz^0\cos\theta}}{r}\sin\theta\psi(\theta,\varphi), \tag{6.4.13}$$

where

$$\psi(\theta,\varphi) = e^{-ikp^0\sin\theta\cos(\varphi-\varphi^0)}\,A(\theta,\varphi). \tag{6.4.14}$$

By integration in (6.4.13) over z^0 from z_1 to z_2, we obtain generalized solutions to the problem for the case of a linear antenna, ranging from $z = z_1$ to $z = z_2$ with a current $I = I(z^0)$ $\left(p = \frac{icI(z^0)\,dz^0}{4\pi\omega\varepsilon}\right)$, for the field of which in the remote band,

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{k\sin\theta}{4\pi i}\cdot\frac{e^{ikr}}{r}\times \times\left\{\int_{z_1}^{z_2} I(z^0)\,e^{-ikz^0\cos\theta}\,dz^0\right\}\psi(\theta,\varphi). \tag{6.4.15}$$

If we integrate in (6.4.1) and (6.4.2) over z^0 from $-\infty$ to ∞, we then obtain the expressions

$$u^0 = -\frac{I\sqrt{\frac{\mu}{\varepsilon}}}{4k}\sum_{n=-\infty}^{\infty} J_n(k\rho_s^0)\,H_n^{(1)}(k\rho_s)\,e^{in(\varphi_s-\varphi_s^0)}, \qquad (6.4.16)$$

$$\rho_s > \rho_s^0,$$

$$u^1 = \frac{I\sqrt{\frac{\mu}{\varepsilon}}}{4k}\sum_{s=\pm 1}\sum_{n=-\infty}^{\infty} X_n^s H_n^{(1)}(k\rho_s)\,J_n(ka_s)\,e^{in(\varphi_s-\varphi_s^0)}, \qquad (6.4.17)$$

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which represent potentials of the primary and secondary fields of a linear infinitely long filament with the electric current I. In approximation of the wave band,

$$u^1 = \frac{I\sqrt{\frac{\mu}{\varepsilon}}}{4k}\sqrt{\frac{2}{\pi k\rho}}\,e^{i\left(k\rho-\frac{\pi}{4}\right)} \times \qquad (6.4.18)$$

$$\times \sum_{s=\pm 1} e^{-iskl_s\cos\varphi}\sum_{n=-\infty}^{\infty} X_n^s(-i)^n J_n(ka_s)\,e^{in(\varphi-\varphi_s^0)},$$

and the only non-zero component E_2 of the complete field will be equal to

$$E_z = k^2 u, \qquad (6.4.19)$$

where $u = u^0 + u^1$.

We can see that expressions (6.4.16), (6.4.17) determine the strict solution to the problem on diffraction of a cylindrical wave $-\frac{I}{4k}\sqrt{\frac{\mu}{\varepsilon}}\,H_0^{(1)}(k\bar{\rho})$ on two ideally conducting parallel circular cylinders of infinite length, if the coefficients X_n^s are the solution to system (6.4.4) - (6.4.6) when $h = 0$, $\theta = \pi/2$. Here $\rho^{-2} = \rho_s^{0\,2} + \rho_s^2 - 2\rho_s^0\rho_s\cos(\phi_s - \phi_s^0)$ (see Fig. 34, a).

Magnetic Dipole. In the case of a longitudinal magnetic dipole we will have $u \equiv 0$ for the primary and secondary fields, but in local coordinates of the s-th cylinder the potential v^0 of the primary field is written in the form,

$$v^0 = \frac{im}{2} \sum_{n=-\infty}^{\infty} e^{in(\varphi_s - \varphi_s^0)} \times$$

$$\times \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} J_n(v\rho_s^0) H_n^{(1)}(v\rho_s), & \rho_s^0 < \rho_s \\ J_n(v\rho_s) H_n^{(1)}(v\rho_s^0), & \rho_s^0 > \rho_s \end{array} \right\} e^{ih(z-z^0)}\, dh \qquad (6.4.20)$$

[in coordinates with the origin at point O the expansion has the form of (6.2.2)]. Therefore, if we seek the potential v^1 of the secondary field in the form of analogous series of Fourier integrals,

$$v^1 = -\frac{im}{2} \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} e^{in(\varphi_s - \varphi_s^0)} \times$$

$$\times \int_{-\infty}^{\infty} x_n^{*s}(h) H_n^{(1)}(v\rho_s) e^{ih(z-z^0)}\, dh, \qquad (6.4.21)$$

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then after substitution of the unknowns $x_n^{*s}(h)$ by new unknowns $X_n^{*s}(h)$ according to the formula,

$$x_n^*(h) = \alpha_n J_n'(va_s) X_n^{*s}(h), \qquad (6.4.22)$$

where the prime denotes the differentiation over the argument, but the coefficient α_n is introduced from the same arguments as in (6.4.7), from boundary conditions (1.7.4) on the surface of each cylinder we find, using formula (2.4.5) for determining $X_n^{*s}(h)$, an infinite system of linear equations of the same type as (6.4.4) with the matrix elements,

$$c_{nm}^{(-s,s)} = \frac{\alpha_m J_m'(va_{-s})}{\alpha_n H_n^{(1)'}(va_s)} H_{m-n}^{(1)}(vl)\, e^{i(m-n)\varphi_{-s,s} + im\varphi_s^0 - in\varphi_s^0} \qquad (6.4.23)$$

and with the free terms,

$$F_n^s(h) = \frac{H_n^{(1)}(v\rho_s^0)}{\alpha_n H_n^{(1)'}(va_s)} \qquad (6.4.24)$$

($v^2 = k^2 - h^2$), which is solvable for each fixed value of v by the method of truncation. In the same way as in the previous case we find that in spherical coordinates with the origin at point O in approximation of the wave band we will have

$$v^1 = -m \frac{e^{ikr - ikz^0\cos\theta}}{r} \times \qquad (6.4.25)$$

$$\times \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} (-i)^n \alpha_n X_n^{*s}(h) J_n'(x_s) e^{in(\varphi - \varphi_s^0) - iskl_s\sin\theta\cos\varphi},$$

where $X_n^{*s}(h)$ must be found from system (6.4.4) with (6.4.23), (6.4.24) with a value of $h = k \cos\theta$. On the basis of (6.4.25) for the components of the complete field we find

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\theta = \\ = k^2 \sqrt{\frac{\mu}{\varepsilon}} m \frac{e^{ikR}}{R} \sin\theta A^*(\theta,\varphi), \tag{6.4.26}$$

where the attenuation factor $A^*(\theta, \phi)$ which takes into account the presence in the field of a magnetic longitudinal dipole of two parallel ideally conducting circular cylinders of infinite length, is equal to /275

$$A^*(\theta, \varphi) = 1 - e^{ikp^0 \sin\theta\cos(\varphi-\varphi^0)} \times \\ \times \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty} (-i)^n \alpha_n X_n^{*s}(h) J_n'(x_s) e^{in(\varphi-\varphi_s^0)-iskl_0\sin\theta\cos\varphi}. \tag{6.4.27}$$

Let us rewrite (6.4.26) in the form,

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\theta = \\ = k^2 \sqrt{\frac{\mu}{\varepsilon}} m \frac{e^{ikr-ikz^0\cos\theta}}{r} \sin\theta\psi^*(\theta, \varphi), \tag{6.4.28}$$

where

$$\psi^*(\theta, \varphi) = e^{-ikp^0\sin\theta\cos(\varphi-\varphi^0)} A^*(\theta, \varphi), \tag{6.4.29}$$

and integrate in (6.4.28) over z^0 from z_1 to z_2. As a result let us obtain generalized solutions for the case of a linear antenna, ranging from z_1 to z_2, for the field of which, in the wave band,

$$E_\varphi = \frac{ik\sin\theta}{4\pi} \cdot \frac{e^{ikr}}{r} \times \\ \times \left\{ \int_{z_1}^{z_2} K(z^0) e^{-ikz^0\cos\theta} dz^0 \right\} \psi^*(\theta, \varphi). \tag{6.4.30}$$

Formula (6.4.30) takes into account the presence in the field of a linear antenna of two parallel ideally conducting circular cylinders of infinite length; in it, $K = K(z^0)$ is the given magnetic current $\left(m = \frac{icK(z^0)\,dz^0}{4\pi\omega\mu}\right)$.

If we integrate in (6.2.2) and (6.4.21) over z^0 from $-\infty$ to ∞, we then find expressions for the potentials v^0, v^1 of the primary and secondary fields which correspond to the case of an infinitely long filament with the magnetic current K:

$$v^0 = -\frac{K\sqrt{\frac{\varepsilon}{\mu}}}{4k} \times \tag{6.4.31}$$

$$\times \sum_{n=-\infty}^{\infty} J_n(k\rho_s^0) H_n^{(1)}(k\rho_s) e^{in(\varphi_s-\varphi_s^0)}, \quad \rho_s > \rho_s^0,$$

$$v^1 = \frac{K\sqrt{\frac{\varepsilon}{\mu}}}{4k} \times \tag{6.4.32}$$

$$\times \sum_{s=\pm 1} \sum_{n=-\infty}^{\infty}{}' \alpha_n X_n^{*s} J_n'(ka_s) H_n^{(1)}(k\rho_s) e^{in(\varphi_s-\varphi_s^0)}.$$

Use of expressions (6.4.31), (6.4.32) will determine the strict solution to the problem of diffraction of the field of the cylindrical wave $\frac{K}{4k} \times \sqrt{\frac{\mu}{\varepsilon}} H_0^{(1)}(k\bar{\rho})$ on two ideally conducting circular cylinders, if the coefficients $X_n^{*s}(h)$ are found from (6.4.4), (6.4.23), (6.4.24) with a value of $h = 0$, $\theta = \pi/2$. For a stray field in the wave band in cylindrical coordinates with the origin at point θ,

$$E_\varphi^1 = \sqrt{\frac{\mu}{\varepsilon}} H_z^1 = k^2 \sqrt{\frac{\mu}{\varepsilon}} v^1, \tag{6.4.33}$$

where

$$v^1 = \frac{Kke^{ik\gamma}}{4\sqrt{\rho}} \sqrt{\frac{2.}{\pi k}} e^{-\frac{i\pi}{4}} \sum_{s=\pm 1} e^{-iskl_s\cos\varphi} \times$$

$$\times \sum_{n=-\infty}^{\infty} (-i)^n \alpha_n X_n^{*s} J_n'(ka_s) e^{in(\varphi-\varphi_s^*)}. \tag{6.4.34}$$

If in (6.4.26) ρ_s^0 tends to a_s ($s = +1$ or $s = -1$), then at the limit we obtain the solution to the problem on the field of the source, equivalent to the unit longitudinal emitting slot located on the s-th cylinder (active antenna) in the presence of another ($-s$-th) cylinder, which plays the role of the passive reflector. A similar type of maximum transition to (6.4.28) leads to generalization of this problem to the case of a source which is equivalent to the

narrow longitudinal slot with a width of $a_s d_s \phi_s^0$ located on the s-th cylinder and ranging in length from $z = z_1$ to $z = z_2$. By subsequent integration of (6.4.28) over the variable ϕ_s^0 in the range from ϕ_1 to ϕ_2 we find a further generalization of the solution to the problem to the case of a rectangular emitting slot of finite width $a_s(\phi_2 - \phi_1)$ with a given distribution of the field in it $E_\phi(a_s, \phi_s^0, z^0)$ in the opening of the slot. After substituting in (6.4.30) the distribution of the magnetic current $K = K(z^0)$ of the distribution of voltage on the slot $E_\phi(a_s, \phi_s^0, z^0) a_s d\phi_s^0$ and integrating then over $\phi_s^0 = \phi^0$ ($s = +1$ or $s = -1$) from ϕ_1 to ϕ_2, we obtain /277

$$E_\varphi = \frac{e^{ikr}}{r} S(\theta) M(\theta, \varphi), \tag{6.4.35}$$

where

$$S(\theta) = \frac{ik \sin\theta}{4\pi} \int_{z_1}^{z_2} V(z^0) e^{-ikz^0 \cos\theta} dz^0, \tag{6.4.36}$$

$$M^*(\theta, \varphi) = \frac{1}{\varphi_2 - \varphi_1} \int_{\varphi_1}^{\varphi_2} [\Phi_0(\theta, \varphi; \varphi^0) + \Phi_s(\theta, \varphi; \varphi_s^0) + \Phi_{-s}(\theta, \varphi; \varphi_{-s}^0)] d\varphi_s^0, \tag{6.4.37}$$

$$\Phi_0 = e^{ikp^* \sin\theta \cos(\varphi - \varphi^*)}, \tag{6.4.38}$$

$$\Phi_s = \sum_{n=-\infty}^{\infty} (-i)^n X_n^{*s}(h) J_n'(x_s) e^{in(\varphi - \varphi_s^0) - iskl_0 \sin\theta \cos\varphi}, \tag{6.4.39}$$

$h = k\cos\theta$, $x_s = ka_s \sin\theta$, but $V(z^0) = E_\varphi(a_s, \varphi_s^0, z^0) a_s(\varphi_2 - \varphi_1)$ is the voltage at the slot ($s = +1$ or $s = -1$).

When the coordinate system $Oxyz$ coincides with the local coordinate system of the s-th cylinder ($s = +1$ or $s = -1$),

$$\Phi_s = \sum_{n=-\infty}^{\infty} (-i)^n X_n^{*s}(h) J_n'(x_s) e^{in(\varphi - \varphi_s^0)}, \tag{6.4.40}$$

$$\Phi_{-s} = \sum_{n=-\infty}^{\infty} (-i)^n X_n^{*-s}(h) J_n'(x_{-s}) \times \times e^{in(\varphi - \varphi_{-s}^0) - iskl_0 \sin\theta \cos\varphi}. \tag{6.4.41}$$

From (6.4.4) and (6.4.23), (6.4.24) it is clear that the unknowns $X_n^{*s}(h)$ depend simultaneously both on the angle ϕ_s^0 and on the angle ϕ_{-s}^0. Therefore, in order to carry out the integration in (6.4.37) over ϕ_s^0 ($s = +1$ or $s = -1$) we must first find the explicit

expression for the dependence of the value $X_n^{*s}(h)$ on ϕ_s^0 and ϕ_{-s}^0. This may be done according to the following scheme. Let us substitute $Y_n^s(h)$ for $X_n^{*s}(h)$ after setting

$$X_n^{*s}(h) = Y_n^s(h)\, e^{in\varphi_s^0}. \tag{6.4.42}$$

As a result for $Y_n^s(h)$ from (6.4.4) we obtain an infinite system of /2 linear equations with the coefficients

$$c_{nm}^{(-s,s)} = \frac{\alpha_m J_m(x_{-s})}{\alpha_n H_n^{(1)}(x_s)} H_{m-n}^{(1)}(kl\sin\theta)\, e^{i(m-n)\varphi_{-s,s}} \tag{6.4.43}$$

instead of (6.4.23) and with right-hand parts

$$F_n^s(h) = \frac{H_n^{(1)}(k\rho_s^0\sin\theta)}{\alpha_n H_n^{(1)}(ka_s\sin\theta)}\, e^{-in\varphi_s^0} \tag{6.4.44}$$

instead of (6.4.24), in which the dependence on the angles ϕ_s^0 and ϕ_{-s}^0 is removed in the right-hand sides of (6.4.44). After solving the system, we find that

$$Y_n^s(h) = \frac{\sum_j A_{jn}F_j^s + \sum_j B_{jn}F_j^{-s}}{\Delta}, \tag{6.4.45}$$

where Δ is the determinant of the system, having the form of (6.3.23), but A_{in} and B_{in} are algebraic supplements to the elements of the i-th column of the determinant after substituting it with the column of free terms (6.4.44) of the system. On the basis of (6.4.44) formula (6.4.45) may be rewritten in the form,

$$Y_n^s(h) = \sum_j \alpha_{jn} e^{-ij\varphi_s^0} + \sum_j \beta_{jn} e^{-ij\varphi_{-s}^0}, \tag{6.4.46}$$

where $\alpha_{jn} = \frac{H_j^{(1)}(k\rho_s^0\sin\theta)}{\alpha_j H_j^{(1)}(ka_s\sin\theta)} A_{jn}$, $\beta_{jn} = \frac{H_j^{(1)}(k\rho_{-s}^0\sin\theta)}{\alpha_j H_j^{(1)}(ka_{-s}\sin\theta)} B_{jn}$. After substitution of (6.4.42) and (6.4.46) into (6.4.40), (6.4.41) for Φ_s we find expressions with an explicit dependence of the on the angles ϕ_s^0 and ϕ_{-s}^0. The angle ϕ_{-s}^0 may be expressed through ϕ_s^0 if we proceed from simple geometric arguments (equally ρ_{-s}^0 may be expressed through ρ_s^0). The functions $H_j^{(1)}(k\rho_{-s}^0\sin\theta)\, e^{ij\varphi_{-s}^0}$ may be expressed through $H_j^{(1)}(k\rho_s^0\sin\theta)\, e^{ij\varphi_s^0}$ and with the aid of the addition theorem (2.4.5) which however leads to more awkward expressions for Φ_s.

By using formulas (6.4.35) - (6.4.46) we can investigate the influence of the passive reflector ($-s$-th cylinder) on the characteristics of the emission from the longitudinal slot, located on the surface of the s-th cylinder, as a function of the radii of the cylinders, the distance between them, the orientation of the slot, etc. (see Fig. 34, b).

Let us note that if the slot is sufficiently narrow, when the distribution of voltage on the slot in it is almost sinusoidal and is given by the formula, /279

$$V(z) = \frac{V_0 \sin(k - |z|)}{\sin kl_1},$$

where V_0 is the voltage in the middle of the slot, and $z_1 = l_1$, $z_2 = l_1$, then

$$S(\theta) = \frac{iV_0'}{2\pi} \cdot \frac{\cos(kl_1 \cos\theta) - \cos kl_1}{\sin kl_1 \sin\theta}.$$

If, moreover, the length of the slot is equal to half the wave length ($2l_1 = \lambda_0/2$), then

$$S(\theta) = \frac{iV_0}{2\pi} \cdot \frac{\cos\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta}.$$

The resistance of the emission from the dipole field in the presence of two cylinders as well as the conductivity of the emission from the slot, located on the surface of the s-th cylinder in the presence of the other ($-s$-th) cylinder ($s = \pm 1$), can be computed numerically by integration according to formulas (1.4.14), (1.4.19), if we carry out the integration there over a sphere of infinitely large radius with the center at the origin of one of the coordinate systems.

From comparison of (6.4.10) with (6.4.18) and (6.4.25) with (6.4.32) it is clear that when $\theta = \pi/2$ (the plane Oxy) and when $r = \rho$, formulas (6.4.10), (6.4.25) are identical to formulas (6.4.18), (6.4.32), respectively. Consequently, in the remote zone of the plane Oxy, the characteristics of the radiation from the longitudinal dipole in the presence of two parallel circular cylinders are similar to the characteristics of the radiation of an infinitely long filament with the current parallel to the axes of the cylinders.

In the case when the distance between cylinders l is sufficiently high in comparision with the radii of the cylinders (a_{-1}, $a_{+1} \neq 0$), in the same manner as in §3 of this chapter, for the

unknown values $X_n^{*s}(h)$ we can obtain the formula,

$$X_n^{*s}(h) = F_n^s + Lb_n^s \frac{LQ^{-s}F^s - F^{-s}}{1 - [Q^{-s}Q^sL^2]}, \qquad (6.4.47)$$

by which the coefficients of scattering of the secondary waves of all successive orders of scattering are approximately summed. Here F_n^s are determined by formula (6.4.24), L is the same as in (6.3.64), and

$$b_n^s = \frac{e^{in(\pi/2-\varphi_{-s,s}+\varphi_s^0)}}{\alpha_n H_n^{(1)'}(va_s)},$$

$$Q^s = \sum_{n=-\infty}^{\infty} (-1)^n \frac{J_n'(va_s)}{H_n^{(1)'}(va_s)}$$

$$F^s = \sum_{n=-\infty}^{\infty} \frac{J_n'(va_s)}{H_n^{(1)'}(va_s)} H_n^{(1)}(v\rho_s^0)\, e^{-in(\varphi_s^0-\varphi_{s,-s}+\pi/2)}, \qquad (6.4.48)$$

$$v^2 = k^2 - h^2.$$

In the wave band in formulas (6.4.48) it follows to set $h = k\sin\theta$. Formula (6.4.47) can be used for analysis of the diffracted field with values of $kl \sin\theta >> 1$, if l is much greater than $a = \max_{s=\pm 1} \{a_s\}$ (a_{-1}, $a_{+1} \neq 0$). This case is studied more completely in [95, 102].

In conclusion, let us note that if we introduce a new substitution of unknown coefficients $x_n^s(h)$, from (6.4.2) and $x_n^{*s}(h)$ from (6.4.21), similar to (6.3.106), (6.3.107), then seeking the approximate values of the diffraction coefficients may be reduced to solving systems such as (6.3.95), (6.3.96) with positive indices n and m.

CHAPTER 7

DIFFRACTION OF ELECTROMAGNETIC WAVES ON SPHERES

§1. Diffraction of a Plane Wave on One Sphere.

In an unbounded uniform and isotropic space with permeabilities ε, μ and a conductivity of $\sigma = 0$, let there be contained a sphere with permeabilities ε_2, μ_2 and conductivity σ_2, occupying (in the spherical system of coordinates r, θ, ϕ) a part of the space $r \leq a$, on which a plane electromagnetic wave (1.3.4) is incident and which has (in the Cartesian coordinates x, y, z)(origin of the system $Oxyz$ coincides with the center of the sphere, Fig. 35) only the non-zero components, /281

$$E_x^0 = \sqrt{\frac{\mu}{\varepsilon}} H_y^0 = Ee^{ikz} \qquad (7.1.1)$$

In spherical coordinates,

$$E_x^0 = Ee^{ikr\cos\theta}. \qquad (7.1.2)$$

The problem is formulated of seeking the secondary fields both outside and inside the sphere. In spherical coordinates this problem is reduced to solving two scalar independent boundary problems for the electric and magnetic Debye potentials u and v, which satisfy the Helmholtz equation given by the boundary condition on the surface of the sphere and the supplementary condition of radiation at infinity, superposed on the potentials of the secondary field scattered on the sphere. Here we can define two cases, when $\sigma_2 \neq \infty$ and when $\sigma_2 = \infty$. In the first of these, the complete electromagnetic field $E = E^0 + E^1$, $H = H^0 + H^1$ outside the sphere (E^1, H^1 is the secondary field, scattered on the sphere) just as the field inside the sphere E^2, H^2 may be found through the respective Debye potentials u, v from formulas (1.6.15) written in spherical coordinates (in each of the regions the electromagnetic field is assumed to be the result of a superposition of fields of electric and magnetic types (1.6.22), (1.6.23), and $k = k_0\sqrt{\mu\varepsilon}$ outside the sphere and $k^2 = k_2^2 = (\omega^2\mu_2\varepsilon_2 + 4i\pi\omega\mu_2\sigma_2)/c^2$ inside it). On the surface of the sphere the Debye potentials must satisfy the conjugate boundary conditions (1.7.10), (1.7.11). In the second case the electromagnetic field inside the sphere is equal to zero ($u^2 \equiv 0$, $v^2 \equiv 0$), but outside the sphere the field is found from formulas (1.6.15) written in spherical coordinates. Here on the surface of the sphere the potentials must satisfy boundary conditions (1.7.13). In both cases the solution to the scalar boundary problems assumes knowledge of the explicit expressions of the electric u^0 and magnetic v^0 Debye poten- /282

tials[1] of the primary field of the incident wave. They are found in the following manner.

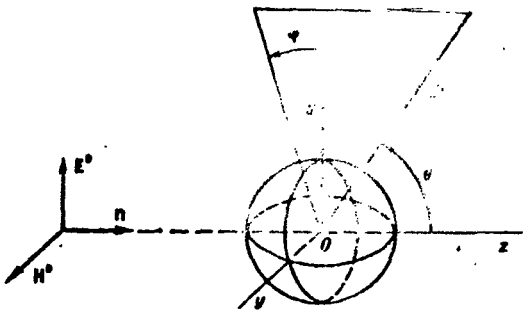

Fig. 35. Diffraction of a Plane Wave on a Sphere.

From (1.5.35), (7.1.2) it is clear that the radial component E_r^0 of the vector of the electric field intensity of the incident wave is coupled with the Cartesian component E_x^0 by the relationship,

$$E_r^0 = E_x^0 \sin\theta\cos\varphi = \frac{\cos\varphi}{ikr}\cdot\frac{\partial E_x^0}{\partial\theta}, \quad (7.1.3)$$

where the function E_x^0 can be represented, on the basis of (3.6.9), by the expansion,

$$E_x^0 = E\sum_{n=0}^{\infty} i^n(2n+1)j_n(kr)P_n(\cos\theta). \quad (7.1.4)$$

Therefore,

$$E_r^0 = \frac{E\cos\varphi}{ikr}\sum_{n=1} i^n(2n+1)j_n(kr)P_n(\cos\theta), \quad (7.1.5)$$

since $P_n^1(\cos\theta) = \dfrac{dP_n(\cos\theta)}{d\theta}$, but $P_0^1(\cos\theta) = 0$. If we seek the electric potential u^0 in the form of a series, /283

$$u^0 = \sum_{n=1}^{\infty} a_n j_n(kr)P_n^1(\cos\theta)\cos\varphi, \quad (7.1.6)$$

then from comparison of (7.1.5) with the expression E_r^0, obtained through (7.1.6) from (1.6.25) and (1.6.22), we find

$$a_n = \frac{Ei^n}{ik}\cdot\frac{2n+1}{n(n+1)}, \quad (7.1.7)$$

since

$$\left(\frac{d^2}{dr^2}+k^2\right)\left\{\begin{matrix}\psi_n(kr)\\ \zeta_n(kr)\end{matrix}\right\} = \frac{n(n+1)}{r^2}\left\{\begin{matrix}\psi_n(kr)\\ \zeta_n(kr)\end{matrix}\right\}, \quad (7.1.8)$$

where in Debye designations,

---

[1]Below, the Debye potentials $\bar{u}$ and $\bar{v}$ (see Ch.1) are denoted through u and v, respectively, which should be taken into account in converting to (1.6.15), (1.6.22), (1.6.23).

$$\psi_n(x) = xj_n(x) = \sqrt{\frac{\pi}{2x}}\, J_{n+1/2}(x),$$
$$\zeta_n(x) = xh_n^{(1)}(x) = \sqrt{\frac{\pi}{2x}}\, H_{n+1/2}^{(1)}(x). \qquad (7.1.9)$$

Therefore, for the electric field of the incident wave,

$$u^0 = \frac{E\cos\varphi}{ik^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\psi_n(kr)P_n^1(\cos\theta). \qquad (7.1.10)$$

Since $H_y^0 = \sqrt{\frac{\varepsilon}{\mu}}E_x^0$, but from (1.5.35)

$$H_r^0 = H_y^0\sin\theta\sin\varphi = -\frac{\sin\varphi}{ikr}\cdot\frac{\partial E_x^0}{\partial\theta},$$

then after representing the potential v^0 in the form of a series,

$$v^0 = \sum_{n=1}^{\infty} b_n j_n(kr)P_n^1(\cos\theta)\sin\varphi$$

from (1.6.25), (1.6.23) we find

$$b_n = \frac{Ei^n}{ik}\cdot\frac{2n+1}{n(n+1)}\sqrt{\frac{\varepsilon}{\mu}},$$

and then for the magnetic field,

$$v^0 = \frac{E\sin\varphi}{ik^2r}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\psi_n(kr)P_n^1(\cos\theta). \qquad (7.1.11)$$ /284

An Ideally Conducting Sphere. In the case of an ideally conducting sphere the electromagnetic field inside the sphere is identically equal to zero, but the Debye potentials of the complete field outside the sphere $u = u^0 + u^1$ and $v = v^0 + v^1$, boundary conditions (1.7.13) will be satisfied on its surface. Therefore, if we seek the potentials of the secondary field in the form of a series such as (3.1.15), by assuming

$$u^1 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{mn} h_n^{(1)}(kr) P_n^m(\cos\theta) e^{im\varphi}, \qquad (7.1.12)$$

$$v^1 = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} B_{mn}^* h_n^{(1)}(kr) P_n^m(\cos\theta) e^{im\varphi}, \qquad (7.1.13)$$

we then arrive at the conclusion that conditions (1.7.13) will be satisfied if in (7.1.12), (7.1.13) the coefficients $B_{mn} \equiv B_{mn}^* \equiv 0$ for all $m \neq \pm 1$, but when $m = \pm 1$,

$$B_{-1,n} = -n(n+1) B_{1n},$$

$$B_{1n} = -\frac{i^n E(2n+1)}{2ikn(n+1)} \cdot \frac{\psi_n'(ka)}{\zeta_n'(ka)},$$

$$B_{-1,n}^* = n(n+1) B_{1n}^*, \qquad (7.1.14)$$

$$B_{1n}^* = \sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{i^n E(2n+1)}{2kn(n+1)} \cdot \frac{\psi_n(ka)}{\zeta_n(ka)},$$

where $n = 1, 2, \ldots,$; therefore,

$$u^1 = -\frac{E\cos\varphi}{ik^2 r} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n'(ka)}{\zeta_n'(ka)} P_n^1(\cos\theta), \qquad (7.1.15)$$

$$v^1 = -\sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{E\sin\varphi}{ik^2 r} \times$$

$$\times \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n(ka)}{\zeta_n(ka)} P_n^1(\cos\theta), \qquad (7.1.16)$$

and the potentials of the secondary field are represented by the expressions, /285

$$\left.\begin{aligned} u &= E\cos\varphi \\ v &= \sqrt{\frac{\varepsilon}{\mu}} E\sin\varphi \end{aligned}\right\} V(r, \theta), \qquad (7.1.17)$$

in which

$$V(r, \theta) = \frac{1}{ik^2 r} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \psi_n(kr) - \frac{\Omega\psi_n(x)}{\Omega\zeta_n(x)} \zeta_n(kr) \right\} P_n^{(1)}(\cos\theta) \qquad (7.1.18)$$

(Ω has the same meaning as before; in the electric case $\Omega = \frac{\partial}{\partial x}$, but in the magnetic case, $\Omega = 1$, $x = ka$).

By determining the field potentials the mathematical solution to the problem is practically finished. The components of the particular electric and magnetic fields may be found through (7.1.17) from formulas (1.6.22) and (1.6.23), but the components of the complete field may be found from formulas (1.6.15) written in spherical coordinates. For the complete field,

$$E_r = E\cos\varphi\left\{\frac{\partial^2}{\partial r^2}+k^2\right\}\left(r\frac{\partial P}{\partial\theta}\right),$$

$$E_\theta = E\cos\varphi\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial^2 P}{\partial\theta^2}\right)+\frac{ik}{\sin\theta}\cdot\frac{\partial Q}{\partial\theta}\right\},$$

$$E_\varphi = -E\sin\varphi\left\{\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial P}{\partial\theta}\right)+ik\frac{\partial^2 Q}{\partial\theta^2}\right\}, \qquad (7.1.19)$$

$$H_r = \sqrt{\frac{\varepsilon}{\mu}}E\sin\varphi\left\{\frac{\partial^2}{\partial r^2}+k^2\right\}\left(r\frac{\partial Q}{\partial\theta}\right),$$

$$H_\theta = \sqrt{\frac{\varepsilon}{\mu}}E\sin\varphi\left\{\frac{ik}{\sin\theta}\cdot\frac{\partial P}{\partial\theta}+\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial^2 Q}{\partial\theta^2}\right)\right\},$$

$$H_\varphi = \sqrt{\frac{\varepsilon}{\mu}}E\cos\varphi\left\{ik\frac{\partial^2 P}{\partial\theta^2}+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial r}\left(r\frac{\partial Q}{\partial\theta}\right)\right\}, \qquad (7.1.20)$$

where through P and Q we denote the Fok potentials [49] of the complete field, coupled with the Debye potentials u and v by the formulas, /286

$$u = E\cos\varphi\frac{\partial P}{\partial\theta}, \quad v = \sqrt{\frac{\varepsilon}{\mu}}E\sin\varphi\frac{\partial Q}{\partial\theta},$$
$$P = P^0 + P^1, \quad Q = Q^0 + Q^1. \qquad (7.1.21)$$

For the primary field

$$P^0 = \frac{i}{k^2 r}\sum_{n=1}^{\infty} i^n\frac{2n+1}{n(n+1)}\psi_n(kr)P_n(\cos\theta),$$
$$Q^0 = \frac{i}{k^2 r}\sum_{n=1}^{\infty} i^n\frac{2n+1}{n(n+1)}\psi_n(kr)P_n(\cos\theta), \qquad (7.1.22)$$

and for the stray field

$$P^1 = -\frac{i}{k^2 r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n'(ka)}{\zeta_n'(ka)} P_n(\cos\theta)\zeta_n(kr),$$
$$Q^1 = -\frac{i}{k^2 r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n(ka)}{\zeta_n(ka)} P_n(\cos\theta)\zeta_n(kr). \qquad (7.1.23)$$

The derivatives of the Legendre polynomials $P_n(\cos\theta)$ which enter into (7.1.19), (7.1.20) are computed for θ from the formulas,

$$\frac{dP_n(\cos\theta)}{d\theta} = -P_n^1(\cos\theta),$$
$$\frac{d^2P_n(\cos\theta)}{d\theta^2} = \cot\theta P_n^1(\cos\theta) - n(n+1)P_n(\cos\theta),$$
$$\lim_{\theta\to 0}\frac{d^2P_n(\cos\theta)}{d\theta^2} = \qquad (7.1.24)$$
$$= \lim_{\theta\to 0}\frac{1}{\sin\theta}\cdot\frac{dP_n(\cos\theta)}{d\theta} = -\frac{n(n+1)}{2},$$
$$\lim_{\theta\to\pi}\frac{d^2P_n(\cos\theta)}{d\theta^2} =$$
$$= -\lim_{\theta\to\pi}\frac{1}{\sin\theta}\cdot\frac{dP_n(\cos\theta)}{d\theta} = -(-1)^n\frac{n(n+1)}{2}.$$

In approximation of the wave band the radial components of the stray field decrease as r^{-2}, and in this approximation they are equal to /287 zero, but the transverse components are equal to

$$E_\theta^1 = \sqrt{\frac{\mu}{\varepsilon}}H_\varphi^1 = \frac{iE\cos\varphi}{kr}e^{ikr}S(\theta),$$
$$-E_\varphi^1 = \sqrt{\frac{\mu}{\varepsilon}}H_\theta^1 = \frac{iE\sin\varphi}{kr}e^{ikr}S^*(\theta), \qquad (7.1.25)$$

where the amplitude functions $S(\theta)$ and $S^*(\theta)$ are determined by the expressions,

$$S(\theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$
$$\times\left\{\frac{\psi_n'(ka)}{\zeta_n'(ka)}\tau_n(\cos\theta) + \frac{\psi_n(ka)}{\zeta_n(ka)}\pi_n(\cos\theta)\right\}, \qquad (7.1.26)$$

$$S^*(\theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \frac{\psi_n'(ka)}{\zeta_n'(ka)} \pi_n(\cos\theta) + \frac{\psi_n(ka)}{\zeta_n(ka)} \tau_n(\cos\theta) \right\}. \qquad (7.1.27)$$

Here

$$\tau_n = \frac{d}{d\theta} P_n^1(\cos\theta) = -\frac{d^2}{d\theta^2} P_n(\cos\theta),$$

$$\pi_n' = \frac{1}{\sin\theta} P_n^1(\cos\theta) = -\frac{1}{\sin\theta} \frac{d}{d\theta} P_n(\cos\theta). \qquad (7.1.28)$$

The law governing the density distribution j of the surface currents, induced on the surface of the sphere, is found from (1.1.7), whence in spherical coordinates when $r = a$,

$$j_\theta = -\frac{c}{4\pi} H_\varphi, \quad j_\varphi = \frac{c}{4\pi} H_\theta \qquad (7.1.29)$$

(H is the vector of the complete magnetic field). Proceeding from (7.1.20) and taking into account that the Wronskian determinant

$$W[\psi_n(x), \quad \zeta_n(x)] = i, \qquad (7.1.30)$$

we find

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$$j_\theta = \frac{Ec\cos\varphi}{4\pi ka} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \frac{i\tau_n(\cos\theta)}{\zeta_n'(ka)} - \frac{\pi_n(\cos\theta)}{\zeta_n(ka)} \right\}, \qquad (7.1.31)$$

$$j_\varphi = \frac{Ec\sin\varphi}{4\pi ka} \sqrt{\frac{\varepsilon}{\mu}} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \frac{i\pi_n(\cos\theta)}{\zeta_n'(ka)} - \frac{\tau_n(\cos\theta)}{\zeta_n(ka)} \right\}. \qquad (7.1.32)$$

The cross section of back scattering (radar cross section) for the sphere, according to the definition, will be equal to

$$\sigma_B = \lim_{r\to\infty} 4\pi r^2 \left|\frac{E_\theta^1}{E_\theta^0}\right|^2, \qquad (7.1.33)$$

where the value E_θ^1 is taken in the direction to the source ($\theta = \pi$, $\phi = 0$), or

$$\sigma_B = \frac{4\pi}{k^2}|S(\pi)|^2. \qquad (7.1.34)$$

From (7.1.25) it is clear that in the wave band the vectors E^1, H^1 are mutually perpendicular. From these same formulas it is clear that the stray field is linearly polarized only in the directions, lying in the planes parallel to the plane $\phi = 0$, π or the plane $\phi = \frac{\pi}{2}, \frac{3\pi}{2}$. In the directions which differ from the above, the field is elliptically polarized and for them the value of the scattering cross section (two-position) is equal to

$$\sigma = \sigma_\theta + \sigma_\varphi, \qquad (7.1.35)$$

where

$$\sigma_\theta = \frac{4\pi}{k^2}\cos^2\varphi\,|S(\theta)|^2, \qquad (7.1.36)$$

$$\sigma_\varphi = \frac{4\pi}{k^2}\sin^2\varphi\,|S^*(\theta)|^2. \qquad (7.1.37)$$

The complete scattering cross section is computed from the formula, /289

$$\sigma_S = \frac{4\pi}{k^2}\operatorname{Re} S(0). \qquad (7.1.38)$$

Infinite series of spherical wave functions, through which all values given above are expressed, are suitable for proving the numerical computations in the case when the radius of the sphere does not substantially exceed the wave length λ_0 ($ka = 2\pi a/\lambda_0$ is of an order of magnitude $\lesssim 10$). The convergence of the series is improved with decrease in the values of the parameter ka, and when $ka \ll 1$ the basic contribution to the value of the sum of the series is made by the term with the summation index $n = 1$. With increase in ka the convergence of the series is worsened and with large values of ka ($ka \ll 1$), to obtain satisfactory numerical results in the series we must approximately take into account the ka of the first terms, which leads to a large amount of computational work. However, in this case we can obtain approximate formulas, which are

suitable for numerical computations if we apply the asymptotic methods of summation to the series, for example, the Watson transform [48, 143] or the Fok method [68, 69, 71-74].

Let us represent the function $V(r, \theta)$ from (7.1.17) in the form

$$V(r, \theta) = \frac{\partial \tilde{W}}{\partial \theta}, \tag{7.1.39}$$

where

$$\tilde{W} = P \tag{7.1.40}$$

in the case of the electric field and

$$\tilde{W} = Q \tag{7.1.41}$$

for the magnetic field, and we can write the amplitude functions $S(\theta)$, $S^*(\theta)$ of the stray field in the form,

$$\begin{aligned} S(\theta) &= -\frac{\partial^2 P_2}{\partial \theta^2} - \frac{1}{\sin\theta} \cdot \frac{\partial Q_2}{\partial \theta}, \\ S^*(\theta) &= -\frac{1}{\sin\theta} \frac{\partial P_2}{\partial \theta} - \frac{\partial^2 Q_2}{\partial \theta^2}, \end{aligned} \tag{7.1.42}$$

where

$$\begin{aligned} P_2 &= \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n'(ka)}{\zeta_n'(ka)} P_n(\cos\theta) = \\ &= \lim_{r\to\infty} (-k^2 r) e^{-ikr} P^1, \\ Q_2 &= \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n'(ka)}{\zeta_n'(ka)} P_n(\cos\theta) = \\ &= \lim_{r\to\infty} (-k^2 r) e^{-ikr} Q^1. \end{aligned} \tag{7.1.43}$$

Then using the Watson transform for the function $\tilde{W}$ we obtain asymptotic expressions which have in the region of the geometric shadow ($0 < \theta < \pi/2$) the form ($x = ka$), /290

$$\tilde{W}(r,\ \theta)=\frac{x^{-\frac{7}{6}}e^{i\left[k\sqrt{r^2-a^2}+\frac{\pi}{3}\right]}}{k^2\sqrt{2r\sin\theta}\,(r^2-a^2)^{1/4}}\sum_s\frac{C_s}{1+e^{2\pi i\nu_s}}\times$$

$$\times\left\{e^{i\left[\nu_s\left(\theta-\arccos\frac{a}{r}+\frac{\pi}{2}\right)-\frac{\pi}{4}\right]}+e^{i\left[\nu_s\left(\frac{\pi}{2}-\arccos\frac{a}{r}-\theta\right)+\frac{\pi}{4}\right]}\right\} \qquad (7.1.44)$$

or

$$\tilde{W}(r,\ \theta)=\frac{x^{-\frac{2}{3}}\sqrt{\pi}e^{i\left[k\sqrt{r^2-a^2}+\frac{\pi}{3}\right]}}{k^2\sqrt{r}\,(r^2-a^2)^{1/4}}\sum_s\frac{C_s}{1+e^{2\pi\nu_s i}}\times \qquad (7.1.45)$$

$$\times e^{i\nu_s\left(\frac{\pi}{2}-\arccos\frac{a}{r}\right)}J_0(\nu_s\theta)$$

depending on the position of the observation point relative to the beam $\theta = 0$. Here through ν_s we denote, just as in the problem of diffraction of a plane wave on a cylinder, the zeros of the functions $\Omega H_{\nu}^{(1)}(x)$, whose approximate values may be found from (2.2.41) or (2.2.42), but the C_s values have the same form as (6.1.37) or (6.1.38), depending on the character of the Debye potential in (7.1.17) for which the asymptotic expressions (7.1.44), (7.1.45) are constructed. In the case of the electric potential the values ν_s and C_s are determined by formulas (2.2.42) and (6.1.38), but in the case of the magnetic potential v the values ν_s and C_s are found from formulas (2.2.41), (6.1.37). In the physical interpretation, expressions (7.1.44), (7.1.45) represent sums of waves of residues (creeping waves), enveloping the sphere. When $x >> 1$, series (7.1.44) converges very rapidly, if the observation point, being located in the region of the geometric shadow behind the sphere, does not belong to the vicinity of the beam $\theta = 0$ (focal beam), along which the function $\sin\theta$, found in the denominator of (7.1.44), vanishes (waves of residues are focused along the beam $\theta = 0$). In the vicinity of the focal beam $\theta = 0$ the asymptotic representation (7.1.44) becomes invalid and unsuitable for use, since it is derived on the assumption that $|\nu_s\sin\theta| >> 1$. The field distribution in the /291
region of the geometric shadow near the focal line $\theta = 0$ for large values of x is found using the asymptotic expression (7.1.45), where $J_0(x)$ is the Bessel function of zero-th order.

In the wave band of the shadow region ($r \to \infty$) in the direction of propagation of the incident plane wave ($\theta = 0$), the exponent in (7.1.45), which until now guarantees convergence of the series, vanishes. Therefore, formula (7.1.45) is unsuitable for studying the behavior of the field in the wave band of the region of the geometric shadow. It is not suitable, for the same reason, for computing the scattering cross section σ_s. The convergence of series (7.1.44) is upset near the boundary which separates the region

of the geometric shadow from the illuminated region, i.e., in the vicinity of $\theta = \pi/2$, where formula (7.1.45), derived on the assump-

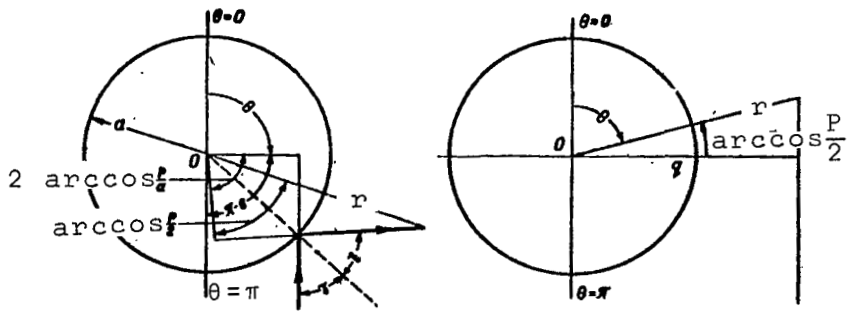

Fig. 36. Reflection From a Sphere According to Geometric Optics.

tion that $|\nu_s\theta|$ is not very high in comparison with unity, is also /292 not suitable.

In the illuminated region, asymptotically ($\pi/2 < \theta < \pi$),

$$\tilde{W}(r, \theta) = w_g(r, \theta) + w(r, \theta), \tag{7.1.46}$$

where

$$w_g(r, \theta) = \frac{e^{ik\sqrt{r^2-q^2}+\frac{\pi i}{2}}}{k(kq)^2} \mp \frac{e^{ik(\sqrt{r^2-p^2}-2a\cos\gamma)+\frac{\pi i}{2}}}{k(kp)^2} \times$$
$$\times \{a^2 \sin 2\gamma [4r \sin\theta \sqrt{r^2 - a^2\sin^2\gamma} - 2a\cos\gamma]^{-1}\}^{1/2} \tag{7.1.47}$$

(here $\sqrt{a^2 - p^2} = a\cos\gamma$, $\sqrt{r^2 - q^2} = r\cos\theta$, where $p = a\sin\gamma$, and $q = r\sin\theta$; Fig. 36), but

$$w(r, \theta) = \frac{x^{-\frac{7}{6}} e^{i\left[k\sqrt{r^2-a^2}-\frac{\pi}{6}\right]}}{k^2\sqrt{2r\sin\theta}(r^2-a^2)^{1/4}} \sum_s \frac{C_s}{1+e^{2\pi i \nu_s}} \times$$
$$\times \left\{ e^{i\left[\nu_s\left(\theta-\arccos\frac{a}{r}+\frac{\pi}{2}\right)+\frac{\pi}{4}\right]} + e^{i\left[\nu_s\left(\frac{5\pi}{2}-\arccos\frac{a}{r}-\theta\right)-\frac{\pi}{4}\right]} \right\}, \tag{7.1.48}$$

if θ does not belong to the vicinity of the focal beam $\theta \blacksquare \pi$, along which $\sin\theta$ vanishes in the denominator of (7.1.48). In the vicinity of the focal beam $\theta = \pi$ the field distribution is found using the expressions,

$$w(r,\ \theta)=\frac{\sqrt{\pi}\,x^{-\frac{2}{3}}e^{i\left[k\sqrt{r^2-a^2}-\frac{\pi}{6}\right]}}{k^2\sqrt{r}\,(r^2-a^2)^{1/4}}\sum_s\frac{C_s}{1+e^{2\pi i\nu_s}}\times \tag{7.1.49}$$

$$\times e^{i\nu_s\left(\frac{3\pi}{2}-\arccos\frac{a}{r}\right)}J_0[\nu_s(\pi-\theta)].$$

Formula (7.1.48) is valid on the assumption that $|\nu_s \sin\theta| >> 1$, whereas formula (7.1.49) can be used only on the assumption that the value of $|\nu_s(\theta - \pi)|$ is not too high in comparison with unity ($x >> 1$). The values of ν_s and C_s in (7.1.48), (7.1.49) have the same meaning as before. Both formulas (7.1.48), (7.1.49) are suitable for computing the field in the wave band. When $r \to \infty$, they assume the form,

$$w(r,\ \theta)=\frac{x^{-\frac{7}{6}}e^{i\left(kr-\frac{\pi}{6}\right)}}{k^2r\sqrt{2\sin\theta}}\sum_s\frac{C_s}{1+e^{2\pi i\nu_s}}\times \tag{7.1.50}$$

$$\times\left\{e^{i\left(\nu_s\theta+\frac{\pi}{4}\right)}+e^{i\nu_s(2\pi-\theta)-\frac{i\pi}{4}}\right\}$$

and

$$w(r,\ \theta)=\frac{\sqrt{\pi}\,x^{-\frac{2}{3}}e^{ikr-i\frac{\pi}{6}}}{k^2r}\times$$

$$\times\sum_s\frac{C_s}{1+e^{2\pi i\nu_s}}e^{\pi i\nu_s}J_0[\nu_s(\pi-\theta)]. \tag{7.1.51}$$

respectively. Function (7.1.47) in the wave band is given by the expression,

$$w_g(r,\ \theta)=\frac{i}{k}\cdot\frac{e^{ikr\cos\theta}}{(kr\sin\theta)^2}\mp\frac{e^{ikr}}{k^2r}\cdot\frac{xe^{-2ix\sin\frac{\theta}{2}+\frac{i\pi}{2}}}{2\left(x\cos\frac{\theta}{2}\right)^2}, \tag{7.1.52}$$

where the subscript is taken in the case of the electric field and the superscript in the case of the magnetic field. /293

Using these asymptotic representations of the function $\tilde{W}(\theta,r)$ and relationships (7.1.39), (7.1.19), (7.1.20), through it we can find the asymptotic expressions for the electric and the magnetic Debye potentials, and through them we can find the expressions of the components of the electromagnetic field. In particular, for

the amplitude functions $S(\theta)$, $S^*(\theta)$ of the stray field in the wave band of the illuminated region, from (7.1.43) and (7.1.46) we obtain (when $r \to \infty$) the asymptotic expressions

$$S(\theta) = \frac{x}{2} e^{-2ix\sin\frac{\theta}{2} - \frac{\pi i}{2}} + \frac{x^{\frac{5}{6}} e^{\frac{i\pi}{12}}}{\sqrt{2\sin\theta}} \times$$
$$\times \sum_s \frac{C_s}{1 + e^{2\pi i \nu_s}} \left\{ e^{i\nu\left(\frac{\pi}{2}+\theta\right)} - ie^{i\nu_s(2\pi-\theta)} \right\} \qquad (7.1.53)$$

and

$$S^*(\theta) = \frac{x}{2} e^{-2ix\sin\frac{\theta}{2} + \frac{\pi i}{2}} + \frac{x^{\frac{5}{6}} e^{\frac{i\pi}{12}}}{\sqrt{2\sin\theta}} \times$$
$$\times \sum_s \frac{C_s}{1 + e^{2\pi i \nu_s}} \left\{ e^{i\nu_s\left(\frac{\pi}{2}+\theta\right)} - ie^{i\nu_s(2\pi-\theta)} \right\}, \qquad (7.1.54)$$

where $x = ka$. In (7.1.53) C_s and ν_s correspond to the case of the electric field, but in (7.1.54) to that of the magnetic field. Expressions (7.1.53), (7.1.54) are derived on the assumption that $x \cos\frac{\theta}{2} \sin\theta >> 1$, therefore they become invalid in the vicinity of the focal beam $\theta = \pi$, when instead of (7.1.53), (7.1.54) it follows to use the asymptotic representations,

$$S(\theta) = \frac{x}{2} e^{-i\left(2x\sin\frac{\theta}{2} + \frac{\pi}{2}\right)} - \frac{\sqrt{\pi}\, x^{4/3} e^{-i\pi/3}}{2} \sum_s \left\{ \left\{ \frac{C_s e^{\pi i \nu_s}}{1 + e^{2\pi i \nu_s}} \times \right.\right.$$
$$\left. \times [J_0[\nu_s(\pi-\theta)] - J_2[\nu_s(\pi-\theta)]] \right\}_e -$$
$$\left. - \left\{ \frac{C_s e^{i\pi\nu_s}}{1 + e^{2\pi i \nu_s}} [J_0[\nu_s(\pi-\theta)] + J_2[\nu_s(\pi-\theta)]] \right\}_m \right\}, \qquad (7.1.55)$$

$$S^*(\theta) = \frac{x}{2} e^{-i\left(2x\sin\frac{\theta}{2} - \frac{\pi}{2}\right)} -$$
$$- \frac{\sqrt{\pi}\, x^{4/3} e^{-i\pi/3}}{2} \sum_s \left\{ \left\{ \frac{C_s e^{\pi i \nu_s}}{1 + e^{2\pi i \nu_s}} \times \right.\right.$$
$$\left. \times [J_0[\nu_s(\pi-\theta)] - J_2[\nu_s(\pi-\theta)]] \right\}_m - \qquad (7.1.56) \quad /294$$
$$\left. - \left\{ \frac{C_s e^{\pi i \nu_s}}{1 + e^{2\pi i \nu_s}} [J_0[\nu_s(\pi-\theta)] + J_2[\nu_s(\pi-\theta)]] \right\}_e \right\}.$$

The asymptotic expressions for the components of the stray field and the amplitude functions in the region of the geometric shadow may be obtained from (7.1.44), (7.1.45) and (7.1.39), (7.1.19), (7.1.20). However, as already noted, they will be unsuitable for

computing the field in the wave band and in the vicinity of $\theta = 0$.

For the density of the surface currents j_θ, j_ϕ, determined by formulas (7.1.31), (7.1.32), when $ka >> 1$ we obtain the asymptotic expressions,

$$j_\theta = -\frac{cE}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\cos\varphi\,\varphi(\theta), \tag{7.1.57}$$

$$j_\varphi = -\frac{cE}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\sin\varphi\,\psi(\theta), \tag{7.1.58}$$

where in the shadow region ($0 < \theta < \pi/2$)

$$\varphi(\theta) \doteq -\frac{i}{\sqrt{\sin\theta}}\sum_s D_s \frac{e^{i\nu_s\left(\frac{\pi}{2}+\theta\right)} + ie^{i\nu_s\left(\frac{\pi}{2}-\theta\right)}}{1+e^{2\pi i\nu_s}},$$

$$\psi(\theta) = \frac{x^{-1/3}e^{-i\pi/3}}{\sqrt{\sin\theta}}\sum_s D_s \frac{e^{i\nu_s\left(\frac{\pi}{2}+\theta\right)} + ie^{i\nu_s\left(\frac{\pi}{2}-\theta\right)}}{1+e^{2\pi i\nu_s}}, \tag{7.1.59}$$

if the point of the surface of the sphere is not located in the vicinity of the terminal $\theta = 0$, and

$$\varphi(\theta) = -i\sqrt{2\pi x}\sum_s D_s \frac{e^{i\left(\frac{\pi}{2}\nu_s+\frac{\pi}{4}\right)}}{1+e^{2\pi i\nu_s}} J_0(\nu_s\theta),$$

$$\psi(\theta) = -\sqrt{2\pi}\,x^{1/6}e^{-2\pi i/3}\sum_s D_s \frac{e^{i\left(\frac{\pi}{2}\nu_s+\frac{\pi}{4}\right)}}{1+e^{2\pi i\nu_s}} J_0(\nu_s\theta), \tag{7.1.60}$$

if the point of the surface of the sphere lies in the vicinity of the terminal $\theta = 0$. In the illuminated region outside the vicinity of the terminal $\theta = \pi$ ($\pi/2 < \theta < \pi$)

$$\varphi(\theta) = 2e^{ix\cos\theta} + \frac{-i}{\sqrt{\sin\theta}} \times$$

$$\times \sum_s D_s \frac{e^{i\nu_s\left(\theta-\frac{\pi}{2}\right)} - ie^{i\nu_s\left(\frac{5\pi}{2}-\theta\right)}}{1+e^{2\pi i\nu_s}}, \tag{7.1.61}$$

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$$\psi(\theta) = -2\cos\theta\,e^{ix\cos\theta} + \frac{x^{-1/3}e^{-\pi i/3}}{\sqrt{\sin\theta}} \times$$

$$\times \sum_s D_s \frac{e^{i\nu_s\left(\theta-\frac{\pi}{2}\right)} - ie^{i\nu_s\left(\frac{5\pi}{2}-\theta\right)}}{1+e^{2\pi i\nu_s}},$$

and in the vicinity of the terminal $\theta = \pi$

$$\varphi(\theta) = 2e^{ix\cos\theta} - i\sqrt{2\pi x}\times$$

$$\times \sum_s D_s \frac{e^{i\left(\frac{3\nu_s}{2} - \frac{\pi}{4}\right)}}{1 + e^{2\pi i\nu_s}} J_0[\nu_s(\pi - \theta)], \qquad (7.1.62)$$

$$\psi(\theta) = 2\cos\theta e^{ix\cos\theta} + \sqrt{2\pi} e^{-\frac{\pi i}{3}} x^{\frac{1}{6}} \times$$

$$\times \sum_s D_s \frac{e^{i\left(\frac{3\nu_s}{2} - \frac{\pi}{4}\right)}}{1 + e^{2\pi i\nu_s}} J_0[\nu_s(\pi - \theta)]$$

(in (7.1.59) - (7.1.62) in the expressions for $\phi(\theta)$, the values of D_s and ν_s correspond to the case of the electric field and in the expressions for $\psi(\theta)$ to the magnetic). The disadvantage of the asymptotic formulas, derived using the Watson transform, is eliminated by asymptotic summation of the series in the remote band by the Fok method. As shown in [68, 69], the use of Fok's ideas does not require separate examination of the problem in the region of the geometric shadow and in the illuminated region. Only the behavior of the field in the vicinity of the focal line $\theta = \pi$, $\theta = 0$ will be investigated especially. Here for the amplitude functions $S(\theta)$, $S^*(\theta)$ from (1.7.24), (7.1.25) we obtain the asymptotic expressions (the wave band),

$$S(\theta) = S_g^e - \left\{ \frac{2M^{5/2}}{\sqrt{\sin\theta}} \left[-e^{ix\theta}\hat{g}(M\theta) + ie^{ix(2\pi-\theta)}\hat{g}(M(2\pi-\theta)) \right] - \right.$$

$$- \frac{1}{M^{1/2}\sin\theta\sqrt{\sin\theta}} \times$$

$$\left. \times \left[ie^{ix\theta}\hat{f}(M\theta) - e^{ix(2\pi-\theta)}\hat{f}(M(2\pi-\theta)) \right] \right\}, \qquad (7.1.63)$$

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$$S^*(\theta) = S_g^m - \left\{ \frac{2M^{5/2}}{\sqrt{\sin\theta}} \times \right.$$

$$\times \left[-e^{ix\theta}\hat{f}(M\theta) + ie^{ix(2\pi-\theta)}\hat{f}(M(2\pi-\theta)) \right] -$$

$$- \frac{1}{M^{1/2}\sin\theta\sqrt{\sin\theta}} \times$$

$$\left. \times \left[ie^{ix\theta}\hat{g}(M\theta) - e^{ix(2\pi-\theta)}\hat{g}(M(2\pi-\theta)) \right] \right\}, \qquad (7.1.64)$$

$$S_g^{e,m} = \mp \frac{x}{2} e^{-i\left(2x\sin\frac{\theta}{2} - \frac{\pi}{2}\right)},$$

where the functions $\hat{f}(x)$ and $\hat{g}(x)$ are determined as in (6.1.69), (6.1.64), $M = \left(\frac{ka}{2}\right)^{1/3}$. As mentioned in [68], in computing the field in the case of large values of the parameter ka, in first approximation we can confine ourselves only to the first two terms in the braces in formulas (7.1.63), (7.1.64). Here however it should be borne in mind that the function $\hat{f}(x)$ with increase in the positive argument drops more rapidly than does $\hat{g}(x)$, with respect to which the expression in the brackets in $S^*(\theta)$ becomes comparable in value to the first two terms. Therefore, in refining the result for the value of $S^*(\theta)$ it follows to also retain the terms which contain the function $\hat{g}(x)$.

Formulas (7.1.63), (7.1.64) are valid in the wave band for all angles of observation θ, with the exception only of the vicinities $\theta = 0$ and $\theta = \pi$.

In the vicinity $\theta = \pi$ instead of (7.1.63), (7.1.64) we obtain the asymptotic expressions,

$$S(\theta) = S_g^e + 2iM^4\sqrt{\pi}\,e^{ix\pi}e^{-i3\pi/4}\sqrt{\frac{\pi-\theta}{\sin(\pi-\theta)}}\times$$
$$\times\left\{(J_1'+iJ_1'')\,\hat{g}(M\theta)+(J_1'-iJ_1'')\,\hat{g}\,(M(2\pi-\theta))+\right.$$
$$+\frac{1}{x\sin\theta}\left[(J_1-iJ_2)\,\hat{f}\,(M\theta)+\right.$$
$$\left.\left.+(J_1+iJ_2)\,\hat{f}\,(M(2\pi-\theta))\right]\right\}, \qquad (7.1.65)$$

$$S^*(\theta) = S_g^m + 2iM^4\sqrt{\pi}\,e^{ix\pi}e^{-i3\pi/4}\sqrt{\frac{\pi-\theta}{\sin(\pi-\theta)}}\times$$
$$\times\left\{(J_1'+iJ_1'')\,\hat{f}\,(M\theta)+(J_1'-iJ_1'')\,\hat{f}\,(M(2\pi-\theta))+\right.$$
$$+\frac{1}{x\sin\theta}\left[(J_1-iJ_2)\,\hat{g}\,(M\theta)+\right.$$
$$\left.\left.+(J_1+iJ_2)\,\hat{g}\,(M(2\pi-\theta))\right]\right\}, \qquad (7.1.66)$$

in which the Bessel functions $J_\eta(\lambda)$ and their derivatives are taken from the argument $\lambda = x(\pi - \theta)$. Formulas (7.1.65), (7.1.66), when $x(\pi - \theta) >> 1$, convert to formulas (7.1.63), (7.1.64).

In the vicinity of the focal beam $\theta = 0$, the asymptotic expressions for the amplitude functions $S(\theta)$, $S^*(\theta)$ have the form,

$$S(\theta) = -2M^4\sqrt{\pi}\,e^{-i3\pi/4}\left\{(J_1' + iJ_1'')\,\hat{g}(-M\theta) + \right.$$

$$+ (J_1' - iJ_1'')\,\hat{g}(M\theta) - \frac{1}{x\sin\theta}\times$$

$$\left.\times\,[(J_1 - iJ_2)\,\hat{f}(-M\theta) + (J_1 + iJ_2)\,\hat{f}(M\theta)]\right\}, \qquad (7.1.67)$$

$$S^*(\theta) = 2M^4\sqrt{\pi}\,e^{i\pi/4}\left\{(J_1' + iJ_1'')\,\hat{f}(-M\theta) + \right.$$

$$+ (J_1' - iJ_1'')\,\hat{f}(M\theta) - \frac{1}{x\sin\theta}\times$$

$$\left.\times\,[(J_1 - iJ_2)\,\hat{g}(-M\theta) + (J_1 + iJ_2)\,g(M\theta)]\right\}, \qquad (7.1.68)$$

where the Bessel functions and their derivatives are taken from the argument $x\theta$. In first approximation for computation of the field in the vicinity $\theta = 0$ we can use the formulas,

$$S(\theta) = 2M^4\sqrt{\pi}\,e^{-i3\pi/4}[(J_2 + iJ_1)\,\hat{g}(-M\theta) +$$

$$+ (J_2 - iJ_1)\,\hat{g}(M\theta)], \qquad (7.1.69) \qquad \underline{/298}$$

$$S^*(\theta) = -2M^4\sqrt{\pi}\,e^{i\pi/4}[(J_2 + iJ_1)\,\hat{f}(-M\theta) +$$

$$+ (J_2 - iJ_1)\,\hat{f}(M\theta)]. \qquad (7.1.70)$$

Numerical computation according to the formulas of the strict theory and according to the asymptotic formulas given above give for $x \geq 5$ a satisfactory agreement of the results, which improve with increase in $x = (ka)$ [68].

The asymptotic expressions (7.1.57) - (7.1.62) for the density of the surface currents induced on the sphere are not suitable for computations in the immediate vicinity of the boundary between the shadow and the illuminated region ($\theta \approx \pi/2$). In this vicinity we can use the asymptotic expressions,

$$j_\theta = -\frac{cE}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\,\frac{\cos\varphi}{6\sqrt{\sin\theta}}\times$$

$$\times\sum_{l=0}^{\infty}(-1)^l\left\{g(\xi_l)\,e^{i\nu_l\left(\theta+\frac{\pi}{2}+2\pi l\right)} + \right.$$

$$\left.+\,ig(\xi_l')\,e^{i\nu_l\left(\frac{\pi}{2}-\theta+2\pi l\right)}\right\},$$

$$j_\varphi = -\frac{cE}{4\pi}\sqrt{\frac{\varepsilon}{\mu}}\,\frac{e^{i\pi/6}\sin\varphi}{x^{1/3}6^{2/3}\sqrt{\sin\theta}}\times$$

$$\times\sum_{l=0}^{\infty}(-1)^l\left\{g(\xi_l)e^{i\nu_l\left(\theta+\frac{\pi}{2}+2\pi l\right)}+\right.$$

$$\left.+ig(\xi_l')e^{i\nu_l\left(\frac{\pi}{2}-\theta+2\pi l\right)}\right\}, \qquad (7.1.71)$$

where $\xi_l=\left(\frac{x}{6}\right)^{1/3}e^{i\pi/3}\left(\theta+\frac{\pi}{2}+2\pi l\right)$, $\xi_l'=\left(\frac{x}{6}\right)^{1/3}e^{i\pi/3}\times\left(\theta-\frac{\pi}{2}+2\pi l\right)$, but $g(\xi)$ is the Fok function, determined here by the formula /299

$$g(\xi)=\int_{-\infty e^{-i\pi/3}}^{\infty e^{-i\pi/3}}\frac{e^{i\xi q}}{\Omega A(q)}\,dq$$

(just as before in the subintegral expression $\Omega = 1$ in the case of a TM type field and $\Omega \square \partial/\partial x$ in the case of a TE type field). In practice the major contribution to the values of the function j_θ and j_ϕ are given by the terms with the summation index $l = 0$, determined by the integrals $g(\xi)$, whose numerical values are found from the tables (the Tables of Fok Functions are found in [169, 170]).

A Sphere With Finite Conductivity. In the case of sphere with finite conductivity along with the secondary field outside the sphere there will also exist (non-zero) a field inside the sphere, determined by the potentials u^2 and v^2. They can be sought in the form of series such as (3.1.13):

$$u^2=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{mn}j_n(k_2r)P_n^m(\cos\theta)e^{im\varphi}, \qquad (7.1.72)$$

$$v^2=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}A_{mn}^*j_n(k_2r)P_n^m(\cos\theta)e^{im\varphi}, \qquad (7.1.73)$$

whose coefficients must be found from the boundary conditions (1.7.10), (1.7.11). If we seek the potentials u^1, v^1 of the field, scattered on the sphere, again in the form of (7.1.12), (7.1.13), then the boundary conditions (1.7.10), (1.7.11) will be satisfied only under the condition that

$$A_{mn}=A_{mn}^*=B_{mn}=B_{mn}^*=0, \quad \text{when } m\neq\pm1,$$

but

$$A_{-1,n} = -n(n+1)A_{1n}, \quad B_{-1,n} = -n(n+1)B_{1n},$$

$$A^*_{-1,n} = n(n+1)A^*_{1n}, \quad B^*_{-1,n} = n(n+1)B^*_{1n}, \qquad (7.1.74)$$

where

$$A_{1n} = -Ei^n \frac{2n+1}{2n(n+1)} \times$$

$$\times \frac{\mu_2}{k\mu_2 \psi_n'(k_2a)\zeta_n(ka) - k_2\mu\psi_n(k_2a)\zeta_n'(ka)},$$

$$B_{1n} = -E\frac{i^n}{ik}\frac{2n+1}{2n(n+1)} \times \qquad (7.1.75) \quad /300$$

$$\times \frac{k\mu_2\psi_n(ka)\psi_n(k_2a) - k_2\mu\psi_n'(ka)\psi_n(k_2a)}{k\mu_2\zeta_n(ka)\psi_n'(k_2a) - k_2\mu\zeta_n'(ka)\psi_n(k_2a)};$$

$$A^*_{1n} = E\frac{i^n}{k}\sqrt{\frac{\varepsilon}{\mu}}\frac{2n+1}{2n(n+1)} \times$$

$$\times \frac{\mu k_2}{k\mu_2\zeta_n'(ka)\psi_n(k_2a) - \mu k_2\zeta_n(ka)\psi_n'(k_2a)},$$

$$B^*_{1n} = -E\frac{i^n}{k}\sqrt{\frac{\varepsilon}{\mu}}\frac{2n+1}{2n(n+1)} \times$$

$$\times \frac{k\mu_2\psi_n(ka)\psi_n(k_2a) - k_2\mu\psi_n(ka)\psi_n'(k_2a)}{k\mu_2\zeta_n'(ka)\psi_n(k_2a) - k_2\mu\zeta_n(ka)\psi_n'(k_2a)}.$$

Then for the field scattered on the sphere,

$$u^1 = -\frac{E\cos\varphi}{ik^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \beta_n P_n^1(\cos\theta)\zeta_n(kr), \qquad (7.1.76)$$

but

$$v^1 = -\frac{E\sin\varphi}{ik^2r}\sqrt{\frac{\varepsilon}{\mu}} \times$$

$$\times \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} a_n P_n^1(\cos\theta)\zeta_n(kr), \qquad (7.1.77)$$

where through α_n, β_n we denote the Mie coefficients [45, 113],

$$\alpha_n = \frac{k\mu_2\psi_n(ka) - k_2\mu\chi_n(k_2a)\psi_n(ka)}{k\mu_2\zeta_n'(ka) - k_2\mu\chi_n(k_2a)\zeta_n(ka)},$$

$$\beta_n = \frac{k_2\mu\psi_n'(ka) - k\mu_2\chi_n(k_2a)\psi_n(ka)}{k_2\mu\zeta_n'(ka) - k\mu_2\chi_n(k_2a)\zeta_n(ka)}, \qquad (7.1.78)$$

but

$$\chi_n(k_2a) = \frac{\psi_n'(k_2a)}{\psi_n(k_2a)} \qquad (7.1.79)$$

By determining the coefficients A_{mn}, A^{*}_{mn}, B_{mn} and B^{*}_{mn} we have completely solved the problem on the scattering of a plane electromagnetic wave on a sphere with finite conductivity. The components /301 of the fields outside and inside the sphere may be found from formulas (7.1.19) - (7.1.20), where for the stray field,

$$P^1 = \frac{-i}{k^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\alpha_n\zeta_n(kr)P_n(\cos\theta),$$

$$Q^1 = \frac{-i}{k^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)}\beta_n\zeta_n(kr)P_n(\cos\theta), \qquad (7.1.80)$$

and for the field inside the sphere

$$P^2 = \frac{1}{k_2^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times \frac{k_2\mu_2\psi_n(k_2r)}{k\mu_2\psi_n'(k_2a)\zeta_n(ka) - k_2\mu\psi_n(k_2a)\zeta_n'(ka)}P_n(\cos\theta),$$

$$Q^2 = \frac{-1}{k_2^2r}\sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \times \qquad (7.1.81)$$

$$\times \frac{\mu k_2^2/k\psi_n(k_2r)}{k\mu_2\zeta_n'(ka)\psi_n(k_2a) - k_2\mu\psi_n'(k_2a)\zeta_n(ka)}P_n(\cos\theta).$$

In the wave band, relationships (7.1.25) are approximately satisfied; here the amplitude functions $S(\theta)$ and $S^{*}(\theta)$ are given by the expansions,

$$S(\theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\{\alpha_n\tau_n(\cos\theta) + \beta_n\pi_n(\cos\theta)\}, \qquad (7.1.82)$$

$$S^{*}(\theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\{\alpha_n\pi_n(\cos\theta) + \beta_n\tau_n(\cos\theta)\}. \qquad (7.1.83)$$

If we take into account that [(7.1.24), (7.1.28)],

$$\pi_n(1) = \tau_n(1) = \frac{1}{2} n(n+1),$$

$$\pi_n(-1) = \tau_n(-1) = \frac{(-1)^n}{2} n(n+1),$$

then

$$S(0) = S^*(0) = \frac{1}{2}\sum_{n=1}^{\infty}(2n+1)\{\alpha_n + \beta_n\},$$

$$S(\pi) = S^*(\pi) = \frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(2n+1)\{\alpha_n + \beta_n\}, \qquad (7.1.84)$$

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and for the scattering cross sections we obtain the expressions

$$\sigma_B = \frac{4\pi}{k^2}|S(\pi)|^2 =$$

$$= \frac{2\pi}{k^2}\left|\sum_{n=1}^{\infty}(-1)^n(2n+1)(\alpha_n + \beta_n)\right|^2, \qquad (7.1.85)$$

$$\sigma_s = \frac{4\pi}{k^2}\operatorname{Re} S(0) = \frac{2\pi}{k^2}\operatorname{Re}\sum_{n=1}^{\infty}(2n+1)(\alpha_n + \beta_n), \qquad (7.1.86)$$

$$\sigma = \frac{4\pi}{k^2}\{\cos^2\varphi\,|S(\theta)|^2 + \sin^2\varphi\,|S^*(\theta)|^2\}. \qquad (7.1.87)$$

§2. Field of an Electric (Magnetic) Dipole in the Presence of a Sphere.

Now let us look at the problem of diffraction of a field emitted by an electric or by a magnetic dipole on a sphere.

Let a unit dipole emitter with a dipole moment $\mathfrak{a}$ (for the electric dipole $\mathfrak{a} = \mathfrak{p}$, and for the magnetic, $\mathfrak{a} = \mathfrak{m}$) be located at a certain point P of a uniform and isotropic unbounded space with electric and magnetic permeabilities ε, μ and with a conductivity $\sigma = 0$, located at a distance l from the center of the sphere having a radius $r = a$, and let, in the coordinate system $Oxyz$ with the origin at the center of the sphere, the dipole moment form a certain angle α with the positive direction of the axis Oz, which passes through point P (the vector $\mathfrak{a}$ lies in the plane Oxz, see Fig. 37). The electromagnetic properties of the sphere are characterized by the constants ε_2, μ_2 and σ_2, where in the general case $\sigma_2 \neq \infty$.

We can formulate the problem of seeking the secondary fields outside and inside the sphere. As usual, the solution to this vector problem consists of integrating the uniform system of Maxwell equations (1.1.4) (for the secondary fields) under the boundary conditions (1.1.5) or (1.1.6) on the surface of the sphere and the supplementary conditions at infinity, superposed on the vectors E^1, H^1 of the stray field. In spherical coordinates it may be reduced to two independent scalar problems relative to the potential functions u and v, which satisfy the uniform Helmholt equation, the boundary conditions (1.7.1), (1.7.11), if $\sigma_2 \neq \infty$, or conditions (1.7.13), if $\sigma_2 = \infty$, and the condition of radiation at infinity for the potentials u^1, v^1 of the field scattered on the sphere [50, 70]. /303

The field of the electric dipole in free space may be found through the Hertz electric vector $\vec{\Pi}^0$ [see (1.3.9)] from relationships (1.2.13), and the field of the magnetic dipole, through the Hertz magnetic vector $\vec{\Pi}^{0*}$ [see (1.3.10)] from relationships (1.2.17). It is obvious that if we represent the dipole moment a in the form of a sum,

$$\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2,$$

where $a_1 = \{0, 0, |a| \cos \alpha\}$, and $a_2 = \{|a| \sin \alpha, 0, 0\}$, then the Hertz vectors corresponding to it may be written in the form of the sum $\vec{\Pi}_1 + \vec{\Pi}_2$, in which

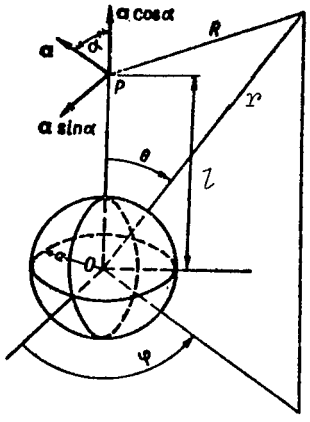

Fig. 37. Excitation of the Sphere by the Dipole Field.

$$\vec{\Pi}_1 = \left\{0, 0, |\mathbf{a}| \cos\alpha \frac{e^{ikR}}{R}\right\},$$

and

$$\vec{\Pi}_2 = \left\{|\mathbf{a}| \sin\alpha \frac{e^{ikR}}{R}, 0, 0\right\}.$$

Therefore, the primary dipole field with the dipole moment a, directed at an angle α to the axis Oz, may be assumed as a result of the superposition of two particular electromagnetic fields: the field E_1^0, H_1^0, produced only by the vertical component of the vector a, and the field E_2^0, H_2^0 produced only by the horizontal component of the vector a. Each of these particular fields is represented by its own potential functions u and v (let us note, that the vertical electric dipole excites an electromagnetic field only of electric type, and the vertical magnetic dipole, only a field of magnetic type). /304

We can find expressions for the potentials u^0, v^0 of the primary field of the electric and magnetic dipoles.

In the case of a vertical electric dipole,

$$\vec{\Pi}^0 = \mathbf{p}_1 e^{ikR}/R, \tag{7.2.1}$$

where $|\mathbf{p}_1| = p_1 = |\mathbf{p}| \cos\alpha$, and $\mathbf{p}_1 = \{0, 0, p_1\}$. Converting to spherical coordinates r, θ, ϕ with the origin at point O (the polar axis is directed along the Oz axis), we find

$$\vec{\Pi}^0 = \{\Pi^0 \cos\theta, -\Pi^0 \sin\theta, 0\},$$

where $\Pi^0 = p_1 e^{ikR}/R$, and ($l = b$)

$$R = \sqrt{r^2 + b^2 - 2br\cos\theta}. \tag{7.2.2}$$

Consequently, in spherical coordinates $\Pi^0 = \Pi^0(r, \theta)$ and $\frac{\partial \Pi^0}{\partial \phi} \equiv 0$ (the primary dipole field possesses axial symmetry). Therefore, if we write (1.2.13) in spherical coordinates and take into account that the relationship,

$$\sin\theta \frac{\partial \Pi^0}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial \Pi^0}{\partial \theta} = \frac{1}{b} \cdot \frac{\partial \Pi^0}{\partial \theta}, \tag{7.2.3}$$

is valid, we find for the components of the primary field, the expressions,

$$\begin{aligned} E_r^0 &= -\frac{1}{br\sin\theta} \cdot \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \Pi^0}{\partial \theta}\right), \\ E_\theta^0 &= \frac{1}{rb} \cdot \frac{\partial}{\partial r}\left(r \frac{\partial \Pi^0}{\partial \theta}\right), \\ H_\varphi^0 &= \frac{ik_0 \varepsilon}{b} \cdot \frac{\partial \Pi^0}{\partial \theta}. \end{aligned} \tag{7.2.4}$$

The components of the electromagnetic field in spherical coordinates may be written through the Debye potential function u^0 (in the case if the field is of electric type) by formulas (1.6.25), (1.6.22). From comparison of them with (7.2.4) ($\partial/\partial\phi \equiv 0$) we find /305

$$u^0 = \frac{1}{b}\Pi^0 = \frac{p_1 e^{ikR}}{bR}. \tag{7.2.5}$$

After using formula (3.6.7), and sitting there cos γ = cos θ we find, the expansion of the Debye potential u^0 of the primary field of the vertical electric dipole into a series of spherical wave functions,

$$u^0 = \frac{ip_1}{kb^2r}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\begin{cases}\psi_n(kr)\zeta_n(kb), & b>r,\\ \psi_n(kb)\zeta_n(kr), & b<r.\end{cases} \qquad (7.2.6)$$

Since the field of the vertical electric dipole is a field of only electric type, then for it $v^0 \equiv 0$.

In the case of a horizontal electric dipole,

$$\vec{\Pi}^0 = \mathbf{p}_2 e^{ikR}/R, \qquad (7.2.7)$$

where $|\mathbf{p}_2| = p_2 = |\mathbf{p}|\sin\alpha$, and $\mathbf{p}_2 = \{p_2, 0, 0\}$. In spherical coordinates,

$$\vec{\Pi}^0 = \{\Pi\cos\varphi,\ 0,\ -\Pi^0\sin\varphi\},$$

where $\Pi^0 = p_2 e^{ikR}/R$, and just as in the previous case, $\Pi^0 = \Pi^0(\theta, r)$, $\frac{\partial\Pi^0}{\partial\phi} = 0$. From (1.2.13) in spherical coordinates we find that the radial components of the dipole field are expressed by the formulas,

$$E_r^0 = \frac{\cos\varphi}{r}\cdot\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\Pi^0}{\partial r} - \frac{\sin\theta}{r}\cdot\frac{\partial\Pi^0}{\partial\theta}\right) - \frac{\cos\varphi}{r}\left(\sin\theta\frac{\partial\Pi^0}{\partial r} + \frac{\cos\theta}{r}\frac{\partial\Pi^0}{\partial\theta}\right), \qquad (7.2.8)$$

$$H_r^0 = \frac{ik_0\varepsilon}{r}\sin\varphi\frac{\partial\Pi^0}{\partial\theta}. \qquad (7.2.9)$$

Since

$$-\frac{\partial\Pi^0}{\partial b} = \cos\theta\frac{\partial\Pi^0}{\partial r} - \frac{\sin\theta}{r}\frac{\partial\Pi^0}{\partial\theta},$$

$$\frac{1}{b}\cdot\frac{\partial\Pi^0}{\partial\theta} = \sin\theta\frac{\partial\Pi^0}{\partial r} + \frac{\cos\theta}{r}\cdot\frac{\partial\Pi^0}{\partial\theta},$$

then instead of (7.2.8) we can write /306

$$E_r^0 = -\frac{\cos\varphi}{r}\cdot\frac{\partial}{\partial\theta}\left(\frac{\partial\Pi^0}{\partial b}+\frac{\Pi^0}{b}\right). \qquad (7.2.10)$$

In spherical coordinates of the components E_r^0, H_r^0 of the dipole field may be written on the basis of (1.6.25) and (1.6.15) in the form

$$E_r^0 = \frac{\partial^2(ru^0)}{\partial r^2}+k^2(ru^0),$$

$$H_r^0 = \frac{\partial^2(rv^0)}{\partial r^2}+k^2(rv^0), \qquad (7.2.11)$$

where u^0 and v^0 are the Debye potentials. Therefore, if we represent the function Π^0 in the form of series (3.6.7)

$$\Pi^0 = \frac{ip_2}{kbr}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r,\end{cases}$$

and substitute this expression of the function Π^0 into the right-hand side of (7.2.9) and (7.2.10), we then find

$$E_r^0 = -\frac{i\cos\varphi\, p_2}{br^2}\times$$

$$\times\sum_{n=1}^{\infty}(2n+1)P_n^1(\cos\theta)\begin{cases}\psi_n'(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n'(kb), & b>r,\end{cases}$$

$$H_r^0 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p_2\sin\varphi}{br^2}\times \qquad (7.2.12)$$

$$\times\sum_{n=1}^{\infty}(2n+1)P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r,\end{cases}$$

and from (7.2.11), if we seek u^0 and v^0 in the form of the series,

$$u^0 = \frac{1}{r}\sum_{n=1}^{\infty}a_n P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r,\end{cases}$$

$$v^0 = \frac{1}{r}\sum_{n=1}^{\infty}b_n P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r,\end{cases} \qquad (7.2.13)$$

on the basis of (7.1.8) we obtain /307

$$E_r^0 = \frac{1}{r^2}\sum_{n=1}^{\infty} a_n n(n+1) P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r.\end{cases}$$

$$H_r^0 = \frac{1}{r^2}\sum_{n=1}^{\infty} b_n n(n+1) P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r.\end{cases} \qquad (7.2.14)$$

As the result, by comparing (7.2.12) with (7.2.14) we establish that the Debye potentials of the primary field of the horizontal electric dipole are represented by the expansions into series of spherical wave functions of the type,

$$u^0 = -\frac{ip_2\cos\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\begin{cases}\psi_n'(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n'(kb), & b>r.\end{cases} \qquad (7.2.15)$$

$$v^0 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p_2\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r.\end{cases} \qquad (7.2.16)$$

Then for the Debye potentials of the primary field of the electric dipole directed at an angle α to the Oz axis, we obtain the expressions,

$$u^0 = \frac{i|\mathbf{p}|\cos\alpha}{kb^2r}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr)\\ \psi_n(kr)\zeta_n(kb)\end{cases} -$$

$$-\frac{i\cos\varphi|\mathbf{p}|\sin\alpha}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\times \qquad (7.2.17)$$

$$\times\begin{cases}\psi_n'(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n'(kb), & b>r,\end{cases}$$

$$v^0 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{|\mathbf{p}|\sin\alpha\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\begin{cases}\psi_n(kb)\zeta_n(kr), & b<r,\\ \psi_n(kr)\zeta_n(kb), & b>r.\end{cases} \qquad (7.2.18)$$ /308

Analogously we find that in the case of a magnetic dipole, directed at an angle α to the Oz axis, the potentials of its primary field will be given by the expressions

$$v^0 = \frac{i|\mathbf{m}|\cos\alpha}{kb^2r}\sum_{n=0}^{\infty}(2n+1)P_n(\cos\theta)\begin{cases}\psi_n(kr)\zeta_n'(kb)\\ \psi_n'(kb)\zeta_n(kr)\end{cases} +$$

$$+\frac{i|\mathbf{m}|\sin\alpha\cos\varphi}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\times$$

$$\times\begin{cases}\psi_n(kr)\zeta_n'(kb), & b>r,\\ \psi_n'(kb)\zeta_n(kr), & b<r,\end{cases} \qquad (7.2.19)$$

$$u^0 = -\sqrt{\frac{\mu}{\varepsilon}}\frac{|\mathbf{m}|\sin\alpha}{br}\sin\varphi\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}P_n^1(\cos\theta)\begin{cases}\psi_n(kr)\zeta_n(kb), & b>r,\\ \psi_n(kb)\zeta_n(kr), & b<r.\end{cases} \qquad (7.2.20)$$

Expression (7.2.17) for the potential u^0 of the field of the electric dipole in free space will be written from now on in the form,

$$u^0 = \frac{i|\mathbf{p}|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)A_{mn}\times$$

$$\times\begin{Bmatrix}\psi_n(kr)\\ \zeta_n(kr)\end{Bmatrix}P_n^m(\cos\theta)e^{im\varphi}, \qquad (7.2.21)$$

where

$$A_{0n} = \frac{\cos\alpha}{kb}\begin{cases}\zeta_n(kb), & b>r,\\ \psi_n(kb), & b<r,\end{cases}$$

$$A_{-1,n} = -n(n+1)A_{1n}, \quad n\geqslant 1,$$

$$A_{1n} = -\frac{\sin\alpha}{2n(n+1)}\begin{cases}\zeta_n'(kb), & b>r,\\ \psi_n'(kb), & b<r,\end{cases} \qquad (7.2.22)$$

$$A_{mn} = 0, \quad \text{if} \quad |m|>1, \quad n=0, 1, \ldots,$$

and expression (7.2.19) for the potential of the magnetic dipole field in free space, in the form /309

$$v^0 = \frac{i|\mathbf{m}|}{br}\sum_{n=0}^{\infty}\sum_{m=+n}^{n}(2n+1)B_{mn}\times$$

(7.2.23)

$$\times \left\{ \begin{array}{l} \psi_n(kr) \\ \zeta_n(kr) \end{array} \right\} P_n^m(\cos\theta)\, e^{im\varphi},$$

where

$$B_{0n} = \frac{\cos\alpha}{kb} \left\{ \begin{array}{ll} \zeta_n(kb), & b > r, \\ \psi_n(kb), & b < r, \end{array} \right.$$

$$B_{-1,n} = -n(n+1) B_{1n},$$

$$B_{1n} = \frac{\sin\alpha}{2n(n+1)} \left\{ \begin{array}{ll} \zeta_n'(kb), & b > r, \\ \psi_n'(kb), & b < r, \end{array} \right.$$

$$B_{mn} = 0, \quad \text{if} \quad |m| > 1, \quad n = 0, 1, \ldots$$

Electric Dipole Over an Ideally Conducting Sphere. In this case the Debye potentials in the region inside the sphere are equal to zero and only the potentials u^1, v^1 of the secondary field, scattered on the sphere, are subject to determination. If they are sought in the form of expansions such as (3.1.15),

$$u^1 = \frac{1}{kr} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn}\, \zeta_n(kr)\, P_n^m(\cos\theta) e^{im\varphi}, \tag{7.2.24}$$

$$v^1 = \frac{1}{kr} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{mn}\, \zeta_n(kr)\, P_n^m(\cos\theta)\, e^{im\varphi}, \tag{7.2.25}$$

then from boundary conditions (1.7.13) for the potentials $u = u^0 + u^1$, $v = v^0 + v^1$ of the resulting field we find that these conditions are satisfied only in the case when as the coefficients of the series (7.2.24), (7.2.25) we take the expressions ($p = |\mathrm{p}|$):

$$a_{mn} = -\frac{ikp}{b}(2n+1)\frac{\psi_n'(ka)}{\zeta_n'(ka)} A_{mn}, \quad b > a = r, \tag{7.2.26}$$

$$b_{mn} = 0, \quad \text{if} \quad m = 0 \quad \text{и} \quad |m| > 1, \tag{7.2.27}$$

$$b_{-1,n} = n(n+1) b_{1n}, \quad n \geqslant 1,$$

$$b_{1n} = -\sqrt{\frac{\varepsilon}{\mu}}\, \frac{kp\sin\alpha}{2bi} \cdot \frac{2n+1}{n(n+1)} \cdot \frac{\psi_n(ka)}{\zeta_n(ka)} \zeta_n(kb).$$

As a result we find that /310

$$u^1 = -\frac{ip}{br} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \frac{\psi_n'(ka)}{\zeta_n'(ka)} \times \tag{7.2.28}$$

$$\times\, \zeta_n(kr)\, A_{mn}\, P_n^m(\cos\theta)\, e^{im\varphi},$$

and

$$v^1 = -\sqrt{\frac{\varepsilon}{\mu}} \sin\varphi \frac{p \sin\alpha}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \times$$

$$\times \frac{\psi_n(ka)}{\zeta_n(ka)} \zeta_n(kb) \zeta_n(kr) P_n^1(\cos\theta). \qquad (7.2.29)$$

In the same way we completely solve the mathematical problem for the Debye potentials of the electric dipole field in the presence of an ideally conducting sphere. Functions (7.2.28), (7.2.29) satisfy all conditions of the problem. For the resulting field in the region $b > r \geq a$,

$$u = \frac{ip}{br} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \times$$

$$\times \left\{ \psi_n(kr) - \frac{\psi_n'(ka)}{\zeta_n'(ka)} \zeta_n(kr) \right\} A_{mn} P_n^m(\cos\theta) e^{im\varphi}, \qquad (7.2.30)$$

$$v = \sqrt{\frac{\varepsilon}{\mu}} \frac{p \sin\alpha \sin\varphi}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \times$$

$$\times \left\{ \psi_n(kr) - \frac{\psi_n(ka)}{\zeta_n(ka)} \zeta_n(kr) \right\} \zeta_n(kb) P_n^1(\cos\theta), \qquad (7.2.31)$$

and in the region $r > b$,

$$u = \frac{ip}{br} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) \times$$

$$\times \left\{ A_{mn}(b<r) - \frac{\psi_n(ka)}{\zeta_n'(ka)} A_{mn}(b \geqslant a) \right\} \times \qquad (7.2.32)^2 \quad /311$$

$$\times \zeta_n(kr) P_n^m(\cos\theta) e^{im\varphi},$$

$$v = \sqrt{\frac{\varepsilon}{\mu}} \frac{p \sin\alpha \sin\varphi}{br} \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \times \qquad (7.2.33)$$

$$\times \left\{ \psi_n(kb) - \frac{\psi_n(ka)}{\zeta_n(ka)} \zeta_n(kb) \right\} \zeta_n(kr) P_n^1(\cos\theta).$$

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[2] $A_{mn}(b \gtrless r)$ represents the coefficients A_{mn} determined when $b \gtrless r$, as in (7.2.22). Below we also write the coefficients B_{mn} from (7.2.23).

The components of the field E, H may now be found through (7.2.30) - (7.2.33) at any point of space from formulas (1.6.15), written in spherical coordinates.

From (1.1.7) and (1.6.15) using (7.1.30) we find that the law governing the density distribution of surface currents induced by the dipole field on the surface of the sphere will be determined by the formulas,

$$j_\theta = \frac{\omega\varepsilon\, pi}{4\pi\, ba}\left\{\sum_{n=0}^{\infty}\sum_{m=\mp -n}^{n}(2n+1)\,A_{mn}(b>a)\times\right.$$

$$\times P_n^{m'}(\cos\theta)\,e^{im\varphi}\,\frac{1}{\zeta_n'(ka)} - \frac{\sin\alpha\cos\varphi}{\sin\theta}\times$$

$$\left.\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{\zeta_n(kb)}{\zeta_n(ka)}\,P_n^1(\cos\theta)\right\}, \qquad (7.2.34)$$

$$j_\varphi = -\frac{\omega\varepsilon\, pi}{4\pi\, ba}\left\{\frac{-i}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\,m\times\right.$$

$$\times A_{mn}(b>a)\,P_n^m(\cos\theta)\,e^{im\varphi}\,\frac{1}{\zeta_n'(ka)} - \sin\alpha\sin\varphi\times$$

$$\left.\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{\zeta_n(kb)}{\zeta_n(ka)}\cdot\frac{d^2}{d\theta^2}P_n(\cos\theta)\right\} \qquad (7.2.35)$$

($j_\theta = -\frac{c}{4\pi}H_\phi$, and $j_\phi = \frac{c}{4\pi}H_\theta$ when $r = a$). In the particular case of the vertical electric dipole, when $\alpha = 0$, $j_\phi = 0$, but

$$j_\theta = \frac{\omega\varepsilon\, pi}{4\pi\, b^2 ka}\sum_{n=0}^{\infty}(2n+1)\,\zeta_n(kb)\,\frac{P_n^1(\cos\theta)}{\zeta_n'(ka)}. \qquad (7.2.36)$$

In the wave band ($r \approx R + b\cos\theta$), approximately, /312

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\,H_\varphi = k^2 p\,\frac{e^{ikR}}{R}\,W_1(\theta,\ \varphi), \qquad (7.2.37)$$

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\,H_\theta = k^2 p\,\frac{e^{ikR}}{R}\,W_2(\theta,\ \varphi), \qquad (7.2.38)$$

where through $W_1(\theta, \phi)$ and $W_2(\theta, \phi)$ we denote the damping coefficients, equal to

$$W_1(\theta, \varphi) = \frac{e^{ikb\cos\theta}}{kb}\left\{ i\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n(2n+1)\times\right.$$

$$\times\left\{A_{mn}(b<r) - \frac{\psi_n(ka)}{\zeta_n'(ka)}A_{mn}(b>a)\right\}\times$$

$$\times\frac{d}{d\theta}P_n^m(\cos\theta)e^{im\varphi} + \frac{\sin\alpha\cos\varphi}{\sin\theta}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$

$$\left.\times\left\{\psi_n(kb) - \frac{\psi_n(ka)}{\zeta_n(ka)}\zeta_n(kb)\right\}(-i)^n P_n^1(\cos\theta)\right\}, \qquad (7.2.39)$$

$$W_2(\theta, \varphi) = \frac{e^{ikb\cos\theta}}{kb}\left\{\frac{i}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-1)^n im(2n+1)\times\right.$$

$$\times\left\{A_{mn}(b<r) - \frac{\psi_n'(ka)}{\zeta_n'(ka)}A_{mn}(b\geqslant a)\right\}P_n^m(\cos\theta)e^{im\varphi} +$$

$$+\sin\alpha\sin\varphi\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)}\left\{\psi_n(kb) - \frac{\psi_n(ka)}{\zeta_n(ka)}\zeta_n(kb)\right\}\times$$

$$\left.\times(-i)^n\frac{d^2P_n(\cos\theta)}{d\theta^2}\right\}. \qquad (7.2.40)$$

In the particular case of the vertical electric dipole, located on the surface of the sphere ($b = a$),

$$E_\varphi = H_\theta = 0,$$

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}H_\varphi = -k^2p\frac{e^{ikR}}{R}W(\theta), \qquad (7.2.41)$$

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where

$$W(\theta) = -\frac{e^{ika\cos\theta}}{k^2a^2}\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{P_n^1(\cos\theta)}{\zeta_n'(ka)}. \qquad (7.2.42)$$

This case is of particular interest in the sense that from the electrodynamic point of view it is equivalent to the case of a unit ring slot, located on a sphere (Fig. 38). The directional diagram of the vertical vibrator, located on the surface of the sphere, may be computed from the formula,

$$|W(\theta)| = \frac{1}{k^2a^2}\left|\sum_{n=1}^{\infty}(-i)^n(2n+1)\frac{P_n^{(1)}(\cos\theta)}{\zeta_n'(ka)}\right|. \qquad (7.2.43)$$

Formula (7.2.43) may be used also for computation of the directional diagrams of the ring emitting slot, located on the surface of an ideally conducting sphere, under the condition that the radius of the circle is small in comparison with the wave length and the radius of the sphere. The graphs showing the function $k^2a^2\,|W(\theta)|$ for the various values of the parameter ka can be found, for example, in [50].

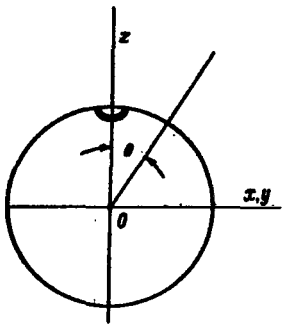

Fig. 38. Ring Slot on the Sphere.

If $ka \gg 1$, then asymptotically,

$$W(\theta) = \sin\theta\, G(\xi), \tag{7.2.44}$$

where the function $G(\xi)$ is determined just as in (6.1.58), where $\xi = -M\cos\theta$, but $M = (ka/2)^{1/3}$. In the case when $ka \to \infty$,

$$W(\theta) = 2\sin\theta \tag{7.2.45}$$

for $0 < \theta < \pi/2$ (the illuminated region) and /314

$$W(\theta) = 0 \tag{7.2.46}$$

for $\pi/2 < \theta < \pi$ (the shadow region). The plane $\theta = \pi/2$ is the geometric boundary between light and shadow, above which an ideally conducting sphere doubles the moments of the electric dipoles. As noted in [50], the value of the parameter $ka = 10$ is still insufficiently high in order for the characteristics of emission from the radial electric dipole, computed according to the precise formulas, to be given even qualitatively by the approximate formulas in (7.2.44). Allowable values of the parameter ka are values of $ka \geq 100$.

If we use formula (1.4.14) we can compute the resistance of the emission from the radial electric dipole, located on the surface of the sphere or the value of Γ determined by formula (1.4.16). In particular, for Γ we obtain

$$\Gamma = \frac{3}{2(ka)^4}\sum_{n=1}^{\infty}\frac{n(n+1)(2n+1)}{|\zeta_n'(ka)|^2} \tag{7.2.47}$$

(graphs of the function Γ can be found in [50]).

In the particular case of a horizontal electric dipole, located on the surface of an ideally conducting sphere, when $b = a$, $a \square \pi/2$, expressions (7.2.39), (7.2.40) vanish and along with them the components of the resulting field in the wave band.

For large values of the parameter ka, function (7.2.42) may be computed according to the approximate (asymptotic) formulas. In the region of the geometric shadow they have the form,

$$W(\theta) = \frac{e^{ika\cos\theta}}{\sqrt{\sin\theta}}\left[e^{ika\left(\theta-\frac{\pi}{2}\right)}g(\xi) + ie^{ika\left(\frac{3\pi}{2}-\theta\right)}g(\xi_1)\right], \qquad (7.2.48)$$

if the values of θ do not belong to the vicinity of the focal beam $\theta = \pi$, or

$$W(\theta) = -2\sqrt{\pi}M^{3/2}e^{i\left(ka+\frac{1}{2}\right)\frac{\pi}{4}+ika\cos\theta}\times$$
$$\times\sqrt{\frac{\pi-\theta}{\sin\theta}}J_1'[\nu_1'(\pi-\theta)]\,g\left(M\frac{\pi}{2}\right), \qquad (7.2.49)$$

if $\theta \approx \pi$ and $M(\pi - \theta) << 1$. Here the function $g(\xi)$ is determined as in formula (6.1.56), $J_1(x)$ is the Bessel function, $\nu_1' = ka + Mt_1'$, where t_1' is the kernel of the equation $w_1'(t) = 0$ [see (6.1.57)], $\xi = M\left(\theta - \frac{\pi}{2}\right)$, $\xi_1 = M\left(\frac{3\pi}{2} - \theta\right)$ (it is assumed that $\frac{1}{60M^2}|t_1|^{5/2} << 1$). /315

In the illuminated region,

$$W(\theta) = \sin\theta\,G(\xi) + \frac{ie^{ika\left(\cos\theta+\frac{3\pi}{2}-\theta\right)}}{\sqrt{\sin\theta}}g(\xi_1), \qquad (7.2.50)$$

where $G(\xi)$ is the same as in (6.1.58) and $\xi = -M\cos\theta < 0$. In (7.2.50) the basic contribution to the value of the function $W(\theta)$ is made only by the first term, whereas the second term gives only a small correction to the region of the shadow boundary.

Formulas (7.2.48) - (7.2.50) refine formulas (7.2.44) - (7.2.46). As mentioned in [50], comparison of the characteristics of radiation for $ka = 10$, computed from the precise formulas and from the asymptotic formulas for the function $W(\theta)$, shows that these latter graphically represent the character of the behavior of the function $W(\theta)$ sufficiently well and give completely satisfactory quantitative results whose divergence does not exceed 15 - 20%.

Magnetic Dipole Over an Ideally Conducting Sphere. Just as in the previous case, the Debye potentials of the field in the region inside the sphere are equal to zero and only the potentials u^1 and v^1 of the secondary field, scattered on the sphere, are subject to determination. If we seek them in the form of series (7.2.24), (7.2.25), then from boundary conditions (1.7.13) on the basis of

(7.2.20), (7.2.23) we find

$$b_{mn} = -\frac{ik|\mathbf{m}|}{b}(2n+1)\frac{\psi_n(ka)}{\zeta_n(ka)}B_{mn}(b \geqslant a) \qquad (7.2.51)$$

for all n and $|m| \leq n$, but

$$a_{mn} = 0, \quad \text{if} \quad |m| \neq 1,$$
$$a_{-1,n} = n(n+1)a_{1n}, \qquad (7.2.52)$$

where

$$a_{1n} = -\sqrt{\frac{\mu}{\varepsilon}}\frac{ik|\mathbf{m}|(2n+1)\sin\alpha}{2bn(n+1)} \times \qquad (7.2.53)$$
$$\times \frac{\psi_n'(ka)}{\zeta_n'(ka)}\zeta_n(kb).$$

Therefore,

$$v^1 = -\frac{i|\mathbf{m}|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\times$$
$$\times \frac{\psi_n(ka)}{\zeta_n(ka)}\zeta_n(kr)B_{mn}(b \geqslant a)P_n^m(\cos\theta)e^{im\varphi}, \qquad (7.2.54)$$

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$$u^1 = \sqrt{\frac{\mu}{\varepsilon}}\frac{|\mathbf{m}|\sin\alpha}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$
$$\times \frac{\psi_n'(ka)}{\zeta_n'(ka)}\zeta_n(kb)\zeta_n(kr)P_n^1(\cos\theta)\sin\varphi. \qquad (7.2.55)$$

Thus we solve the mathematical problem on seeking the Debye potentials of the magnetic dipole field in the presence of a sphere. For the resulting field in the region $b > r \geq a$,

$$u = -\sqrt{\frac{\mu}{\varepsilon}}\frac{|\mathbf{m}|\sin\alpha\sin\varphi}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$
$$\times\left\{\psi_n(kr) - \frac{\psi_n(ka)}{\zeta_n'(ka)}\zeta_n(kr)\right\}\zeta_n(kb)P_n^1(\cos\theta), \qquad (7.2.56)$$

$$v = \frac{i|\mathbf{m}|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\left\{\psi_n(kr) - \frac{\psi_n(ka)}{\zeta_n(ka)}\zeta_n(kr)\right\}\times$$
$$\times B_{mn}(b \geqslant a)P_n^m(\cos\theta)e^{im\varphi}, \qquad (7.2.57)$$

and in the region $b < r$

$$u = -\sqrt{\frac{\mu}{\varepsilon}}\,\frac{|m|\sin\alpha\sin\varphi}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$

$$\times\left\{\psi_n(kb)-\frac{\psi_n'(ka)}{\zeta_n'(ka)}\zeta_n(kb)\right\}\zeta_n(kr)\,P_n^1(\cos\theta), \qquad (7.2.58)$$

$$v = \frac{i|m|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\times$$

$$\times\left\{B_{mn}(b<r)-\frac{\psi_n(ka)}{\zeta_n(ka)}B_{mn}(b\geqslant a)\right\}\zeta_n(kr)\,P_n^m(\cos\theta)\,e^{im\varphi}. \qquad (7.2.59)$$

The components of the field may be found from (1.6.15), written in spherical coordinates. In the wave band, approximately,

$$H_\varphi = \sqrt{\frac{\varepsilon}{\mu}}\,E_\theta = k^2|m|\frac{e^{ikR}}{R}\tilde{W}_1(\theta,\ \varphi), \qquad (7.2.60)$$ /317

$$H_\theta = -\sqrt{\frac{\varepsilon}{\mu}}\,E_\varphi = k^2|m|\frac{e^{ikR}}{R}\tilde{W}_2(\theta,\ \varphi), \qquad (7.2.61)$$

where the damping coefficients are equal to

$$\tilde{W}_1(\theta,\ \varphi) = \frac{e^{ikb\cos\theta}}{kb}\left\{\sin\alpha\sin\varphi\sum_{n=1}^{\infty}(-i)^n\frac{2n+1}{n(n+1)}\times\right.$$

$$\times\left\{\psi_n(kb)-\frac{\psi_n(ka)}{\zeta_n'(ka)}\zeta_n(kb)\right\}\frac{d^2P_n(\cos\theta)}{d\theta^2}+$$

$$+\frac{i}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}im(2n+1)\left\{B_{mn}(b<r)-\right. \qquad (7.2.62)$$

$$\left.\left.-\frac{\psi_n(ka)}{\zeta_n(ka)}B_{mn}(b\geqslant a)\right\}(-i)^n P_n^m(\cos\theta)\,e^{im\varphi}\right\},$$

$$\tilde{W}_2(\theta,\ \varphi) = \frac{e^{ikb\cos\theta}}{kb}\left[\frac{\sin\alpha\cos\varphi}{\sin\theta}\sum_{n=1}^{\infty}(-i)^n\frac{2n+1}{n(n+1)}\times\right.$$

$$\left.\times\left\{\psi_n(kb)-\frac{\psi_n'(ka)}{\zeta_n'(ka)}\zeta_n(kb)\right\}P_n^1(\cos\theta)+\right.$$

$$+i\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\left\{B_{mn}(b<r)-\frac{\psi_n(ka)}{\zeta_n(ka)}B_{mn}(b\geqslant a)\right\}\times \qquad (7.2.63)$$

$$\times(-i)^n\frac{d}{d\theta}P_n^m(\cos\theta)\,e^{im\varphi}\Big].$$

In particular, if the dipole is located on the surface of the sphere ($b = a$) and $\alpha = \pi/2$ (the case of a horizontal magnetic dipole), then (7.2.60), (7.2.61) may be written in the form,

$$H_\varphi=\sqrt{\frac{\varepsilon}{\mu}}E_\theta=k^2|\mathbf{m}|\sin\varphi\frac{e^{ikR}}{R}\tilde{W}_2(\theta),$$

$$H_\theta=-\sqrt{\frac{\varepsilon}{\mu}}E_\varphi=-k^2|\mathbf{m}|\cos\varphi\frac{e^{ikR}}{R}\tilde{W}_1(\theta), \qquad (7.2.64)$$

where /318

$$\tilde{W}_2(\theta)=-\frac{e^{ika\cos\theta}}{ka}\left\{i\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{(-i)^n}{\zeta_n'(ka)}\tau_n(\cos\theta)+\right.$$

$$\left.+\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{(-i)^n}{\zeta_n(ka)}\pi_n(\cos\theta)\right\}, \qquad (7.2.65)$$

$$\tilde{W}_1(\theta)=-\frac{e^{ika\cos\theta}}{ka}\left\{i\sum_{n=1}^{\infty}(-i)^n\frac{2n+1}{n(n+1)}\cdot\frac{1}{\zeta_n'(ka)}\times\right.$$

$$\left.\times\pi_n(\cos\theta)+\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{(-i)^n}{\zeta_n(ka)}\tau_n(\cos\theta)\right\}. \qquad (7.2.66)$$

An ideally conducting sphere with a horizontal magnetic dipole on its surface in the electrodynamic sense is equivalent to a unit radiating "dumb-bell" slot, cut in the sphere (Fig. 39). The characteristic emission from such a slot may be determined from formulas (7.2.65), (7.2.66). This case was investigated in [50].

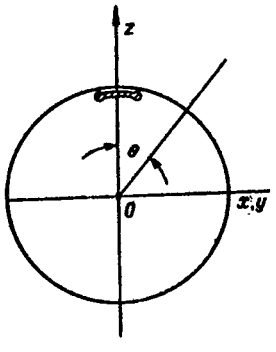

Fig. 39. "Dumb-Bell" Slot on a Sphere.

Series (7.2.65), (7.2.66) can be conveniently used for values of the parameter ka which do not exceed 10. If $ka \gg 1$, then approximately,

$$\tilde{W}_1(\theta)=-\frac{i}{M}F(\xi), \qquad (7.2.67)$$

$$\tilde{W}_2(\theta)=-G(\xi),$$

where the functions $F(\xi)$, $G(\xi)$ are determined as in (6.1.53) and (6.1.58), respectively, $\xi = -M\cos\theta$, $M = (ka/2)^{1/3}$. Formulas (7.2.67) give a continuous transition from the illuminated region to the shadow region, and when $ka \to \infty$, $0 < \theta < \pi/2$, /319

$$\tilde{W}_1(\theta) = -2\cos\theta,$$
$$\tilde{W}_2(\theta) = -2, \qquad (7.2.68)$$

and when $ka \to \infty$ in the region $\pi/2 < \theta < \pi$,

$$\tilde{W}_1(\theta) = 0,$$
$$\tilde{W}_2(\theta) = 0. \qquad (7.2.69)$$

The plane $\theta = \pi/2$ is the interface between the illuminated region and that of the shadow.

Just as in the case of the electric dipole, formulas (7.2.67) give satisfactory results for values of $ka \gtrsim 100$ (when $ka = 10$ they do not reflect, even qualitatively, the true characteristics of the emission [50]). For values of $ka \geq 10$ the approximate values of the characteristics $\tilde{W}_1(\theta)$ and $\tilde{W}_2(\theta)$ may be computed using the asymptotic formulas having the form:

in the shadow region

$$\tilde{W}_1(\theta) = u^{(1)}(\theta) - \frac{i}{ka\sin\theta}W(\theta),$$
$$\tilde{W}_2(\theta) = u^{(2)}(\theta) - \frac{1}{ka\,M\sin\theta}u(\theta), \qquad (7.2.70)$$

where outside the vicinity of the focal beam $\theta = \pi$,

$$u^{(1)}(\theta) = \frac{-ie^{ika\cos\theta}}{M\sqrt{\sin\theta}}\left[e^{ika\left(\theta-\frac{\pi}{2}\right)}f(\xi) - ie^{ika\left(\frac{3\pi}{2}-\theta\right)}f(\xi_1)\right],$$
$$u^{(2)}(\theta) = \frac{-e^{ika\cos\theta}}{\sqrt{\sin\theta}}\left[e^{ika\left(\theta-\frac{\pi}{2}\right)}g(\xi) - ie^{ika\left(\frac{3\pi}{2}-\theta\right)}g(\xi_1)\right], \qquad (7.2.71)$$
$$u(\theta) = \frac{-e^{ika\cos\theta}}{\sqrt{\sin\theta}}\left[e^{ika\left(\theta-\frac{\pi}{2}\right)}f(\xi) + ie^{ika\left(\frac{3\pi}{2}-\theta\right)}f(\xi_1)\right],$$

and in the vicinity of the beam $\theta = \pi$, when $M(\pi - \theta) \ll 1$,

$$u^{(1)}(\theta) = 2\sqrt{\pi}M^{1/2}e^{i\left(ka+\frac{1}{2}\right)\frac{\pi}{2}+ika\cos\theta}\times$$

$$\times \sqrt{\frac{\pi-\theta}{\sin\theta}}\, J_0''[\nu_1^0(\pi-\theta)]\, f\left(M\frac{\pi}{2}\right),$$

$$u^{(2)}(\theta) = 2\sqrt{\pi}\, M^{3/2}\, e^{i\left(ka-\frac{1}{2}\right)\frac{\pi}{2}+ika\cos\theta} \times$$

$$\times \sqrt{\frac{\pi-\theta}{\sin\theta}}\, J_0''[\nu_1'(\pi-\theta)]\, g\left(M\frac{\pi}{2}\right), \qquad (7.2.72)$$

$$u(\theta) = 2\sqrt{\pi}\, M^{3/2}\, e^{i\left(ka+\frac{1}{2}\right)\frac{\pi}{2}+ika\cos\theta} \times$$

$$\times \sqrt{\frac{\pi-\theta}{\sin\theta}}\, J_1[\nu_1^0(\pi-\theta)]\, f\left(M\frac{\pi}{2}\right),$$

where $f(\xi)$ is determined as in (6.1.51), and $g(\xi)$ as in (6.1.56); $\nu_1^0 = ka + Mt_1^0$, where t_1^0 is the kernel of the equation $w_1(t) = 0$, the value of ν_1' is determined just as before $\left(\frac{1}{60M^2}|t_1|^{5/2} << 1\right)$.

When $\theta = \pi$,

$$\tilde{W}_1(\pi) = -\tilde{W}_2(\pi). \qquad (7.2.73)$$

For large positive values of ξ function $f(\xi)$ tends to zero more rapidly than does $g(\xi)$. For this reason, in the first of the formulas in (7.2.70) the term $-iW(\theta)/ka\sin\theta$ is included, which (for values of $\xi \sim 1$ in order of magnitude) is M^2 times smaller than the first; however in proportion to distance from the shadow it gradually becomes comparable with the first and near the terminal $\theta = \pi$, becomes the dominant term. In contradiction to this, the term $-u(\theta)/kaM\sin\theta$ has no substantial significance in computing the function $\tilde{W}_2(\theta)$ [50].

In the illuminated region,

$$\tilde{W}_1(\theta) = -\frac{i}{M}F(\xi) + \tilde{W}_1''(\theta),$$

$$\tilde{W}_2(\theta) = -G(\xi) + \tilde{W}_2''(\theta), \qquad (7.2.74)$$

where

$$\tilde{W}_1'' = -\frac{e^{ika\left(\cos\theta+\frac{3\pi}{2}-\theta\right)}}{M\sqrt{\sin\theta}}\, f(\xi_1),$$

$$\tilde{W}_2'' = \frac{ie^{ika\left(\cos\theta+\frac{3\pi}{2}-\theta\right)}}{\sqrt{\sin\theta}}\, g(\xi_1). \qquad (7.2.75)$$

The functions $F(\xi)$ and $G(\xi)$ are determined here as in (6.1.53) and (6.1.58), respectively, $\xi = -M\cos\theta < 0$. In (7.2.74) the basic terms are the first ones. The functions $\tilde{W}_1''(\theta)$ and $\tilde{W}_2''(\theta)$ give only a small correction in the region of the boundary of the shadow. Comparison of the characteristics of the emission for $ka = 10$, obtained from the precise formulas and from the asymptotic formulas given above for the functions $\tilde{W}_1(\theta)$, $\tilde{W}_2(\theta)$, shows that even when $ka = 10$ they graphically reflect the character of the behavior of these functions quite fully. From the quantitative point of view the divergence of the results does not exceed 15 - 20%. /321

In the case of a horizontal magnetic dipole in the presence of a sphere, the resistance from the radiation may be computed from formula (1.4.14). In particular, for the horizontal magnetic dipole, lying on the surface of an ideally conducting sphere, from (1.4.16) for the value of Γ we find

$$\Gamma = \Gamma_1 + \Gamma_2, \tag{7.2.76}$$

where

$$\Gamma_1 = \frac{3}{2(ka)^2}\sum_{n=1}^{\infty}\frac{2n+1}{2|\zeta_n(ka)|^2},$$

$$\Gamma_2 = \frac{3}{2(ka)^2}\sum_{n=1}^{\infty}\frac{2n+1}{2|\zeta_n'(ka)|^2}.$$

The graphs of the functions Γ_1, Γ_2 are shown in [50].

It is easy to see that for the vertical magnetic dipole, located on the surface of an ideally conducting sphere, when $\alpha = 0$ and $b = a$, the functions $\tilde{W}_1(\theta, \phi)$ and $\tilde{W}_2(\theta, \phi)$ from (7.2.60), (6.2.61) vanish.

In conclusion let us note that the magnetic dipole located at point P at a distance b from the center of the sphere and oriented at an angle α to the Oz axis, induces currents on its surface whose surface density is determined by the expressions:

$$j_\theta = -\frac{ck|\mathbf{m}|}{4ba\pi}\left\{\sin\alpha\sin\varphi\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{\zeta_n(kb)}{\zeta_n'(ka)}\times\right.$$

$$\times\tau_n(\cos\theta) - \frac{1}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} im\frac{2n+1}{\zeta_n(ka)}\times \tag{7.2.77}$$

$$\left.\times B_{mn}(b \geqslant a)\,P_n^m(\cos\theta)\,e^{im\varphi}\right\},$$

$$j_\varphi = -\frac{ck\,|m|}{4ba\,\pi}\left\{\sin\alpha\cos\varphi\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\cdot\frac{\zeta_n(kb)}{\zeta_n'(ka)}\times\right.$$

$$\times\,\pi_n(\cos\theta)-\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\frac{B_{mn}(b>a)}{\zeta_n(ka)}\times \qquad (7.2.78)$$

$$\left.\times\,\frac{d}{d\theta}P_n^m(\cos\theta)\,e^{im\varphi}\right\}.$$

Dipole Emitter Over a Sphere With Finite Conductivity. In this case, regardless of the type of dipole, along with the existence of a secondary field scattered on the sphere there will exist a non-zero field inside the sphere as well. Therefore, the scalar problems for the Debye potentials consist now of seeking the functions u, v both outside the sphere and inside it under the conjugate conditions of (1.7.10), (1.7.11) on the surface of the sphere. If the dipole is electric and its moment p forms an arbitrary angle α with the polar axis Oz, and the Debye potentials of the secondary field in the region outside the sphere are sought in the form of series (7.2.24), (7.2.25) and inside the sphere, in the form of series such as (3.1.13),

$$u^{(2)} = \frac{1}{k_2 r}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\tilde{A}_{mn}\,\psi_n(k_2 r)\,P_n^m(\cos\theta)\,e^{im\varphi}, \qquad (7.2.79)$$

$$v^{(2)} = \frac{1}{k_2 r}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\tilde{A}_{mn}^{*}\,\psi_n(k_2 r)\,P_n^m(\cos\theta)\,e^{im\varphi}, \qquad (7.2.80)$$

then we find that boundary conditions (1.7.10), (1.7.11) will be satisfied only in the case if

$$a_{mn} = -\frac{ipk}{b}(2n+1)\beta_n A_{mn}\,(b\geqslant a),$$

$$\tilde{A}_{mn} = \frac{pk}{b}\times$$

$$\times\frac{k\mu_2}{k\mu_2\psi_n'(k_2a)\zeta_n(ka)-k_2\mu\psi_n(k_2a)\zeta_n'(ka)}A_{mn}\,(b\geqslant a), \qquad (7.2.81)$$

$$b_{mn} = \tilde{A}_{mn}^{*} = 0, \quad \text{if} \quad m=0 \quad \text{or} \quad |m|>1;$$

$$b_{n,-1} = n(n+1)\,b_{1n}, \quad \tilde{A}_{-1,n}^{*} = n(n+1)\,\tilde{A}_{1n}^{*},$$

$$b_{1n} = -\sqrt{\frac{\varepsilon}{\mu}}\,\frac{kp\sin\alpha}{2ib}\cdot\frac{2n+1}{n(n+1)}\,a_n\zeta_n(kb),$$

$$\tilde{A}^{*}_{1n} = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{kp\sin\alpha}{ib}\cdot\frac{2n+1}{n(n+1)}\times \qquad (7.2.82)$$

$$\times\frac{ik_2\mu}{k\mu_2\psi_n(k_2a)\zeta_n'(ka) - k_2\mu\psi_n'(k_2a)\zeta_n(ka)}\,\zeta_n(kb).$$

If we take the magnetic dipole as the source of the primary field, and the Debye potentials of the secondary field in the respective regions are sought in the form of series (7.2.24), (7.2.25) and (7.2.79), (7.2.80), then the boundary conditions on the surface of the sphere will be satisfied only under the condition that

$$b_{mn} = -\frac{i|m|k}{b}(2n+1)\,a_n B_{mn}(b \geqslant a),$$

$$\tilde{A}^{*}_{mn} = \frac{|m|k}{b}\times \qquad (7.2.83)$$

$$\times\frac{-k_2\mu}{k_2\mu\zeta_n(ka)\psi_n'(k_2a) - k\mu_2\zeta_n'(ka)\psi_n(k_2a)}\,B_{mn}(b \geqslant a),$$

and

$$a_{mn} = \tilde{A}_{mn} = 0, \quad \text{if} \quad m \neq 0 \ \text{or} \ |m| > 1,$$

$$a_{-1,n} = n(n+1)\,a_{1n}, \quad \tilde{A}_{-1,n} = n(n+1)\,\tilde{A}_{1n},$$

$$a_{1n} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{k|m|\sin\alpha}{2bi}\cdot\frac{2n+1}{n(n+1)}\,\beta_n\zeta_n(kb),$$

$$A_{1n} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{k|m|\sin\alpha}{b}\cdot\frac{2n+1}{n(n+1)}\times \qquad (7.2.84)$$

$$\times\frac{k\mu_2}{\mu k_2\psi_n(k_2a)\zeta_n'(ka) - \mu_2 k\zeta_n(ka)\psi_n'(k_2a)}\,\zeta_n(kb).$$

As a result for the field of the electric dipole in the presence of a sphere with finite conductivity, the Debye potentials are determined by the expansions:

in the region $b > r \geq a$,

$$u = \frac{ip}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\{\psi_n(kr) - \beta_n\zeta_n(kr)\}\times$$

$$\times A_{mn}(b \geqslant a)\,P_n^m(\cos\theta)\,e^{im\varphi}, \qquad (7.2.85)$$

$$v = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p\sin\alpha\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\{\psi_n(kr)-\alpha_n\zeta_n(kr)\}P_n^1(\cos\theta)\zeta_n(kb), \tag{7.2.86}$$

where through α_n, β_n we denote the Mie coefficients (7.1.78), (7.1.79);

in the region $r > b$,

$$u = \frac{ip}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\{A_{mn}(r>b)-$$

$$-\beta_n A_{mn}(b>a)\}\zeta_n(kr)P_n^m(\cos\theta)e^{im\varphi}, \tag{7.2.87}$$

$$v = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p\sin\alpha\sin\varphi}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$

$$\times\{\psi_n(kb)-\alpha_n\zeta_n(kb)\}\zeta_n(kr)P_n^1(\cos\theta), \tag{7.2.88}$$

and finally in the region $r \leq a$,

$$u = \frac{pk^2\mu_2}{k_2 br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\gamma_n A_{mn}(b\geqslant a)\times$$

$$\times\psi_n(k_2r)P_n^m(\cos\theta)e^{im\varphi}, \tag{7.2.89}$$

$$v = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{ikp\sin\alpha\cdot\mu\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\delta_n\zeta_n(kb)P_n^1(\cos\theta)\psi_n(k_2r), \tag{7.2.90}$$

where

$$\gamma_n = \frac{1}{\mu_2 k\psi_n'(k_2a)\zeta_n(ka)-k_2\mu\psi_n(k_2a)\zeta_n'(ka)},$$

$$\delta_n = \frac{1}{k\mu_2\psi_n(k_2a)\zeta_n'(ka)-k_2\mu\psi_n'(k_2a)\zeta_n(ka)}. \tag{7.2.91}$$

For a magnetic dipole field in the presence of a sphere with finite conductivity we have: /325

in the region $b > r \geq a$,

$$u = -\sqrt{\frac{\mu}{\varepsilon}}\,\frac{|\mathbf{m}|\sin\alpha\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\{\psi_n(kr)-\beta_n\zeta_n(kr)\}\,\zeta_n(kb)\,P_n^1(\cos\theta), \qquad (7.2.92)$$

$$v = \frac{i|\mathbf{m}|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\{\psi_n(kr)-\alpha_n\zeta_n(kr)\}\times$$

$$\times B_{mn}(b\geqslant a)\,P_n^m(\cos\theta)\,e^{im\varphi}, \qquad (7.2.93)$$

in the region $r > b$

$$u = -\sqrt{\frac{\mu}{\varepsilon}}\,\frac{|\mathbf{m}|\sin\alpha\sin\varphi}{br}\times$$

$$\times\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\{\psi_n(kb)-\beta_n\zeta_n(kb)\}\,\zeta_n(kr)\,P_n^1(\cos\theta), \qquad (7.2.94)$$

$$v = \frac{i|\mathbf{m}|}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\{B_{mn}(r>b)-$$

$$-\alpha_n B_{mn}(b\geqslant a)\}\,\zeta_n(kr)\,P_n^1(\cos\theta)\,e^{im\varphi}, \qquad (7.2.95)$$

and inside the sphere $r \leq a$

$$\mu = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{|\mathbf{m}|ik^2\mu_2}{k_2br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\times$$

$$\times\gamma_n\psi_n(k_2r)\,\zeta_n(kb)\,P_n^1(\cos\theta)\cos\varphi\sin\alpha, \qquad (7.2.96)$$

$$v = \frac{|\mathbf{m}|k\mu}{br}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\delta_n B_{mn}(b\geqslant a)\times$$

$$\times\psi_n(k_2a)\,P_n^m(\cos\theta)\,e^{im\varphi}. \qquad (7.2.97)$$

In the same way we completely solve the problem for the Debye potentials. Using (7.2.85) - (7.2.97) and the formulas in (1.6.15), written in spherical coordinates, the components of the electromagnetic field E, H may be found at any point of space.

In the theory of the diffraction of a dipole source field on a sphere with finite conductivity, of special interest is the /326

problem on the propagation of radio waves around the Earth. Here the region usually studied is that of $b > r \geq a$, and the parameters ka, kr and kb are assumed quite large. In this case, the series given above become unsuitable for numerical computations. However, we can obtain approximate computational formulas from them which are suitable for computation with large values of these parameters. The analytical results from the field of the dipole emitter in the presence of a sphere with finite conductivity are shown in the works of Fok [70]. The results from the numerical computation of the problem on propagation of radio waves around the Earth are given in [67], where the following formulas were used ($\varepsilon = \mu = 1$):

for the vertical electric dipole,

$$E_r = -H_\varphi = k^2 p \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q),$$
$$E_\theta = k^2 p \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \cdot \frac{\partial}{\partial y_1} V(x, y_1, y_2, q), \qquad (7.2.98)$$
$$E_\varphi = H_r = H_\theta = 0;$$

for the vertical magnetic dipole,

$$H_r = E_\varphi = k^2 |\mathfrak{m}| \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q'),$$
$$H_\theta = k^2 |\mathfrak{m}| \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \cdot \frac{\partial}{\partial y_1} V(x, y_1, y_2, q'), \qquad (7.2.99)$$
$$E_r = E_\theta = H_\varphi = 0;$$

for the horizontal electric dipole,

$$\dot{E}_\varphi = H_r = -k^2 p \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q') \sin\varphi,$$
$$E_r = -H_\varphi = -k^2 p \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \times$$
$$\times \frac{\partial}{\partial y_2} V(x, y_1, y_2, q) \cos\varphi, \qquad (7.2.100)$$
$$H_\theta = -k^2 p \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \cdot \frac{\partial}{\partial y_1} V(x, y_1, y_2, q') \sin\varphi,$$
$$E_\theta = 0;$$

for the horizontal magnetic dipole /327

$$H_\varphi = -E_r = -k^2 |\mathbf{m}| \frac{e^{ika\theta}}{a\sqrt{\theta \sin\theta}} V(x, y_1, y_2, q) \sin\varphi;$$

$$H_r = E_\varphi = -k^2 |\mathbf{m}| \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \cdot \frac{\partial}{\partial y_2} V(x, y_1, y_2, q') \cos\varphi, \qquad (7.2.101)$$

$$E_\theta = k^2 |\mathbf{m}| \frac{ie^{ika\theta}}{Ma\sqrt{\theta \sin\theta}} \cdot \frac{\partial}{\partial y_1} V(x, y_1, y_2, q) \sin\varphi,$$

$$H_\theta = 0,$$

where as everywhere above, it is assumed that the dipole moment p or m lies in the plane $\phi = 0$. The function V, which enters into expressions (7.2.98), (7.2.99) (damping coefficient), depends on three dimensionless coordinates x, y_1, y_2 and on the complex parameter q or q'. Here h_1 is the height of the point of emission, h_2 is the height of the point of observation, s is the distance between them, determined according to the arc of the circumference of the sphere (Fig. 40), and

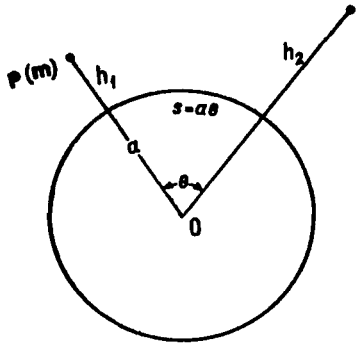

Fig. 40. Determination of the Function V.

$$x = \frac{Ms}{a^*}, \quad y = \frac{kh}{M},$$

$$q = iM\sqrt{\varepsilon - 1 + i\frac{4\pi\sigma}{\omega}} \Big/ \left(\varepsilon + i\frac{4\pi\sigma}{\omega}\right), \qquad (7.2.102)$$

$$q' = iM\sqrt{\varepsilon - 1 + i\frac{4\pi\sigma}{\omega}}$$

(ε, σ represent the permeability and the conductivity of the soil), /328 where $M = \frac{ka^*}{2}^{1/3} = \frac{\pi a^*}{\lambda_0}^{1/3}$ is the large parameter of the problem. Through a^* here we denote the so-called effective radius of the Earth, which takes into account the variation in the refractive index of the Earth's atmosphere in that case when it depends linearly on altitude. For a uniform atmosphere, $a^* = a$. The damping coefficient can be represented by the integral,

$$V(x, y_1, y_2, q) = e^{-\frac{i\pi}{4}} \sqrt{\frac{x}{\pi}} \int_\Gamma e^{ixt} F(t, y_1, y_2, q)\, dt, \qquad (7.2.103)$$

where just as before, Γ is the profile in the plane of the complex variable t from $e^{\frac{i2\pi}{3}} \cdot \infty$ to 0 and from 0 to ∞ (Fig. 22). For $y_2 > y_1$,

$$F(t, y_1, y_2, q) = w(t - y_2)\Phi(t, y_1, q), \tag{7.2.104}$$

where

$$\Phi(t, y_1, q) = v(t - y_1) - \frac{v'(t) - qv(t)}{w'(t) - qw(t)} w(t - y_1) =$$

$$= \frac{i}{2}\left\{ w_2(t - y_1) - \frac{w_2'(t) - qw_2(t)}{w'(t) - qw(t)} w(t - y_1)\right\}, \tag{7.2.105}$$

and $w(t)$ is the Fok-Airy function.

Formulas (7.2.98) - (7.2.101) are suitable for computation in the region of the shadow (for all θ) and in the region of half shadow, but in the illuminated region they give a correct result only for sliding incidence of the beam.

In conclusion, let us cite approximate formulas for computation of the field scattered on a sphere with finite conductivity in the case of a plane wave incident on the sphere $E_x^0 = -H_y^0 = e^{-ikr\cos\theta}$, when the value of the parameter ka is high. These formulas for the diffraction field near the surface of the sphere at small distances from it (in comparison with the radius of the sphere) have the form [67],

$$E_r^1 = -H_\varphi^1 = \frac{e^{ika\left(\theta - \frac{\pi}{2}\right)}}{\sqrt{\sin\theta}} V_1(z, y, q)\cos\varphi, \tag{7.2.106}$$

$$E_\varphi^1 = H_r^1 = \frac{e^{ika\left(\theta - \frac{\pi}{2}\right)}}{\sqrt{\sin\theta}} V_1(z, y, q')\sin\varphi,$$

$$E_\theta^1 = -\frac{ie^{ika\left(\theta - \frac{\pi}{2}\right)}}{M\sqrt{\sin\theta}} \cdot \frac{\partial}{\partial y} V_1(z, y, q)\sin\varphi,$$

$$H_\theta^1 = \frac{ie^{ika\left(\theta - \frac{\pi}{2}\right)}}{M\sqrt{\sin\theta}} \cdot \frac{\partial}{\partial y} V_1(z, y, q')\cos\varphi,$$

where through V_1 we denote the damping coefficient for the field of the incident plane wave, determined in [70] in the form of an integral, /329

$$V_1(z, y, q) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{izt} \Phi(t, y, q)\, dt, \qquad (7.2.107)$$

where Γ is the profile just as in (7.2.103), and $z = \left(\frac{ka}{2}\right)^{1/3} \left(\theta - \frac{\pi}{2}\right)$, $y = \left(\frac{ka}{2}\right)^{-1/3} k(r - a)$. Formulas (7.2.106) are applicable in the region of the shadow, half-shadow and in that part of the illuminated region where the angle of slide of the incident wave is small. The parameters q and q' are determined just as before.

§3. Diffraction of a Plane Wave on Two Spheres.

In an unbounded uniform and isotropic space with permeabilities ε, μ and a conductivity $\sigma = 0$, let there be found two spheres which occupy, in the spherical local coordinate systems r_s, θ_s, ϕ_s ($s = \pm 1$), regions of space $r_{+1} \leq a_{+1}$ and $r_{-1} \leq a_{-1}$ (local coordinate systems are introduced in such a way that their polar axes coincide also for any point in space $\phi_{+1} = \phi_{-1} = \phi$), and let a plane electromagnetic wave (1.3.4) be incident on the sphere; this wave propagates in the direction of the unit vector $\mathbf{n}$ which forms an arbitrary angle α with the axis of rotation that is common for the spheres, and is polarized in such a way that its electric vector $\mathbf{E}^0$ is directed at an angle β to the plane $\phi = 0$. It is assumed that the electromagnetic properties of the spheres are identical and are determined by the permeabilities ε_2, μ_2 and the conductivity σ_2. We pose the problem of seeking the secondary electromagnetic field, scattered on spheres, and also the field which arises inside them in the case of finite conductivity σ_2. A strict solution to the problem consists of integrating the uniform system of Maxwell equations (1.1.4) ($\rho = 0$, $\mathbf{J}^{(ext)} = 0$) under the boundary conditions (1.1.5) or (1.1.6) on the surface of each of the spheres and the supplementary condition of radiation at infinity for the components of the vectors of scattering of the field. In spherical coordinates the solution to problem may be represented by formulas (1.6.15) through the Debye potentials u, v, each of which is found as the solution to the respective scalar boundary problem formulated for the Helmholtz equation under the boundary conditions (1.7.10), (1.7.11), if $\sigma_2 \neq \infty$ or conditions (1.7.13) if $\sigma_2 = \infty$, on the surface of each sphere. As the basic reference system we can take one of the local coordinate systems or the spherical system of coordinates r, θ, ϕ with the origin at point O, lying on the line of the centers of the spheres midway between the centers (Fig. 41; $\phi_s = \phi$, $s \pm 1$). /330

Below it is assumed that the phase coefficient $(\mathbf{n}, \mathbf{r})$ of the plane wave (1.3.4) is determined in the coordinate system with the origin at point O.

For solution to the scalar boundary problems we must have explicit expressions for the potentials u^0 and v^0 of the primary field in local coordinates of the s-th sphere ($s = \pm 1$). Formulas (7.1.10), (7.1.11), derived earlier for the field of the plane wave (1.3.4) propagating along the polar axis of the spherical system of coordinates and polarized such that $E^0 \parallel$ pl. $\phi = 0$, can not be directly used in this case. They must be converted to a suitable form. Conversion of formulas (7.1.10) and (7.1.11) consists of reexpansion of the series into spherical wave functions written in local coordinates of the s-th sphere ($s = \pm 1$) taking into account the given direction of propagation of the wave front and its polarization. For this purpose we introduce into study new local systems of spherical coordinates r_s', θ_s', ϕ_s' ($r_s' = r_s$, $s = \pm 1$), in which the polar axes are directed along the vector n (to the side of the wave propagation), and as the planes $\phi_s' = 0$ we take the planes that are parallel to the vector E^0. In these coordinates,

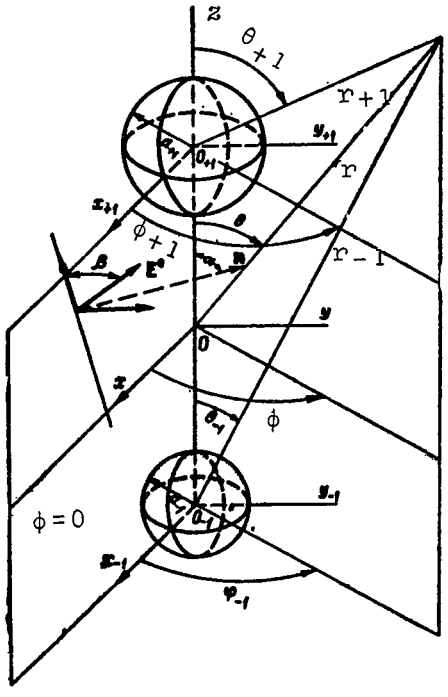

Fig. 41. Diffraction of a Plane Wave on Two Spheres.

$$E^0_{x_s'} = Ee^{ik(\mathrm{nr}_{0s}) + ik(\mathrm{nr}_s')} \qquad (7.3.1)$$

(just as before, here r_{0s} is the radius-vector which connects the point O_s with the origin of the coordinate system $Oxyz$; $(\mathrm{nr}_{0s}) = sl_0 \cos\alpha$, $l_0 = l/2$, $s = \pm 1$). Therefore the potentials u^0 and v^0 of the primary field in local coordinates r_s', θ_s', ϕ_s' are written on the basis of (7.1.10) and (7.1.11) in the following form:

$$u^0 = \frac{Ee^{iksl_0\cos\alpha}}{ik^2 r_s'} \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \psi_n(kr_s') P_n^1(\cos\theta_s') \cos\varphi_s,$$

$$v^0 = \frac{Ee^{iksl_0\cos\alpha}}{ik^2 r_s'} \sqrt{\frac{\varepsilon}{\mu}} \times \qquad (7.3.2)$$

$$\times \sum_{n=1}^{\infty} i^n \frac{2n+1}{n(n+1)} \psi_n(kr_s') P_n^1(\cos\theta_s') \sin\varphi_s'.$$

Hence we obtain the unknown expansions of the potentials u^0 and v^0 of the primary field according to the spherical wave functions of the s-th sphere (in the coordinates r_s, θ_s, ϕ_s; $s = \pm 1$), if we

apply the addition theorems (3.4.21), (3.4.22) to the spherical functions $p_n^1(\cos\theta_s') \times \begin{matrix}\cos\\ \sin\end{matrix}(\phi_s')$:

$$u^0 = \frac{Ee^{iksl_s\cos\theta}}{ik^2r_s^*}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n\frac{2n+1}{n(n+1)}\times \tag{7.3.3}$$
$$\times L_{m1}^n(\psi_3,\ \theta_3,\ \varphi_3)\psi_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi},$$

$$v^0 = \frac{Ee^{iksl_s\cos\alpha}}{ik_s^2r_s}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} i^n\frac{2n+1}{n(n+1)}\times \tag{7.3.4}$$
$$\times L_{m1}^{*n}(\psi_3,\ \theta_3,\ \varphi_3)\psi_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi}.$$

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The quantities $L_{m1}^n(\psi_3,\ \theta_3,\ \phi_3)$ and $L_{m1}^{*n}(\psi_3,\ \theta_3,\ \phi_3)$ are determined here, just as in (3.4.24) with the Euler angles $\psi_3 = \pi/2 - \beta$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$.

The diffraction problems are solved in the following manner.

Ideally Conducting Spheres. In the case of ideally conducting spheres only the potentials u^1 and v^1 of the secondary field scattered on the spheres are subject to determination. They are sought in the form of the sum,

$$u^1 = \sum_{s=\pm1} u_s, \quad v^1 = \sum_{s=\pm1} v_s, \tag{7.3.5}$$

where the diffraction series

$$u_s = \frac{E}{ik^2r_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} a_{mn}^s\zeta_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi}, \tag{7.3.6}$$

$$v_s = \frac{E}{ik^2r_s}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_{mn}^s\zeta_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi} \tag{7.3.7}$$

with the still undetermined coefficients a_{mn}^s and b_{mn}^s ($s = \pm 1$) represent an expansion of the potentials u_s and v_s of the secondary field produced by the presence in space of the s-th sphere, according to the spherical wave functions of this sphere. The coefficients a_{mn}^s and b_{mn}^s are found from the boundary conditions (1.7.13) on the surface of each sphere. If we introduce a substitution of unknowns a_{mn}^s and b_{mn}^s by the new unknowns A_{mn}^s and B_{mn}^s according to the formulas[3],

[3] The prime in the functions $\psi_n(x)$ and $\zeta_n(x)$ means differentation over the argument.

$$a_{mn}^{s} = \left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} \psi_n'(ka_s)\, A_{mn}^{s},$$

$$b_{mn}^{s} = \left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} \psi_n(ka_s)\, B_{mn}^{s}, \quad s = \pm 1, \tag{7.3.8}$$

then using the addition theorems (3.7.10) for the spherical wave functions and formulas (7.3.3), (7.3.4) for the potentials u^0, v^0, from the boundary conditions we find the infinite systems of linear equations

$$A_{mn}^{s} + \sum_{q=|m|}^{\infty} \alpha_{mnq}^{(-s,s)} A_{mq}^{-s} = f_{mn}^{s}, \tag{7.3.9}$$

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$$B_{mn}^{s} + \sum_{q=|m|}^{\infty} \beta_{mnq}^{(-s,s)} B_{mq}^{-s} = \varphi_{mn}^{s},$$

$$n = |m|, \ |m|+1, \ \ldots, \ |m| \geqslant 1; \tag{7.3.10}$$

$$q, \ n = 1, 2, \ \ldots, \ \text{if} \quad m = 0; \quad s = \pm 1,$$

for the unknowns A_{mn}^{s} and B_{mn}^{s} with the matrix elements,

$$\alpha_{mnq}^{(-s,s)} = \left\{\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right\}^{1/2} \frac{\psi_q'(ka_{-s})}{\zeta_n'(ka_s)}\, Q_{mnmq}^{(1)}, \tag{7.3.11}$$

$$\beta_{mnq}^{(-s,s)} = \left\{\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right\}^{1/2} \frac{\psi_q(ka_{-s})}{\zeta_n(ka_s)}\, Q_{mnmq}^{(1)}, \tag{7.3.12}$$

with the right-hand sides ($\psi_3 = \pi/2 - \beta$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$)

$$f_{mn}^{s} = -\frac{i^n(2n+1)}{n(n+1)} \left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2} \times$$

$$\times L_{m1}^{n}(\psi_3, \theta_3, \varphi_3) \frac{e^{ik(\mathbf{n}\mathbf{r}_{0s})}}{\zeta_n'(ka_s)},$$

$$\varphi_{mn}^{s} = -\frac{i^n(2n+1)}{n(n+1)} \left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2} L_{m1}^{*n}(\psi_3, \theta_3, \varphi_3) \frac{e^{ik(\mathbf{n}\mathbf{r}_{0s})}}{\zeta_n(ka_s)}. \tag{7.3.13}$$

Their infinite dimensional determinants Δ_m may be written (for each m, $|m| \leq n$) in a form similar to (6.3.23):

$$\begin{vmatrix} 1 & \alpha_{m,m}^{-} & 0 & \alpha_{m,m+1}^{-} & 0 \\ \alpha_{m,m}^{+} & 1 & \alpha_{m,m+1}^{+} & 0 & \alpha_{m,m+2}^{+} \\ 0 & \alpha_{m+1,m}^{-} & 1 & \alpha_{m+1,m+1}^{-} & 0 \\ \alpha_{m+1,m}^{+} & 0 & \alpha_{m+1,m+1}^{+} & 1 & \alpha_{m+1,m+2}^{+} \end{vmatrix}$$

$$\Delta_m = \begin{vmatrix} 0 & \alpha^-_{m+2,m} & 0 & \alpha^-_{m+2,m+1} & 1 \\ \alpha^+_{m+2,m} & 0 & \alpha^+_{m+2,m+1} & 0 & \alpha^+_{m+2,m+2} \\ 0 & \alpha^-_{m+3,m} & 0 & \alpha^-_{m+3,m+1} & 0 \\ \alpha^+_{m+3,m} & 0 & \alpha^+_{m+3,m+1} & 0 & \alpha^+_{m+3,m+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

$$\begin{vmatrix} \alpha^-_{m,m+2} & 0 & \alpha^-_{m,m+3} & 0 & \alpha^-_{m,m+4} & \cdots \\ 0 & \alpha^+_{m,m+3} & 0 & \alpha^+_{m,m+4} & 0 & \cdots \\ \alpha^-_{m+1,m+2} & 0 & \alpha^-_{m+1,m+3} & 0 & \alpha^-_{m+1,m+4} & \cdots \\ 0 & \alpha^+_{m+1,m+3} & 0 & \alpha^+_{m+1,m+4} & 0 & \cdots \\ \alpha^-_{m+2,m+2} & 0 & \alpha^-_{m+2,m+3} & 0 & \alpha^-_{m+2,m+4} & \cdots \\ 1 & \alpha^+_{m+2,m+3} & 0 & \alpha^+_{m+2,m+4} & 0 & \cdots \\ \alpha^-_{m+3,m+2} & 1 & \alpha^-_{m+3,m+3} & 0 & \alpha^-_{m+3,m+4} & \cdots \\ 0 & \alpha^+_{m+3,m+3} & 1 & \alpha^+_{m+3,m+4} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{vmatrix} \qquad (7.3.14)$$

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where

$$\alpha^-_{n,q} = \alpha^{(-1,1)}_{mnq}, \quad \alpha^+_{n,q} = \alpha^{(+1,-1)}_{mnq} \qquad (7.3.15)$$

for system (7.3.9) and

$$\alpha^-_{n,q} = \beta^{(-1,+1)}_{mnq}, \quad \alpha^+_{n,q} = \beta^{(+1,-1)}_{mnq} \qquad (7.3.16)$$

for system (7.3.10) (n, q = 1, 2, ..., if m = 0 and n, q = $|m|$, $|m|$ + 1, ..., if $m \neq 0$). The quantities Q_{mnmq} in (7.3.11) and (7.3.12) are determined just as in (3.7.10). Hence it is clear that when $kl \to \infty$ all $\alpha^-_{n,q}$ and $\alpha^+_{n,q}$ tend to zero, and then the determinant (7.3.14) tends to unity.

Both systems (7.3.9) and (7.3.10) may be written in normal form

$$x_i + \sum_{j=1}^{\infty} c_{ij} x_j = f_i; \quad i = 1, 2, \ldots, \qquad (7.3.17)$$

if we set

$$x_{2l-1} = A^{+1}_{m,m'+l-1}, \quad f_{2l-1} = f^{+1}_{m,m'+l-1},$$
$$x_{2l} = A^{-1}_{m,m'+l-1}, \quad f_{2l} = f^{-1}_{m,m'+l-1}, \qquad (7.3.18)$$

$$c_{2j-1,2l} = \alpha^{(-1,+1)}_{m,m'+j-1,m'+l-1}, \quad c_{2j,2l-1} = \alpha^{(+1,-1)}_{m,m'+j-1,m'+l-1}, \tag{7.3.19}$$

for (7.3.9) and

$$c_{2j-1,2l-1} = 0, \quad c_{2j,2l} = 0$$

$$x_{2l-1} = B^{+1}_{m,m'+l-1}, \quad f_{2l-1} = \varphi^{+1}_{m,m'+l-1}, \tag{7.3.20}$$

$$x_{2l} = B^{-1}_{m,m'+l-1}, \quad f_{2l} = \varphi^{-1}_{m,m'+l-1},$$

$$c_{2j-1,2l} = \beta^{(-1,+1)}_{m,m'+j-1,m'+l-1}, \quad c_{2j,2l-1} = \beta^{(+1,-1)}_{m,m'+j-1,m'+l-1}, \tag{7.3.21}$$

$$c_{2j-1,2l-1} = 0, \quad c_{2j,2l} = 0$$

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for system (7.3.10); i, j = 1, 2, ...; $m' = |m|$, if $m \neq 0$ and $m' = 1$, if $m = 0$.

The matrix elements of systems (7.3.9), (7.3.10) have the property of parity relative to the parameter m:

$$\alpha^{(-s,s)}_{m,n,q} = \alpha^{(-s,s)}_{-m,n,q}, \quad \beta^{(-s,s)}_{mnq} = \beta^{(-s,s)}_{-m,n,q} \tag{7.3.22}$$

and for each m form a completely continuous form while satisfying condition (1.9.19), and their right-hand sides (7.3.13) are mutually related by the expressions

$$f^s_{-m,n}(\psi_3, \theta_3, \varphi_3) = (-1)^{n+1} f^s_{mn}(-\psi_3, \pi - \theta_3, \varphi_3),$$

$$\varphi^s_{-m,n}(\psi_3, \theta_3, \varphi_3) = (-1)^{n+1} \varphi^s_{mn}(-\psi_3, \pi - \theta_3, \varphi_3) \tag{7.3.23}$$

and for each m they satisfy condition (1.9.20). In fact, on the basis of (3.3.22),

$$P_n^{-m}(\cos\theta) P_q^{-m}(\cos\theta) = \frac{(n-m)!\,(q-m)!}{(n+m)!\,(q+m)!} P_n^m(\cos\theta) P_q^m(\cos\theta).$$

Therefore, after first setting $m_1 = m_2 = m$ into (3.4.13), and then $m_1 = m_2 = -m$ and comparing the obtained expressions for $n_1 = n$, $n_2 = q$, we will have the equation,

$$b^{(q,-m,n,-m)}_{\sigma} = \frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!} b^{(qmnm)}_{\sigma},$$

on the basis of which, from (3.7.11), we obtain

$$Q_{-m,n,-m,q} = \frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!} Q_{mnmq}. \qquad (7.3.24)$$

Using (7.3.24) from (7.3.11), (7.3.12) we also obtain formulas (7.3.22).

Based on (3.4.24), expression (7.3.13) for f^{s}_{mn} can be written in the form,

$$f^{s}_{mn} = -\frac{i^{n+1}(2n+1)}{2\sqrt{n(n+1)}} \cdot \frac{e^{ik(nr_{0s})}}{\zeta'_n(ka_s)} i^{-m} \times \\ \times \{T^{n}_{-m,-1}(\psi_3, \theta_3, \varphi_3) + T^{n}_{-m,1}(\psi_3, \theta_3, \varphi_3)\}, \qquad (7.3.25)$$

whence it is clear that with substitution in f^{s}_{mn} of the index m by $-m$, expression (7.3.25) will vary in first order as a function of the variation in expressions (3.4.6), (3.4.7) which determine the generalized spherical function T^{n}_{mk}. From the fact that /336

$$P^{n}_{-mk}(\theta) = (-1)^{n+k+m} P^{n}_{mk}(\pi - \theta),$$

there follows the equation,

$$T^{n}_{-mk}(\psi_3, \theta_3, \varphi_3) = (-1)^{n+k+m} T^{n}_{mk}(-\psi_3, \pi - \theta_3, \varphi_3),$$

which means that the substitution in the function $T^{n}_{mk}(\psi_3, \theta_3, \phi_3)$ of the index m by the index $-m$ is equivalent to multiplying T^{n}_{mn} $(\psi_3, \theta_3, \phi_3)$ by the coefficient $(-1)^{n+k+m}$ with the simultaneous substitution of the argument ψ by $-\psi_3$, and θ_3 by $\pi - \theta_3$. This also proves the validity of the first of the relationships in (7.3.23). In precisely the same way we establish the validity of the second equation in (7.3.23) as well.

From (3.4.9) it follows that the generalized spherical functions T^{n}_{mk} are limited in absolute value for all n and m, $k = -n, \ldots, n$, and $|T^{n}_{mk}| \leq 1$. Therefore, for all values of $m(|m| \leq n)$ in (7.3.25) we will find the inequality

$$|f^{s}_{\pm m,n}| \leqslant \frac{2n+1}{\sqrt{n(n+1)}\,\zeta'_n(ka_s)}$$

or

$$|f^{s}_{\pm m,n}| \leqslant \text{const}\, g^{n}, \qquad (7.3.26)$$

where $0 < g < 1$. Inequality (7.3.26) means that the right-hand sides of system (7.3.9) satisfy condition (1.9.20) for any value of $m(|m| \leq n)$. This statement is expanded also to the right-hand sides of system (7.3.10)

The property of parity of the matrix elements of systems (7.3.9), (7.3.10) relative to the index m and the possibility of satisfying condition (1.9.20) for the right-hand sides of the systems, regardless of the sign of m, permits in studying the system for solvability to confine ourselves to looking only at the case of positive values of m.

Let $m \geq 0$. Let us set into (3.4.13) $m_1 = m_2 = m$, $n_1 = n$, $n_2 = q$ and multiply both parts of (3.4.13) by $P_\sigma(\cos\theta)\sin\theta d\theta$. Then after integrating them over θ from 0 to π using the computations in (3.3.34) we find that

$$|b_\sigma^{(qmnm)}| \leqslant \text{const}_1 \sqrt{\frac{\sigma}{nq}} \frac{(n+m)!\,(q+m)!}{n!\,q!}. \tag{7.3.27}$$

Using (7.3.27) for computing the absolute value of the quantity Q_{mnmq} from (7.3.11), (7.3.12) and taking into account that

$$\sum_{\sigma=|n-q|}^{n+q} \sqrt{\sigma}\,|h_\sigma^{(1)}(kl)| < \text{const}_2 \sqrt{n+q}\,|h_{n+q}^{(1)}(kl)|, \tag{7.3.28}$$

we find the inequality, /337

$$|Q_{mnmq}| < \text{const}_3 \frac{(q-m)!\,(n+m)!}{n!\,(q-1)!}\,|h_{n+q}^{(1)}(kl)|, \tag{7.3.29}$$

on the basis of which we find in turn the inequality for the matrix elements $\alpha_{mnq}^{(-s,s)}$ of system (7.3.9)

$$|\alpha_{\pm m,nq}^{(-s,s)}| < \text{const}_1 \left\{\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right\}^{1/2} \times \\ \times \frac{(q-m)!\,(n+m)!}{n!\,(q-1)!} \left|\frac{\psi_q'(ka_{-s})}{\zeta_n'(ka_s)}\,h_{n+q}^{(1)}(kl)\right|. \tag{7.3.30}$$

A similar type of inequality is also obtained for the matrix elements of system (7.3.10). Using formulas (3.5.5), (6.3.24), applied to (7.3.30) on the basis of discussions similar to those cited, for example in [84], [97], we come to the conclusion that

$$|\alpha_{\pm m,nq}^{(-s,s)}| < \text{const}\,\frac{(n+q+2m)!}{n!\,(q+2m)!}\left(\frac{a_{-s}}{l}\right)^{q+2m}\left(\frac{a_s}{l}\right)^n, \tag{7.3.31}$$

$$s = \pm 1$$

for each $m \geq 0$ ($m \leq n$, q). Since

$$\sum_{n,q=m}^{\infty}\left[\frac{(n+q+2m)!}{n!\,(q+2m)!}\left(\frac{a_s}{l}\right)^n\left(\frac{a_{-s}}{l}\right)^{q+2m}\right]^2<\infty \qquad (7.3.32)$$

under the condition that $l > a_{+1} + a_{-1}$ (the spheres do not touch), it follows then that the argument given above is valid, i.e., that system (7.3.9) possesses a completely continuous form. This also, to an equal degree, pertains to system (7.3.10). Therefore, the Gilbert alternative on the solvability of the infinite systems is also applicable to this case. We can show that the uniform systems which correspond to the heterogeneous systems (7.3.9), (7.3.10) have no non-trivial solutions. In fact, we can look, for example, at the uniform system,

$$a^s_{mn}+\frac{\psi_n'(ka_s)}{\zeta_n'(ka_s)}\sum_{q=|m|}^{\infty}Q_{mnmq}\,a^{-s}_{mq}=0, \qquad (7.3.33)$$
$$s=\pm 1,\ n=m,\ m+1,\ \ldots,$$

which corresponds to the heterogeneous system (7.3.9), where a^s_{mn} is coupled with A^s_{mn} by formulas (7.3.8). System (7.3.33) corresponds to the internal boundary problem on seeking the potential u^1, that satisfies the homogeneous Helmholtz equation, the zero boundary conditions on the surface of each sphere, and the condition of radiation at infinity. By virtue of the uniqueness of the solution to such a problem the function u^1 must be, at each point of the region being studied, identically zero and therefore from (7.3.5) we find

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$$\sum_{s=\pm1}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}a^s_{mn}\,\zeta_n(kr_s)\,P^m_n(\cos\theta_s)\,e^{im\varphi}\equiv 0.$$

In the coordinates, for example of the s-th sphere, this expression may be written using the addition theorem (3.7.10) in the form,

$$\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{a^s_{mn}\,\zeta_n(kr_s)+\psi_n(kr_s)\times\right.$$
$$\left.\times\sum_{q=|m|}^{\infty}Q_{mnmq}\,a^{-s}_{mq}\right\}P^m_n(\cos\theta_s)\,e^{im\varphi}\equiv 0$$

(in the region, for which $l > r_s$), whence

$$a^s_{mn}\,\zeta_n(kr_s)+\psi_n(kr_s)\sum_{q=|m|}^{\infty}Q_{mnmq}\,a^{-s}_{mn}\equiv 0 \qquad (7.3.34)$$

or, if we take (7.3.33) into account,

$$\{\zeta_n(kr_s)\psi_n(kr_s) - \psi_n(kr_s)\,\zeta_n'(kr_s)\}\,a_{mn}^s \equiv 0,$$

which is possible on the basis of (7.1.30) only under the condition that $a_{mn}^s \equiv 0$. In precisely the same manner we can show that the uniform system which corresponds to the heterogeneous (7.3.10) has only trival solutions. Consequently, for systems (7.3.9), (7.3.10) the first part of the Gilbert alternative is valid, and for each $m(|m| \leq n)$ systems (7.3.9), (7.3.10) have unique solutions $\{A_{mn}^s\}_n$, $\{B_{mn}^s\}_n$ which satisfy the conditions

$$\sum_{n=|m|}^{\infty} |A_{mn}^s|^2 < \infty, \qquad (7.3.35)$$

$$\sum_{n=|m|}^{\infty} |B_{mn}^s|^2 < \infty. \qquad (7.3.36)$$

Using inequality (7.3.31) and the similar inequality for $\beta_{mnq}^{(-s,s)}$ we can establish that systems (7.3.9), (7.3.10) are in the general case quasi-regular and convert to regular systems in proportion to the increase in kl (for the given ka_s, $s = \pm 1$). /339

By discovering the coefficients A_{mn}^s and B_{mn}^s the mathematical problem for the Debye potentials u^1 and v^1 of the secondary field scattered on the spheres is completely solved, since by using the inequalities (7.3.35), (7.3.36) given above and the asymptotic formulas for the adjoint Legendre functions and the spherical Bessel functions we can establish the existence of a solution in the form of (7.3.5) at any point of the examined region outside the spheres ($r_s > a_s$, $s = \pm 1$). The components of the electromagnetic field E^1, H^1 scattered on the spheres are found from formulas (1.6.15) written in one of the spherical coordinate systems where on the basis of (7.3.8) the potentials of the stray field will have the form

$$u^1 = \frac{E}{ik^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{s=\pm 1}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \times$$

$$\times \frac{\psi_n'(ka_s)}{r_s} A_{mn}^s \zeta_n(kr_s) P_n^m(\cos\theta_s) e^{im\varphi}, \qquad (7.3.37)$$

$$v^1 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{E}{ik^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\sum_{s=\pm 1}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \times$$

$$\times \frac{\psi_n(ka_s)}{r_s} B_{mn}^s \zeta_n(kr_s) P_n^m(\cos\theta_s) e^{im\varphi}.$$

The potentials of the complete field ($u = u^0 + u^1$, $v = v^0 + v^1$) at the points for which $l > r_s$, by using the addition theorem (3.7.10), can be written in coordinates of the s-the sphere in the form,

$$u = \frac{E}{ik^2 r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} a_{mn}^s \left\{ \zeta_n(kr_s) - \right.$$

$$\left. - \frac{\zeta_n'(ka_s)}{\psi_n'(ka_s)} \psi_n(kr_s) \right\} P_n^m(\cos\theta_s) e^{im\varphi},$$

$$v = \sqrt{\frac{\varepsilon}{\mu}} \frac{E}{ik^2 r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} b_{mn}^s \left\{ \zeta_n(kr_s) - \right. \qquad (7.3.38)$$

$$\left. - \frac{\zeta_n(ka_s)}{\psi_n(ka_s)} \psi_n(kr_s) \right\} P_n^m(\cos\theta_s) e^{im\varphi}$$

[here we take into account the relationship between the coefficients a_{mn}^s and a_{mn}^{-s}, b_{mn}^s and b_{mn}^{-s}, expressed by equations (7.3.9), (7.3.10)]. /340
Therefore, on the basis of (1.1.8) and formulas (1.6.15), written in local spherical coordinates of the s-th sphere, as well as (7.1.30), (7.3.38) the law governing the density distribution of the surface currents on the surface of the s-th sphere will be given by the expressions,

$$j_\varphi^s = j_\varphi^{ms} + j_\varphi^{es},$$

$$j_\theta^s = j_\theta^{ms} + j_\theta^{es}, \qquad (7.3.39)$$

where through j^{es}, j^{ms} we denote the components of the vectors j^{es} and j^{ms} ($j^s = j^{es} + j^{ms}$) of the density of the surface currents induced, respectively, by the electric (u) and the magnetic (v) fields; then on the s-th sphere

$$j_\varphi^{ms} = \frac{E\omega\varepsilon}{4\pi k^2 a_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left[\frac{(n-m)!}{(n+m)!} \right]^{1/2} \times$$

$$\times B_{mn}^s \frac{d}{d\theta_s} P_n^m(\cos\theta_s) e^{im\varphi},$$

$$j_\varphi^{es} = -\frac{E\omega\varepsilon}{4\pi k^2 a_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} m \left[\frac{(n-m)!}{(n+m)!} \right]^{1/2} \times$$

$$\times A_{mn}^s \frac{P_n^m(\cos\theta_s)}{\sin\theta_s} e^{im\varphi},$$

$$j_\theta^{ms} = -\frac{iE\omega\varepsilon}{4\pi k^2 a_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m\left[\frac{(m-n)!}{(n+m)!}\right]^{1/2}\times$$

$$\times B_{mn}^{s}\frac{P_n^m(\cos\theta_s)}{\sin\theta_s}e^{im\varphi},$$

$$j_\theta^{es} = \frac{iE\omega\varepsilon}{4\pi k^2 a_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2}\times \tag{7.3.40}$$

$$\times A_{mn}^{s}\frac{d}{d\theta_s}P_n^m(\cos\theta_s)e^{im\varphi}$$

In the wave band for which approximately $r_s = r - sl_0\cos\theta$, $\theta_s = \theta$ /341 ($l \ll r$), the components of the stray field are given by the approximate equations

$$E_\theta^1 = \sqrt{\frac{\mu}{\varepsilon}}H_\varphi^1 = E\frac{e^{ikr}}{kr}A(\theta,\ \varphi),$$

$$E_\varphi^1 = -\sqrt{\frac{\mu}{\varepsilon}}H_\theta^1 = E\frac{e^{ikr}}{kr}A^*(\theta,\ \varphi), \tag{7.3.41}$$

where

$$A(\varphi,\ \theta) = -i\sum_{n,m}(-i)^n e^{im\varphi}\times$$

$$\times\sum_{s=\pm 1}\left\{[P_n^{m'}(\cos\theta) + iskl_0\sin\theta\, P_n^m(\cos\theta)]a_{mn}^s + \right. \tag{7.3.42}$$

$$\left. + \frac{m}{\sin\theta}P_n^m(\cos\theta)\,b_{mn}^s\right\}e^{-iskl_0\cos\theta},$$

$$A^*(\varphi,\ \theta) = \sum_{n,m}(-i)^n e^{im\varphi}\sum_{s=\pm 1}\left\{\frac{m}{\sin\theta}P_n^m(\cos\theta)\,a_{mn}^s + \right.$$

$$\left. + i[P_n^{m'}(\cos\theta) + iskl_0\sin\theta\, P_n^m(\cos\theta)\,b_{mn}^s\right\}e^{-iskl_0\cos\theta}, \tag{7.3.43}$$

$$P_n^{m'}(\cos\theta) = \frac{d}{d\theta}P_n^m(\cos\theta).$$

Just as in the case of one sphere, a field scattered on two spheres in approximation of the wave band has no components in the direction of propagation of the incident plane wave, but only has transverse components.

From the formulas in (7.3.41) it is obvious that when $\alpha \neq 0$ and $\alpha \neq \pi$ there generally exist no such directions in which the stray field would be linearly polarized. When $\alpha = 0$ or $\alpha = \pi$, i.e., in the case when the plane wave is propagated along the axis of symmetry which is common for the spheres, in the positive or negative directions of the Oz axis, we can show the directions which depend on the parameter β, in which the stray field (7.3.41) is found to be linearly polarized. For example, let $\alpha = 0$. Then, as follows from (3.4.10) and the formulas in (3.4.24), we find that

$$L_{11}^{n} = e^{-i\beta}/2, \quad L_{-1,1}^{n} = -\frac{1}{2}\frac{(n+1)!}{(n-1)!}e^{i\beta},$$

$$L_{11}^{*n} = e^{-i\beta}/2i, \quad L_{-1,1}^{*n} = \frac{1}{2i}\frac{(n+1)!}{(n-1)!}e^{i\beta}, \qquad (7.3.44)$$

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on the basis of which from (7.3.13) we find the relationships

$$f_{-1,n}^{s} = -e^{2i\beta} f_{1n}^{s}, \quad \varphi_{-1,n}^{s} = e^{2i\beta}\varphi_{1n}^{s}. \qquad (7.3.45)$$

If we take (7.3.15) into account then by using (7.3.45) we find that when $\alpha = 0$ all coefficients A_{mn}^{s} and B_{mn}^{s} with the index $|m| > 1$, $m = 0$ are found to be equal to zero, and

$$A_{-1,n}^{s} = -A_{1n}^{s}e^{2i\beta}, \quad B_{-1,n}^{s} = B_{1n}^{s}e^{2i\beta}.$$

Therefore, in such case the amplitude functions (7.3.42), (7.3.43) may be written in the form,

$$A(\theta, \varphi) = \cos(\varphi - \beta)e^{i\beta}\tilde{A}(\theta),$$
$$A^{*}(\theta, \varphi) = \sin(\varphi - \beta)e^{i\beta}\tilde{A}^{*}(\theta), \qquad (7.3.46)$$

where

$$\tilde{A}(\theta) = i\,2\sum_{n=1}^{\infty}(-i)^{n}\sum_{s=\pm 1}\{[\tau_n(\cos\theta) + isk l_0 \sin^2\theta\pi_n(\cos\theta)]a_{1n}^{s} + i\,\pi_n(\cos\theta)b_{1n}^{s}\}e^{-isk l_0\cos\theta}, \qquad (7.3.47)$$

$$\tilde{A}^{*}(\theta) = -i\,2\sum_{n=1}^{\infty}(-i)^{n}\sum_{s=\pm 1}\{\pi_n(\cos\theta)a_{1n}^{s} + i[\tau_n(\cos\theta) + isk l_0\sin^2\theta\pi_n(\cos\theta)]b_{1n}^{s}\}e^{-isk l_0\cos\theta}.$$

Hence it is clear that in the directions lying in the planes parallel to the plane $\phi - \beta = 0$ ($\phi - \beta = \pi$) or the plane $\phi - \beta = \pi/2$ ($\phi - \beta = 3\pi/2$), the stray field is linearly polarized, whereas in all other directions it will be elliptically polarized. The results are similar in nature also in the case when $\alpha = \pi$.

For the particular case under study the value of the scattering cross section in the direction θ, ϕ, determined from the formula (1.4.5), on the basis of (7.3.41), (7.3.46), (7.3.47) will be

$$\sigma = \frac{4\pi}{k^2}\{\cos^2(\varphi - \beta)|\tilde{A}(\theta)|^2 + \sin^2(\varphi - \beta)|\tilde{A}^*(\theta)|^2\}, \qquad (7.3.48)$$

and in the directions θ, $\phi = \beta$ and θ, $\phi = \pi/2 + \beta$,

$$\sigma = \frac{4\pi}{k^2}|\tilde{A}(\theta)|^2,$$

$$\sigma = \frac{4\pi}{k^2}|\tilde{A}^*(\theta)|^2$$

respectively. The back scattering cross section (in the direction /343 to the source) is found from (7.3.48) when $\theta = \pi$ and $\phi = 0$ and is found to be equal to

$$\sigma_B = \frac{4\pi}{k^2}\{\cos^2\beta|\tilde{A}(\pi)|^2 + \sin^2\beta|\tilde{A}^*(\pi)|^2\}$$

or

$$\sigma_B = \frac{4\pi}{k^2}|\tilde{A}(\pi)|^2, \qquad (7.3.49)$$

since on the basis of (7.1.24)

$$\tilde{A}(\pi) = \tilde{A}^*(\pi) = \sum_{n=1}^{\infty} i^n \sum_{s=\pm 1} n(n+1)\{a_{1n}^s - ib_{1n}^s\}e^{iskl_0}. \qquad (7.3.50)$$

Thus, we find that the back scattering cross section for the two spheres in the case when the plane wave is propagated along the axis of symmetry which is common to the spheres, is independent of the angle β, which forms the vector E^0 with the plane $\phi = 0$. This is natural on the strength of the symmetry of the barrier relative to the direction of propagation of the wave. When $\beta = 0$, the radar cross section for this case is also found from formula (7.3.49).

According to the definition, the complete scattering cross section is found from formula (1.4.7), which has the form

$$\sigma_s = \frac{1}{k^2}\int_0^{2\pi}\int_0^{\pi}[\cos^2(\varphi-\beta)\,|\tilde{A}(\theta)|^2 + \sin^2(\varphi-\beta)\,|\tilde{A}^*(\theta)|^2]\sin\theta\,d\theta\,d\varphi, \tag{7.3.51}$$

where the amplitude functions are determined just as in (7.3.47).

In the general case, for which $\alpha \neq 0$ or $\alpha \neq \pi$ for computation of the scattering cross sections we find formulas of a more complex type. Thus, the value of the scattering cross section in the direction θ, ϕ is found from the formula

$$\sigma = \frac{4\pi}{k^2}\{|A(\theta,\ \varphi)|^2 + |A^*(\theta,\ \varphi)|^2\}, \tag{7.3.52}$$

in which $A(\theta, \phi)$ and $A^*(\theta, \phi)$ are the amplitude functions given by expressions (7.3.42) (7.3.43). In the direction to the source, when $\theta = \pi - \alpha$, and $\phi = 0$, from (7.3.52) we find the values of the back scattering cross section. Here we will have /344

$$A(\pi-\alpha,\ 0) = -i\sum_{n,m} i^{n-m}e^{-im\beta}\times$$

$$\times\sum_{s=\pm 1}\left\{[P_n^{m'}(\cos\alpha) + iskl_0\sin\alpha\,P_n^m(\cos\alpha)]\,d_{mn}^s + \right.$$

$$\left. + \frac{m}{\sin\alpha}P_n^m(\cos\alpha)\,\beta_{mn}^s\right\}e^{iskl_0\cos\alpha}. \tag{7.3.53}$$

$$A^*(\pi-\alpha,\ 0) = \sum_{n,m} i^{n-m}e^{-im\beta}\times$$

$$\times\sum_{s=\pm 1}\left\{\frac{m}{\sin\alpha}P_n^m(\cos\alpha)\,\alpha_{mn}^s + i[P_n^{m'}(\cos\alpha) + \right.$$

$$\left. + iskl_0\sin\alpha\,P_n^m(\cos\alpha)]\,\beta_{mn}^s\right\}e^{iskl_0\cos\alpha}, \tag{7.3.54}$$

where α_{mn}^s and β_{mn}^s, coupled with the coefficients a_{mn}^s and b_{mn}^s by the relationships,

$$\alpha_{mn}^s = e^{im(\pi/2-\beta)}a_{mn}^s, \quad \beta_{mn}^s = e^{im(\pi/2-\beta)}b_{mn}^s,$$

are independent of the angle β. In the case if the wave is propagated in a direction which forms a right angle with the axis of symmetry of the spheres ($\alpha = \pi/2$), $\theta = \alpha = \pi/2$ and $\phi = 0$, then for the back scattering cross section we find instead of (7.3.53), (7.3.54) the expressions

$$A\left(\frac{\pi}{2},\ 0\right) = -i\sum_{n,m} i^{n-m} e^{-im\beta} \times$$

$$\times \sum_{s=\pm 1} \{[P_n^{m'}(0) + iskl_0 P_n^m(0)]\, a_{mn}^s + mP_n^m(0)\, \beta_{mn}^s\}, \tag{7.3.55}$$

$$A^*\left(\frac{\pi}{2}, \theta\right) = \sum_{n,m} i^{n-m} e^{-im\beta} \times$$

$$\times \sum_{s=\pm 1} \{mP_n^m(0)\, a_{mn}^s + i[P_n^{m'}(0) + iskl_0 P_n^m(0)]\, \beta_{mn}^s\}. \tag{7.3.56}$$

Let e be the unit vector in the direction of polarization of the electric field of the incident wave ($E^0 = eEe^{ik(nr)}$). Then, by definition, in the case if the transmitting and the receiving antennas have an identical linear polarization, the radar cross section of the barrier from the two spheres may be found from the formula, /345

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|(E^1 e)|^2}{|(E^0 e)|^2} \tag{7.3.57}$$

or

$$\sigma = \lim_{r\to\infty} 4\pi r^2 \frac{|(E^1 E^0)|^2}{|E^0|^4}, \tag{7.3.58}$$

where in spherical coordinates with the origin at point O,

$$\begin{aligned} \mathbf{e} &= e_r \mathbf{i}_r + e_\theta \mathbf{i}_\theta + e_\varphi \mathbf{i}_\varphi, \\ e_r &= \cos\beta\cos\alpha\cos\varphi\sin\theta + \sin\beta\sin\theta\sin\varphi + \\ &\quad + \cos\beta\cos\theta\sin\alpha, \\ e_\theta &= \cos\beta\cos\alpha\cos\varphi\cos\theta + \cos\theta\sin\beta\sin\varphi - \\ &\quad - \cos\beta\sin\alpha\sin\theta, \\ e_\varphi &= -\cos\beta\cos\alpha\sin\varphi + \sin\beta\cos\varphi \end{aligned} \tag{7.3.59}$$

and where θ, ϕ are taken in the direction to the source ($\theta = \pi - \alpha$, $\phi = 0$). Since when $r \to \infty$ the scalar product $(E^1 e) = E^1_\phi e_\phi + E^1_\theta e_\theta$, then for the general case,

$$\sigma = \frac{4\pi}{k^2}\left|\cos\beta\, A(\theta,\ \varphi) - \sin\beta\, A^*(\theta,\ \varphi)\right|_{\substack{\theta=\pi-\alpha \\ \varphi=0}}. \tag{7.3.60}$$

In particular, if $\beta = 0$, then

$$\sigma = \frac{4\pi}{k^2}|A(\pi - \alpha,\ 0)|^2,$$

where

$$A(\pi-\alpha,\ 0)=-i\sum_{n=1}^{\infty}\sum_{m=-n}^{n}i^{n-m}\times$$

$$\times\sum_{s=\pm1}\left\{[P_n^{m'}(\cos\alpha)+isk l_0 P_n^m(\cos\alpha)\sin\alpha]\alpha_{mn}^s+\right.$$

$$\left.+\frac{m}{\sin\alpha}P_n^m(\cos\alpha)\beta_{mn}^s\right\}e^{isk l_0\cos\alpha}$$

If $\beta = \pi/2$, then /346

$$\sigma=\left|A^*\left(\frac{\pi}{2},\ 0\right)\right|^2,$$

where

$$A^*\left(\frac{\pi}{2},\ 0\right)=\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-1)^m i^n\times$$

$$\times\sum_{s=\pm1}\{mP_n^m(0)\alpha_{mn}^s+i[P_n^{m'}(0)+isk l_0 P_n^m(0)]\beta_{mn}^s\}.$$

With numerical computations the approximate values of the first few coefficients $A_{mn}^{\pm1}$ and $B_{mn}^{\pm1}$ may be found (for each m) with a given degree of accuracy of computation by comparing the successive solutions to the finite systems obtained from (7.3.9) and (7.3.10) with their truncation. As the original value of N it is feasible to take a number of the same order as ka, where $a = \max\limits_{s=\pm1}\{a_s\}$; $a_{-1}, a_{+1} \neq 0$. In this case n and q assume values of 1, 2, ..., N.

The truncated systems have the form,

$$\left.\begin{aligned}A_{mn}^{+1}+\sum_{q=|m|}^{N}\alpha_{mnq}^{(-1,+1)}A_{mq}^{-1}&=f_{mn}^{+1},\\ A_{mn}^{-1}+\sum_{q=|m|}^{N}\alpha_{mnq}^{(+1,-1)}A_{mq}^{+1}&=f_{mn}^{-1},\end{aligned}\right\}\qquad(7.3.61)$$

$$\left.\begin{aligned}B_{mn}^{+1}+\sum_{q=|m|}^{N}\beta_{mnq}^{(-1,+1)}B_{mq}^{-1}&=\varphi_{mn}^{+1},\\ B_{mn}^{-1}+\sum_{q=|m|}^{N}\beta_{mnq}^{(+1,-1)}B_{mq}^{+1}&=\varphi_{mn}^{-1},\end{aligned}\right\}\qquad(7.3.62)$$

in which m is fixed and $|m| \leq n = 1, 2, \ldots, N$. In normal form [see (7.3.17)] the truncated system is written in the form,

$$x_i + \sum_{l=1}^{2N} c_{il} x_l = f_i, \quad i = 1, 2, \ldots, 2N. \quad N = 1, 2, \ldots, \tag{7.3.63}$$

where c_{ij} and x_i, f_i are coupled with $\alpha_{mnq}^{(-s,s)}$, $\beta_{mnq}^{(-s,s)}$ and A_{mn}^{s}, B_{mn}^{s}, f_{mn}^{s}, ϕ_{mn}^{s} by formulas (7.3.18)-(7.3.21). Hence it is clear that in the case when $m = 0$ or $|m| = 1$, seeking the approximate values of /34 A_{mn}^{s} and B_{mn}^{s} leads to a solution of systems of $2N$ linear equations with $2N$ unknows $A_{mn}^{\pm 1}$ or $B_{mn}^{\pm 1}$ ($A_{mn}^{s} = B_{mn}^{s} = 0$ when $|m| > n$; $s = \pm 1$). If $|m| = p \geq 2$, then seeking the approximate values $A_{mn}^{\pm 1}$ and $B_{mn}^{\pm 1}$ leads to a solution of systems of $2(N - p + 1)$ linear equations with $2(N - p + 1)$ unknowns $A_{mn}^{\pm 1}$ or $B_{mn}^{\pm 1}$, in which $n \geq p$. It is obvious that in each specific problem for finding the first approximate values of the unknowns $A_{mn}^{\pm 1}$ and $B_{mn}^{\pm 1}$ ($m = -n, \ldots, n$; with $n = 1, 2, \ldots, N$) it is necessary to solve $2N + 1$ systems such as (7.3.63), whose order decreases from the $2N$-th with values of $m = 0$ or $|m| = 1$ to the second when $|m| = N$.

The order of systems (7.3.61), (7.3.62) (and consequently (7.3.63) as well) may be reduced two times, if we exclude from (7.3.61) A_{mn}^{+1} or A_{mn}^{-1} and from (7.3.62) B_{mn}^{+1} or B_{mn}^{-1}. As a result instead of one system of $2(N - p + 1)$ equations with $2(N - p + 1)$ unknowns (when $|m| = p \geq 2$) or a system of $2N$ unknowns (when $|m| = 0, 1$) computation of the unknowns A_{mn}^{s} and B_{mn}^{s} is reduced to solving two mutually independent systems of $(N - p + 1)$ equations with $(N - p + 1)$ unknowns (when $|m| = p \geq 2$) or systems of N equations with N unknowns (when $|m| = 0, 1$), having the form,

$$y_{mn}^{s} + \sum_{q=|m|}^{N} c_{mnq}^{s} y_{mq}^{s} = F_{mn}^{s} \tag{7.3.64}$$

($s = \pm 1$; $n, q = |m|, |m| + 1, \ldots, N$, if $m \neq 0$ and $n, q = 1, 2, \ldots, N$, if $m = 0$), where

$$c_{mnq}^{s} = \sum_{r=|m|}^{N} c_{mnr}^{(-s,s)} c_{mrq}^{(s,-s)},$$

$$F_{mn}^{s} = q_{mn}^{s} - \sum_{q=|m|}^{N} c_{mnq}^{(-s,s)} q_{mq}^{(-s)},$$

$c_{mnq}^{(-s,s)} = \alpha_{mnq}^{(-s,s)}$, $q_{mn}^{s} = f_{mn}^{s}$, $y_{mn}^{s} = A_{mn}^{s}$ or $c_{mnq}^{(-s,s)} = \beta_{mnq}^{(-s,s)}$, $q_{mn}^{s} = \varphi_{mn}^{s}$, $y_{mn}^{s} = B_{mn}^{s}$ respectively. However conversion to the systems in (7.3.64) involves substantial complication of the structure of the matrix elements and the right-hand sides of the systems, thus leading to additional difficulties in computing the solution to the problem.

The number of equations in the truncated systems may be reduced by two times (with retention of the previous value of N, the order of truncation) without complicating the expressions for the matrix elements of the systems and their right-hand sides in the case when the diffracting spheres have equal radii. Then the matrix elements /348 of the system satisfy the relationships,

$$\alpha_{mnq} = \alpha_{mnq}^{(-1,+1)} = (-1)^{n+q}\,\alpha_{mnq}^{(+1,-1)},$$
$$\beta_{mnq} = \beta_{mnq}^{(-1,+1)} = (-1)^{n+q}\,\beta_{mnq}^{(+1,-1)}. \qquad (7.3.65)$$

In fact, the function $P_\sigma(\cos\theta_{-s,s})$ which appears in Q_{mnmq} is equal to unity when $\theta_{-1,+1} = 0$, and when $\theta_{+1,-1} = \pi$ it is equal to $(-1)^\sigma$, where σ has the same parity as the sum $n + q$, since the Clebsch-Gordan coefficients which appear in the expressions for the coefficients $b_\sigma^{(qmnm)}$ [see (3.4.15)] are non-zero only for those values of n, q, σ, whose sum $n + q + \sigma$ is even. Therefore,

$$Q_{mnmq}(\theta_{-s,s}) = (-1)^{n+q} Q_{mnmq}(\theta_{s,-s}), \quad s = \pm 1. \qquad (7.3.66)$$

On the basis of (7.3.66), from (7.3.11), (7.3.12) we also obtain when $a_s = a$ ($s = \pm 1$) the relationships (7.3.65). Applying (7.3.65), for example, to system (7.3.61) we obtain

$$A_{mn}^{+1} + \sum_{q=|m|}^{N} \alpha_{mnq} A_{mq}^{-1} = f_{mn}^{+1},$$
$$A_{mn}^{-1} + \sum_{q=|m|}^{N} (-1)^{n+q} \alpha_{mnq} A_{mq}^{+1} = f_{mn}^{-1}, \qquad (7.3.67)$$

whence by dividing both sides of the second equation by $(-1)^n$ and subsequent term-by-term addition and subtraction of the equations we obtain two systems.

$$x_{mn} + \sum_{q=|m|}^{N} (-1)^q \alpha_{mnq} x_{mq} = f_{mn}, \qquad (7.3.68)$$

$$y_{mn} - \sum_{q=|m|}^{N} (-1)^q \alpha_{mnq} y_{mq} = \bar{f}_{mn} \qquad (7.3.69)$$

with the unknowns

$$x_{mn} = A_{mn}^{+1} + (-1)^n A_{mn}^{-1}, \quad y_{mn} = A_{mn}^{+1} - (-1)^n A_{mn}^{-1} \tag{7.3.70}$$

and with right-hand sides

$$f_{mn} = f_{mn}^{+1} + (-1)^n f_{mn}^{-1}, \quad \bar{f}_{mn} = f_{mn}^{+1} - (-1)^n f_{mn}^{-1}. \tag{7.3.71}$$

Each of the systems (7.3.68), (7.3.69) is solved independently of the other and is a system of N-th order with values of $|m| = 0, 1$ and a system of $(N - p + 1)$-th order with values of $|m| = p \geq 2$.

The conversion of the system (7.3.61) to systems (7.3.68), (7.3.69), carried out above, by reducing the order of the system, does not lead to any complication of the matrix elements and the right-hand parts. /349

In a similar manner, from (7.3.62) when $a_s = a$ ($s = \pm 1$) we obtain the systems,

$$u_{mn} + \sum_{q=|m|}^{N} (-1)^q \beta_{mnq} u_{mq} = \varphi_{mn}, \tag{7.3.72}$$

$$z_{mn} - \sum_{q=|m|}^{N} (-1)^q \beta_{mnq} z_{mq} = \bar{\varphi}_{mn} \tag{7.3.73}$$

for the unknowns B_{mn}^{-1} and B_{mn}^{+1}, in which

$$u_{mn} = B_{mn}^{+1} + (-1)^n B_{mn}^{-1}, \quad z_{mn} = B_{mn}^{+1} - (-1)^n B_{mn}^{-1}, \tag{7.3.74}$$

$$\varphi_{mn} = \varphi_{mn}^{+1} + (-1)^n \varphi_{mn}^{-1}, \quad \bar{\varphi}_{mn} = \varphi_{mn}^{+1} - (-1)^n \varphi_{mn}^{-1}. \tag{7.3.75}$$

Through the solutions to systems (7.3.68) - (7.3.75), the approximate values of the unknowns A_{mn}^{s} and B_{mn}^{s} are found from the formulas,

$$A_{mn}^{+1} = \frac{1}{2}(x_{mn} + y_{mn}), \quad A_{mn}^{-1} = \frac{(-1)^n}{2}(x_{mn} - y_{mn}),$$

$$B_{mn}^{+1} = \frac{1}{2}(u_{mn} + z_{mn}), \quad B_{mn}^{-1} = \frac{(-1)^n}{2}(u_{mn} - z_{mn}). \tag{7.3.76}$$

The infinite systems (7.3.9), (7.3.10) and the finite (truncated) systems (7.3.61), (7.3.62) obtained from them are simplified substantially in that particular case when the plane wave which is incident on the diffracting spheres is propagated along the Oz axis,

i.e., when either $\alpha = 0$ or $\alpha = \pi$. In this case, as noted before, all values of A^s_{mn} and B^s_{mn} are equal to zero when $|m| \neq 1$. Furthermore, for example when $\alpha = 0$, between $A^s_{\pm 1, n}$ and $B^s_{\pm 1, n}$ we find the relationship $A^s_{-1,n} = -e^{2i\beta} A^s_{1n}$, $B^s_{-1,n} = e^{2i\beta} B^s_{1n}$, due to which the truncated systems (7.3.61), (7.3.62) (and when $\alpha_s = \alpha$, $s = \pm 1$ systems (7.3.68) - (7.3.78) as well) should be solved only with positive values of the index $m = 1$. Here in the systems

$$f^s_{1n} = -Ee^{iskl_s \cos\alpha} \frac{2n+1}{n(n+1)} L^n_{1,1}(\psi_s, 0, \varphi_s) \frac{i^n}{\zeta'_n(ka_s)},$$

$$\varphi^s_{1n} = -Ee^{iksl_s \cos\alpha} \frac{2n+1}{n(n+1)} L^{*n}_{1,1}(\psi_s, 0, \varphi_s) \frac{i^n}{\zeta_n(ka_s)}. \qquad (7.3.77)$$

Now let us note the following. We shall solve the problem by the method of perturbuations. Then /350

$$u^1 = \sum_{s=\pm 1} \sum_{p=1}^{\infty} {}^s u^p, \quad v^1 = \sum_{s=\pm 1} \sum_{p=1}^{\infty} {}^s v^p, \qquad (7.3.78)$$

where through ${}^s u^p$ and ${}^s v^p$ we denote the Debye potentials of the secondary field of p-th order of scattering, excited by the s-th sphere by the effect on it of the wave of $(p-1)$-th order of scattering, coming from the $-s$-th sphere. If we seek them in the form of series of spherical wave functions which satisfy the condition of radiation of infinity,

$$\begin{aligned} {}^s u^p &= \frac{1}{kr_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \times \\ &\times {}^s A^p_{mn} \psi'_n(ka_s) \zeta_n(kr_s) P^m_n(\cos\theta_s) e^{im\varphi}, \\ {}^s v^p &= \frac{1}{kr_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{(n-m)!}{(n+m)!} \Bigg\}^{1/2} \times \qquad (7.3.79) \\ &\times {}^s B^p_{mn} \psi_n(ka_s) \zeta_n(kr_s) P^m_n(\cos\theta_s) e^{im\varphi}, \end{aligned}$$

then

$$u^1 = \sum_{s=\pm 1} u_s, \quad v^1 = \sum_{s=\pm 1} v_s,$$

where

$$u_s = \frac{1}{kr_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}\times$$

$$\times\left(\sum_{p=1}^{\infty}{}^{s}A_{mn}^{p}\right)\psi_n'(ka_s)\,\zeta_n(kr_s)\,P_n^m(\cos\theta_s)\,e^{im\varphi},$$

$$v_s = \frac{1}{kr_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}\times \tag{7.3.80}$$

$$\times\left(\sum_{p=1}^{\infty}{}^{s}B_{mn}^{p}\right)\psi_n(ka_s)\,\zeta_n(kr_s)\,P_n^m(\cos\theta_s)\,e^{im\varphi}.$$

The coefficients ${}^{s}A_{mn}^{p}$ and ${}^{s}B_{mn}^{p}$ of the diffraction series (7.3.80) /351
(the scattering coefficients of p-th order) are found from the boundary conditions,

$$\frac{\partial}{\partial r_s}[r_s({}^{s}u^{p}+{}^{-s}u^{p-1})]=0,\quad r_s=a_s,\quad s=\pm 1,$$

$${}^{s}v^{p}+{}^{-s}v^{p-1}=0;\quad r_s=a_s,\quad s=\pm 1, \tag{7.3.81}$$

whence by using the addition theorem (3.7.10), for them we find the recurrent relationships,

$${}^{s}A_{mn}^{1}=f_{mn}^{s},$$

$${}^{s}A_{mn}^{p}=-\sum_{q}\alpha_{mnq}^{(-s,s)}\,{}^{s}A_{mq}^{p-1},\quad p\geqslant 2, \tag{7.3.82}$$

$${}^{s}B_{mn}^{1}=\varphi_{mn}^{s},$$

$${}^{s}B_{mn}^{p}=-\sum_{q}\beta_{mnq}^{(-s,s)}\,{}^{s}B_{mq}^{p-1},\quad p\geqslant 2 \tag{7.3.83}$$

($n, q = |m|, |m|+1, \ldots$, if $m \neq 0$, and $n, q = 1, 2, \ldots$, if $m = 0$ and $s = \pm 1$), in which $\alpha_{mnq}^{(-s,s)}$, $\beta_{mnq}^{(-s,s)}$, f_{mn}^{s} and ϕ_{mn}^{s} are the same as in (7.3.9), (7.3.10). Relationships (7.3.82), (7.3.83) permit, as usual, computing the scattering coefficients of p-th order through the preceding ones and finally through the scattering coefficients of first order (i.e., through the free terms f_{mn}^{s} and ϕ_{mn}^{s} of systems (7.3.9), (7.3.10), respectively).

If we apply to systems (7.3.9), (7.3.10) the method of successive approximations (formally) after taking as the initial approximations the values $A^{s}_{mn,0} = 0$, $B^{s}_{mn,0} = 0$, their dominant solutions will be described by the formulas,

$$A^{s}_{mn} = \lim_{t\to\infty} A^{s}_{mn,t} = \sum_{p=1}^{\infty} {}^{s}A^{p}_{mn}\,.$$

$$B^{s}_{mn} = \lim_{t\to\infty} B^{s}_{mn,t} = \sum_{p=1}^{\infty} {}^{s}B^{p}_{mn}\,, \qquad (7.3.84)$$

since, for example from (7.3.9), we obtain

$$A^{s}_{mn,1} = f^{s}_{mn}\,,$$

$$A^{s}_{mn,2} = A^{s}_{mn,1} - \sum_{q}^{\infty} \alpha^{(-s,s)}_{mnq} A^{-s}_{mq,1}\,,$$

. . . . . (7.3.85) /352

$$A^{s}_{mn,t} = A^{s}_{mn,1} - \sum_{q}^{\infty} \alpha^{(-s,s)}_{mnq} A^{-s}_{mq,t-1}\,,\ t \geqslant 2,$$

. . . . . .

where $A^{s}_{mq,t} = {}^{s}A^{t}_{mq}$. Hence it is clear that the solutions to systems (7.3.9) and (7.3.10) are the values of A^{s}_{mn} and B^{s}_{mn} which represent the sums of all respective successive coefficients of the diffraction series determining the potentials of the secondary fields of all successive orders of scattering, reflected by the s-th sphere in its interaction with the $-s$-th sphere.

The series which appear in the recurrent formulas (7.3.82) - (7.3.83), converge well only under the condition that the spheres are widely spaced, when the distance l between them is much greater than the wave length of excitation ($kl >> 1$), and the radii of the spheres are small in comparison with it ($ka_s << 1$). In this case the potentials of the secondary fields will satisfy the relationships,

$$|{}^{s}u^{1}| \gg |{}^{s}u^{2}| \gg |{}^{s}u^{3}| \gg \dots,$$

$$|{}^{s}v^{1}| \gg |{}^{s}v^{2}| \gg |{}^{s}v^{3}| \gg \dots,$$

which means that in studying the field we can confine ourselves to the secondary waves of only the first order of scattering.

Computation of the approximate values of the diffraction coefficients A_{mn}^{s} and B_{mn}^{s} from systems (7.3.9) and (7.3.10) assumes a finiteness of the number of mathematical operations required for this, i.e., a finiteness of the values of the indices n and q in the systems ($n, q = 1, 2, \ldots, N$, if $m = 0$ and $n, q = |m| + 1, \ldots, N$, if $m \neq 0$). Proceeding from this, we can obtain analytical expressions for computing the approximate values of the coefficients A_{mn}^{s} and B_{mn}^{s} in the case of widely separated spheres, having a comparatively simple form. In (7.3.11) and (7.3.12) let $kl >> 1$ and $kl >> \sigma$ ($|n - q| \leq \sigma \leq n + q$). Then after using the asymptotic formula (3.5.6) for ths spherical Bessel function $h_{\sigma}^{(1)}(kl)$ we obtain the approximate equation,

$$Q_{mnmq} = \frac{2i^{n-q-1}e^{ikl}}{kl\,N_{mn}} \sum_{\sigma=|n-q|}^{n+q} b_{\sigma}^{(nmqm)} P_{\sigma}(\cos\theta_{-s,s}) =$$

$$= \begin{cases} 0, & m \neq 0, \\ \dfrac{2i^{n-q-1}e^{ikl}}{kl\,N_{mn}} P_n(\cos\theta_{-s,s})\,P_q(\cos\theta_{-s,s}), & m = 0, \end{cases} \qquad (7.3.86)$$

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if we take into account that when $\theta_{-1,1} = 0$ or when $\theta_{+1,1} = \pi$, all functions $P_n^m(\cos\theta_{-s,s}) = 0$, when $m \neq 0$, and if furthermore we bear formula (3.4.13) in mind. On the basis of (7.3.86), approximately,

$$A_{mn}^{s} = f_{mn}^{s} \text{ and } B_{mn}^{s} = \varphi_{mn}^{s}, \qquad (7.3.87)$$

if $m \neq 0$. If $m = 0$, then $f_{0n}^{s} = 0$ for all n, since from (3.4.24) when $m = 0$ it follows that $L_{01}^{n} = \frac{i}{2}\sqrt{n(n+1)}\,P_{01}^{n}(\alpha)\,(e^{i\phi_s} + e^{-i\phi_s}) = 0$ when $\phi_3 = \frac{3\pi}{2}$. Then also $A_{0n}^{s} = 0$ due to the lack in the homogeneous system corresponding to the heterogeneous system (7.3.9), of non-trivial solutions for all m.

From (3.4.24) when $m = 0$ it follows also that

$$L_{01}^{*n} = \frac{1}{2}\sqrt{n(n+1)}\,P_{01}^{n}(\alpha)(e^{i\varphi_s} - e^{-i\varphi_s}) =$$

$$= -i\sqrt{n(n+1)}\,P_{01}^{n}(\alpha) = -P_n^1(\cos\alpha).$$

Therefore to find the coefficients B^s_{0n} we must solve system (7.3.10) with the matrix elements which on the basis of (7.3.86) have the form,

$$\beta^{(-s,s)}_{0nq} = L\alpha^s_n\beta^s_q, \tag{7.3.88}$$

where

$$L = 2e^{ikl}/ikl,$$

$$\alpha^s_n = i^n P_n(\cos\theta_{-s,s})/\zeta_n(ka_s),$$

$$\beta^s_n = -i^n\psi_n(ka_{-s})P_n(\cos\theta_{-s,s}), \tag{7.3.89}$$

and with right-hand sides of the type

$$\varphi^s_{0n} = Ee^{iskl_0\cos\alpha}i^n\frac{(2n+1)P^1_n(\cos\alpha)}{n(n+1)\zeta_n(ka_s)}. \tag{7.3.90}$$

Repeating the arguments given in §3, Chapter 6 pertaining to a similar situation, we find the expression

$$B^s_{0n} = \varphi^s_{0n} + L\alpha^s_n\frac{LQ(s)F(-s) - F(s)}{1 - L^2Q(s)Q(-s)} \tag{7.3.91}$$

under the condition, that the inequality /354

$$|L^2Q(s)Q(-s)| < 1, \tag{7.3.92}$$

is satisfied; it will always be satisfied with the proper choice of the parameter kl. It is obvious that if $\alpha = 0$ (or $\alpha = \pi$) then $B^s_{0n} = 0$. Here

$$Q(s) = \sum_{n=1}^{\infty}\alpha_n^{-s}\beta^s_n = \sum_{n=1}^{\infty}(-1)^n\frac{\psi_n(ka_{-s})}{\zeta_n(ka_{-s})}, \tag{7.3.93}$$

$$F(s) = \sum_{n=1}^{\infty}\beta^s_n\varphi^{-s}_{0n} = Ee^{iskl_0\cos\alpha}\times$$

$$\times\sum_{n=1}^{\infty}\frac{(2n+1)\psi_n(ka_{-s})P^1_n(\cos\alpha)}{n(n+1)\zeta_n(ka_{-s})}P_n(\cos\theta_{-s,s}). \tag{7.3.94}$$

Therefore, in this case the Debye potentials of the secondary field are written in the form,

$$u^1 = \frac{1}{k} \sum_{s=\pm 1} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{f^s_{mn}}{r_s} \psi_n'(ka_s) \times$$

$$\times \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \zeta_n(kr_s) P_n^m(\cos\theta_s) e^{im\varphi}, \qquad (7.3.95)$$

$$v^1 = \frac{L}{k} \sum_{s=\pm 1} \frac{LQ(s)F(-s) - F(s)}{r_s[1 - L^2 Q(s)Q(-s)]} \times$$

$$\times \sum_{n=1}^{\infty} i^n \frac{\psi_n(ka_s)}{\zeta_n(ka_s)} P_n(\cos\theta_{-s,s}) \zeta_n(kr_s) P_n(\cos\theta_s) +$$

$$+ \frac{1}{k} \sum_{s=\pm 1} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{\varphi^s_{mn}}{r_s} \psi_n(ka_s) \times$$

$$\times \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \zeta_n(kr_s) P_n^m(\cos\theta_s) e^{im\varphi}. \qquad (7.3.96)$$

The expressions for the components of the electromagnetic fields are obtained in the usual manner from (1.6.15), written in one of the spherical systems of coordinates.

Use of the asymptotic formula (3.5.6) assumes that $kl >> 1$, $kl >> \sigma$ where $|n - q| \leq \sigma \leq n + q$, and $n, q = 1, 2, \ldots, N \approx \max_{s=\pm 1} \{ka_s\}$; $a_{-1}, a_{+1} \neq 0$. Consequently, as one of the criteria for satisfaction of which we can use formulas (7.3.95) and (7.3.96), we have the relationship, /355

$$l \gg 2a, \quad a = \max_{s=\pm 1} \{a_s\}. \qquad (7.3.97)$$

From (7.3.95) and (7.3.96) it is clear that when $kl >> 1$ and $l >> 2a$, the effect of multiple scattering of secondary waves on the interacting spheres appears only for a magnetic type field; it is taken into account by the first of the multiple sums in (7.3.96), whose coefficient represents the approximate value of the sum of the diffraction coefficients of the secondary waves of all successive orders of scattering, beginning from the second. The second sum in (7.3.96) just as (7.3.95) gives the secondary field of first order of scattering.

With dimensions of the radii of the diffracting spheres (ka on the order of several units and less) that are small in comparison with the wave length, the numerical solution to the problem may be obtained by using available Tables of Spherical Bessel Functions [182], Tables of Adjoint Legendre Functions [180], and others, and the simplest calculating machines (without resorting to electronic

computers). The Clebsch-Gordan coefficients (§4, Ch. 3) which appea in the expressions for the matrix elements of the systems through the coefficients Q_{mnmq}, may be computed from the formulas (3.4.16), (3.4.17) or from any other by which they can be expressed (as noted above there are various forms of representing the Clebsch-Gordan coefficients [24 -27]). For computation of the generalized spherical functions $P^n_{m,\pm 1}(\theta)$, which appear in the right-hand sides of the systems through the coefficients L^n_{m1} and L^{*n}_{m1}, instead of (3.4.7) we can use more convenient forms to express them: for example, the formula [24],

$$P^n_{mk}(z) = i^{-m-k}\left\{\frac{(n-m)!\,(n-k)!}{(n+m)!\,(n+k)!}\right\}^{1/2}\left(\frac{1+z}{1-z}\right)^{\frac{m+k}{2}}\times$$

$$\times\sum_{j=\max(m,k)}^{n}\frac{(-1)^j(n+j)!}{(n-j)!\,(j-k)!\,(j-m)!}\left(\frac{1-z}{2}\right)^j,$$

which uniquely determines the function $P^n_{mk}(z)$ on the complex z-plane with a cross section along the real axis of the beams $(-\infty, -1)$ and $(1, \infty)$ (here we choose that branch of the function $\left(\frac{1+z}{1-z}\right)^{\frac{m+k}{2}}$ which takes positive values on the branch cut $(-1, +1)$ of the real axis; $z = \cos\theta$), or the formula [24], /356

$$P^n_{mk}(z) = i^{m-k}\{(n-m)!\,(n+m)!\,(n-k)!\,(n+k)!\}^{1/2}\times$$

$$\times\left(\frac{1-z}{1+z}\right)^{\frac{m-k}{2}}\left(\frac{1+z}{2}\right)^n\times$$

$$\times\sum_{j=M}^{N}\frac{(-1)^j}{j!\,(n-m-j)!\,(n+k-j)!\,(m-k+j)!}\left(\frac{1-z}{1+z}\right)^j,$$

in which $M = \max(0, k-m)$, $N = \min(n+k, n-m)$. The particular values of the generalized spherical functions can be computed using the formulas [24],

$$P^n_{nk}(z) = \frac{i^{n-k}}{2^n}\left\{\frac{(2n)!}{(n-k)!\,(n+k)!}\right\}^{1/2}(1-z)^{\frac{n-k}{2}}(1+z)^{\frac{n+k}{2}},$$

whence

$$P^n_{n,n}(z) = \left(\frac{1+z}{2}\right)^n,$$

$$P^n_{n,-n}(z) = (-1)^n \left(\frac{1-z}{2}\right)^n,$$

$$P^n_{n,0}(z) = \frac{i^n \sqrt{(2n)!}}{2^n n!} (1-z^2)^{\frac{n}{2}}$$

[see also formula (3.4.10)].

Spheres With Finite Conductivity. In the case when the spheres possess finite conductivity ($\sigma_2 \neq \infty$), the problem of scattering of a plane electromagnetic wave on two spheres is theoretically solved according to the previous scheme. Thus, if we seek the potentials u^1 and v^1 of the field scattered on the spheres in the form of series (7.3.5), (7.3.6) with the coefficients a^s_{mn} and b^s_{mn} connected to the quantities A^s_{mn} and B^s_{mn} by the formulas in (7.3.8), and the potentials of the fields inside each of the spheres, in the form of the series,

$$u^2_s = \frac{1}{k_2 r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \bar{a}^s_{mn} \psi_n(k_2 r_s) P^m_n(\cos\theta_s) e^{im\varphi}, \qquad (7.3.98)$$

$$v^2_s = \frac{1}{k_2 r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \bar{b}^s_{mn} \psi_n(k_2 r_s) P^m_n(\cos\theta_s) e^{im\varphi}, \qquad (7.3.99)$$

where /357

$$\bar{a}^s_{mn} = \left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} \zeta'_n(k_2 a_s) \bar{A}^s_{mn},$$

$$\bar{b}^s_{mn} = \left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} \zeta_n(k_2 a_s) \bar{B}^s_{mn}, \quad s = \pm 1 \qquad (7.3.100)$$

(in the region outside the spheres $k = \omega\sqrt{\varepsilon\mu}/c$, and inside the spheres $k_2^2 = (\omega^2 \varepsilon_2 \mu_2 + i 4\pi\sigma_2\mu_2\omega)/c$), then from the conjugate conditions of the solutions (1.7.10), (1.7.11) on the surface of each sphere we find with the aid of the addition theorem (3.7.10) the infinite systems of linear equations for the unknowns A^s_{mn} and B^s_{mn}:

$$A^s_{mn} + \sum_{q=|m|}^{\infty} \alpha^{(-s,s)}_{mnq} A^{-s}_{mq} = f^s_{mn}, \qquad (7.3.101)$$

$$B^s_{mn} + \sum_{q=|m|}^{\infty} \beta^{(-s,s)}_{mnq} B^{-s}_{mq} = \varphi^s_{mn} \qquad (7.3.102)$$

with the matrix elements

$$\alpha^{(-s,s)}_{mnq} = \frac{\Delta_n(s)}{{}_1\Delta_n(s)} \cdot \frac{\psi'_q(ka_{-s})}{\psi'_n(ka_s)} Q_{mnmq},$$

$$\beta^{(-s,s)}_{mnq} = \frac{\Delta^*_n(s)}{{}_1\Delta^*_n(s)} \cdot \frac{\psi_q(ka_{-s})}{\psi_n(ka_s)} Q_{mnmq} \qquad (7.3.103)$$

and with right-hand sides

$$f^s_{mn} = \frac{E}{ik} e^{ik(\mathbf{nr}_{os})} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times L^n_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\Delta_n(s)}{\psi'_n(ka_s)\,{}_1\Delta_n(s)}, \qquad (7.3.104)$$

$$\varphi^s_{mn} = \sqrt{\frac{\varepsilon}{\mu}} \frac{E}{ik} e^{ik(\mathbf{nr}_{os})} i^n \frac{2n+1}{n(n+1)} \times$$

$$\times L^*_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\Delta^*_n(s)}{\psi_n(ka_s)\,{}_1\Delta^*_n(s)},$$

through the solutions to which the quantities $\bar{A}^s_{mn}$ and $\bar{B}^s_{mn}$ are found /358 then from the formulas,

$$\bar{A}^s_{mn} = -\frac{i\mu_2 k}{{}_2\Delta_n(s)} \cdot \frac{\psi'_n(ka_s)}{\zeta'_n(k_2a_s)} A^s_{mn},$$

$$\bar{B}^s_{mn} = -\frac{i\mu k_2}{{}_2\Delta_n(s)} \cdot \frac{\psi_n(ka_s)}{\zeta_n(k_2a_s)} B^s_{mn}. \qquad (7.3.105)$$

Here

$$\Delta_n(s) = \begin{vmatrix} \mu_2 k \psi_n(ka_s) & \psi'_n(ka_s) \\ \mu k_2 \psi(k_2a_s) & \psi'_n(k_2a_s) \end{vmatrix},$$

$$\Delta^*_n(s) = \begin{vmatrix} \mu k_2 \psi_n(ka_s) & \psi'_n(ka_s) \\ \mu k_2 \psi_n(k_2a_s) & \psi'_n(k_2a_s) \end{vmatrix}, \qquad (7.3.106)$$

$${}_1\Delta_n(s) = \begin{vmatrix} \mu_2 k \zeta_n(ka_s) & \zeta'_n(ka_s) \\ \mu k_2 \psi_n(k_2a_s) & \psi'_n(k_2a_s) \end{vmatrix},$$

$$ {}_1\Delta_n^*(s) = \begin{vmatrix} \mu k_2 \zeta_n(ka_s) & \zeta_n'(ka_s) \\ \mu k_2' \psi_n(k_2 a_s) & \psi_n'(k_2 a_s) \end{vmatrix}, $$

$$ {}_2\Delta_n(s) = \begin{vmatrix} \mu k_2 \psi_n'(ka_s) & \dot{\psi}_n(ka_s) \\ \mu_2 k \psi_n'(k_2 a_s) & \psi_n(k_2 a_s) \end{vmatrix}, $$

$$ {}_2\Delta_n^*(s) = \begin{vmatrix} \mu_2 k \psi_n'(ka_s) & \psi_n(ka_s) \\ \mu k_2 \psi_n'(k_2 a_s) & \psi_n(k_2 a_s) \end{vmatrix}. $$

Just as before, $\psi_3 = \pi/2 - \beta$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$.

The infinite systems (7.3.101), (7.3.102) possess the same properties as systems (7.3.9), (7.3.10). For each m their solutions may be found by the method of truncation. The infinite dimensional determinants of the systems are representable in the form of (7.3.14). By introducing the designations in (7.3.18) - (7.3.21), systems (7.3.101), (7.3.102) are reduced to the normal form (7.3.17). The approximate values of the quantities A_{mn}^s, B_{mn}^s and $\bar{A}_{mn}^s$, $\bar{B}_{mn}^s$ can be found by solving finite (truncated) systems such as (7.3.61), (7.3.62) or (7.3.63) according to the same scheme as in the previous problem.

In approximation of the wave band for the field scattered on /359
the spheres, relationships (7.3.41) are satisfied, in which the amplitude functions $A(\theta, \phi)$ and $A^*(\theta, \phi)$, retaining the former form, will differ from (7.3.42), (7.3.43) by their composition, the coefficients A_{mn}^s and B_{mn}^s which appear in their expressions [through a_{mn}^s and b_{mn}^s, see (7.3.8)], are now computed from (7.3.101), (7.3.102). The form of the formulas that are used to compute the scattering cross sections, does not vary.

The problem on diffraction of a plane electromagnetic wave on any finite number of spheres (with $\sigma_2 \neq \infty$ or with $\sigma_2 = \infty$) can also be solved according to the same scheme in the case when they form a linear system (grid) with a common axis of symmetry, as well as the problem on the diffraction of a plane electromagnetic wave on a sphere of arbitrary conductivity with non-concentric spherical inclusions in the case when they possess a common axis of symmetry.

§4. Diffraction of the Field of a Dipole Emitter on Two Spheres.

In an unbounded uniform and isotropic space with electric and magnetic permeabilities ε, μ and a conductivity $\sigma = 0$ let there be found at a distance l from one another the spheres $r_{-1} \leq a_{-1}$ and $r_{+1} \leq a_{+1}$ with identical electromagnetic constants ε_2, μ_2 but at

point P of the space at distances l_{-1} and l_{+1}, respectively from the centers of the spheres, let there be found a unit oscillational dipole with the moment a(a = p, if the dipole is electric and a = m if it is magnetic) directed at an angle α to the plane P which passes through the point P and the line of the centers of the spheres. The projection of the vector a on the plane P forms the angle β with the positive direction of the Oz axis, coinciding with the axis of symmetry of the spheres, but the direction from the center of the s-th sphere to the point P determined by the vector l_s (s = $\pm$ 1), forms the angle α_s with the positive direction of the Oz axis. We formulate the problem of seeking the secondary electromagnetic field scattered on the spheres and the fields inside the spheres (if $\sigma_2 = \infty$, then E and H inside each of the spheres are identically equal to zero).

If we expand the vector a to two components of the vector a_1 and a_2, where $\mathsf{a}_1 = \mathsf{a}\cos\alpha$, but $\mathsf{a}_2 = \mathsf{a}\sin\alpha$ (the vector a_1 lies in the plane P and the vector a_2 is perpendicular to it at the point P), then the solution to the vector problem with arbitrary orientation of the moment a of the dipole relative to the plane P may be substituted by the solution to the particular mutually independent electromagnetic problems: the problem on determining the field produced by the dipole lying in the plane P and the problem on determining the field produced by the dipole normal to the plane P at point P. Solution to the general problem may be found then in the /360 form of a superposition of solutions to these two problems. In spherical coordinates the particular solutions are reduced to scalar boundary problems of seeking the Debye potentials u and v. Their solution assumes knowledge of the explicit expressions of the Debye potentials of the primary dipole field. Let us find them. For this purpose we introduce into study the local spherical coordinate systems r_s, θ_s, ϕ_s' (s = $\pm$ 1) with the origin at point O_s, superposed with the center of the s-th sphere and with the polar axis $O_s z_s$ which coincides with the Oz axis directed along the line of the centers of the spheres. Besides these coordinates, in the future we shall use the spherical coordinate system r, θ, ϕ as well with the origin at point O, located on the line of the centers of the spheres in its middle (Fig. 42), whose polar axis is also directed along the line of the centers of the spheres. It is assumed that $\phi_s = \phi$, s = $\pm$ 1, and the plane ϕ = 0 coincides with the plane Oxz which passes through the point P with dipole in it and the line of the centers of

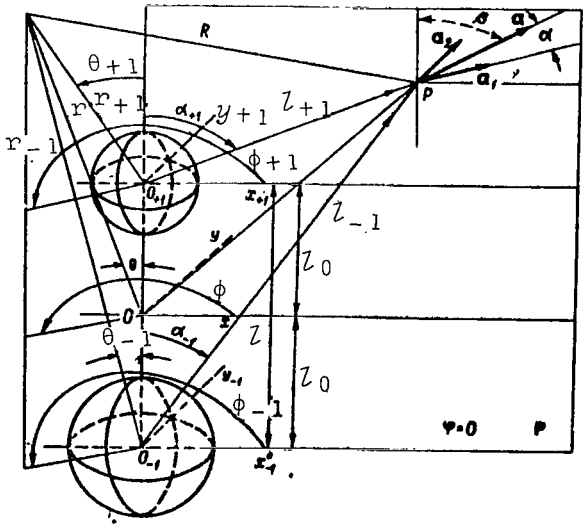


Fig. 42. Excitation of Two Spheres by the Dipole Field.

the spheres. It is obvious that if the dipole is located on the polar axis of the introduced local spherical coordinate systems r_s, θ_s, ϕ_s ($s = \pm 1$), then the explicit form of the potential Debye functions of the primary dipole field will be given by the series of spherical wave functions written in local coordinates of the s-th sphere by the proper formulas from §2 of this chapter, where they were derived for such a case. If this dipole is found to be located at point P, which does not lie on the polar axis of the local systems of spherical coordinates, then the unknown representations of the potential Debye functions of the primary field can be obtained from the formulas in §2 of this chapter by a slight conversion of each term of the series which determines the respective Debye potential. Here we can delineate the cases when the dipole moment a lies on the plane P and when it is perpendicular to the plane P at the point P. Let us look at the first of these (it corresponds to the first of the two particular problems about which we spoke above): a $\in$ pl. P. We can introduce into study the new local systems of Cartesian coordinates x_s', y_s', z_s' ($s = \pm 1$) and the spherical systems of local coordinates r_s', θ_s', ϕ_s' ($r_s' = r_s$) that are coupled with them by the usual conversion formulas, in such a way that their origins coincide with O_s ($s = \pm 1$), that the axes $O_s z_s'$ would pass through the point P in the direction of the proper vectors 1_s, and so that the planes $O_s x_s' z_s'$ ($\phi_s' = 0$) coincide with the plane P. Then in each of these local spherical coordinate systems the Debye potentials u^0, v^0 of the electric dipole are written on the basis of formulas (7.2.18) and (7.2.21) in the form of the expansions, /361

$$u^0 = \frac{ip}{l_s r_s'} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) A_{mn}(\gamma_s, l_s) \times$$

$$\times P_n^m(\cos\theta_s') \begin{Bmatrix} \psi_n(kr_s') \\ \zeta_n(kr_s') \end{Bmatrix} e^{im\varphi_s}, \qquad (7.4.1)$$

$$v^0 = \sqrt{\frac{\varepsilon}{\mu}} \frac{p \sin\gamma_s}{l_s r_s'} \sin\varphi_s' \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \times$$

$$\times P_n^1(\cos\theta_s') \begin{cases} \psi_n(kl_s)\zeta_n(kr_s'), & l_s < r_s, \\ \psi_n(kr_s')\zeta_n(kl_s), & l_s > r_s', \end{cases} \qquad (7.4.2)$$

where $\gamma_s = \beta - \alpha_s$, $s = \pm 1$ (see Fig. 42).

Now let us convert the spherical functions which appear in (7.4.1) and (7.4.2) into the coordinates r_s, θ_s, ϕ_s ($\phi_s = \phi$, $s = \pm 1$). To do so let us use the addition formulas for the spherical functions (3.4.21) and (3.4.22) in which it follows to set $\theta_1 = \theta_s'$, $\phi_1 = \phi_s'$, $\theta_2 = \theta_s$, $\phi_2 = \phi_s = \phi$ and as the Euler angles we must take the values $\psi_3 = \pi/2$, $\theta_3 = \alpha_s$ and $\phi_s = 3\pi/2$. On the basis of (3.4.21), (3.4.22) expansions (7.4.1), (7.4.2) in the coordinates r_s, θ_s, ϕ_s are written in the form,

$$u^0 = \frac{ip\cos\gamma_s}{kl_s^2 r_s}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)L_{mn}(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\begin{matrix}\zeta_n(kl_s)\psi_n(kr_s)\\ \psi_n(kl_s)\zeta_n(kr_s)\end{matrix}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s} + \frac{ip\sin\gamma_s}{l_s r_s}\times$$

$$\times\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}L_{m1}^n(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\begin{matrix}\zeta_n'(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n'(kl_s)\end{matrix}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s},\quad \begin{matrix}l_s>r_s,\\ l_s<r_s,\end{matrix} \qquad (7.4.3)$$

$$v^0 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p\sin\gamma_s}{l_s r_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}\times$$

$$\times L_{m1}^{*n}(\psi_3,\ \theta_3,\ \varphi_3)\left\{\begin{matrix}\zeta_n(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n(kl_s)\end{matrix}\right\}\times$$

$$\times P_n^m(\cos\theta_s)e^{im\varphi_s},\quad \begin{matrix}l_s>r_s,\\ l_s<r_s.\end{matrix}$$

In the same way we obtain expansions of the spherical wave functions from the coordinate system r_s, θ_s, ϕ_s of the potential Debye functions of the primary field of the magnetic dipole located at point P and possessing the moment m, lying in the plane P. They have the form

$$u^0 = -\sqrt{\frac{\mu}{\varepsilon}}\,\frac{|\mathbf{m}|\sin\gamma_s}{l_s r_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}\times$$

$$\times L_{m1}^{*n}(\psi_3,\ \theta_3,\ \varphi_3)\left\{\begin{matrix}\zeta_n(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n(kl_s)\end{matrix}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s},$$

$$v^0 = \frac{i|\mathbf{m}|\cos\gamma_s}{kl_s^2 r_s}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)L_{mn}(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\begin{matrix}\psi_n(kr_s)\zeta_n(kl_s)\\ \psi_n(kl_s)\zeta_n(kr_s)\end{matrix}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s} + \qquad (7.4.4)$$

$$+\frac{i|\mathbf{m}|\sin\gamma_s}{l_s r_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}L_{m1}^n(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\begin{matrix}\zeta_n'(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n'(kl_s)\end{matrix}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s},$$

where the upper line is used when $l_s > r_s$, and the lower when $l_s < r_s$ and where as the Euler angles in the expressions for L_{mn}, L^n_{m1}, L^{*n}_{m1} we take the former values $\psi_3 = \pi/2$, $\theta'_3 = \alpha_3$ and $\phi_3 = 3\pi/2$.

In the case when the dipole moment located at point P is perpendicular to the plane P (this case corresponds to the second particular problem), the local coordinate systems x'_s, y'_s, z'_s and the local spherical coordinate systems r'_s, θ'_s, ϕ'_s corresponding to them are introduced in such a way that the polar axes $O_s z'_s$ ($s = \pm 1$) again pass through the point P in the direction of the vectors l_s and so that the plane $O_s x'_s z'_s$ (the plane $O_s x'_s z'_s$ coincides with the plane $O_s y_s z_s$) is normal to the plane P (here the axes $O_s x'_s$ of the new systems will coincide with the axes $O_s y_s$ of the previous local systems of coordinates; the position of the coordinate axes of the system $O_s x_s y_s z_s$ is determined in the system $O_s x'_s y'_s z'_s$ by the Euler angles $\psi_3 = 0$, $\theta_3 = \alpha_s$, $\phi_3 = 3\pi/2$). Then, if we first write the Debye potentials of the primary dipole field in the form of expansions according to the formulas in §2 of this chapter in spherical coordinates of the local systems r'_s, θ'_s, ϕ'_s ($r'_s = r_s$), after converging the spherical functions to coordinates of the system r_s, θ_s, ϕ_s using the addition formulas (3.4.21), (3.4.22) in which $\psi_3 = 0$, and $\theta_3 = \alpha_s$, $\phi_3 = 3\pi/2$, we find the desired representations of the Debye potentials u^0, v^0 of the primary dipole field, expressed through the spherical wave functions of the coordinate system of the s-th sphere. Here for the electric dipole they will have the form,

$$u^0 = \frac{ip}{l_s r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{2n+1}{n(n+1)} L^n_{m1}(\psi_3, \theta_3, \varphi_3) \times$$
$$\times \left\{ \begin{matrix} \zeta'_n(kl_s)\psi_n(kr_s) \\ \zeta_n(kr_s)\psi'_n(kl_s) \end{matrix} \right\} P^m_n(\cos\theta_s) e^{im\varphi_s},$$
$$v^0 = \sqrt{\frac{\varepsilon}{\mu}} \frac{p}{l_s r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{2n+1}{n(n+1)} L^{*n}_{m1}(\psi_3, \theta_3, \varphi_3) \times \qquad (7.4.5)$$
$$\times \left\{ \begin{matrix} \zeta_n(kl_s)\psi_n(kr_s) \\ \zeta_n(kr_s)\psi_n(kl_s) \end{matrix} \right\} P^m_n(\cos\theta_s) e^{im\varphi_s}$$
$$(\psi_3 = 0, \quad \theta_3 = \alpha_s, \quad \varphi_3 = 3\pi/2),$$

and for the magnetic dipole, /364

$$u^0 = -\sqrt{\frac{\mu}{\varepsilon}} \frac{|m|}{l_s r_s} \sum_{n=1}^{\infty} \sum_{m=-n}^{n} \frac{2n+1}{n(n+1)} \times$$

$$\times L^{*n}_{m1}(\psi_3,\ \theta_3,\ \varphi_3)\left\{\begin{array}{l}\zeta_n(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n(kl_s)\end{array}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s},$$

$$v^0 = \frac{i|m|}{l_s r_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}L^n_{m1}(\psi_3,\ \theta_3,\ \varphi_3)\times \tag{7.4.6}$$

$$\times\left\{\begin{array}{l}\zeta_n'(kl_s)\psi_n(kr_s)\\ \zeta_n(kr_s)\psi_n'(kl_s)\end{array}\right\}P_n^m(\cos\theta_s)e^{im\varphi_s}$$

$$(\psi_3 = 0,\ \theta_3 = \alpha_s;\ \varphi_3 = 3\pi/2).$$

Formulas (7.4.3) - (7.4.6) of the Debye potentials of the primary field of the electric and magnetic dipoles in the form of expansions of spherical wave functions written in the local coordinates of the s-th sphere give the possibility of solving the scalar boundary problems for the potentials u^1 and v^1 of the secondary field according to the same scheme as in the case of a plane wave.

Electric Dipole. Let us first solve the particular problem for the case when the dipole has the moment p lying in the plane P and forming an angle β with the positive direction of the Oz axis. The potentials of the primary dipole field are determined by the formulas in (7.4.3). Therefore if we seek the potentials u^1 and v^1 of the secondary field in the form,

$$u^1 = ip\sum_{s=\pm1}u_s,\quad v^1 = \sqrt{\frac{\varepsilon}{\mu}}\,p\sum_{s=\pm1}v_s, \tag{7.4.7}$$

where the functions

$$u_s = \frac{1}{kr_s}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a^s_{mn}\zeta_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi_s}, \tag{7.4.8}$$

$$v_s = \frac{1}{kr_s}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}b^s_{mn}\zeta_n(kr_s)P_n^m(\cos\theta_s)e^{im\varphi_s} \tag{7.4.9}$$

represent the potentials of the secondary field produced by the presence in space of the s-th sphere, then from boundary conditions (1.7.13) on the surface of each sphere we obtain, after using the addition theorem (3.7.10), infinite systems of linear equations /365 relative to the diffraction coefficients a^s_{mn} and b^s_{mn}. After their substitution by the new coefficients A^s_{mn} and B^s_{mn} according to formulas (7.3.8), respectively for A^s_{mn} and B^s_{mn} we obtain systems (7.3.9), (7.3.10) with the matrix elements (7.3.11) and (7.3.12) and with right-hand sides

$$f^s_{mn} = f^{s1}_{mn} + f^{s2}_{mn} \tag{7.4.10}$$

in the system (7.3.9), where

$$f^{s1}_{mn} = -\frac{\cos\gamma_s}{l_s^2}(2n+1)\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2} \times$$
$$\times L_{mn}(\psi_3, \theta_3, \varphi_3)\frac{\zeta_n(kl_s)}{\zeta'_n(ka_s)}, \tag{7.4.11}$$

$$f^{s2}_{mn} = -\frac{k\sin\gamma_s}{l_s}\frac{(2n+1)}{n(n+1)}\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2} \times$$
$$\times L^n_{m1}(\psi_3, \theta_3, \varphi_3)\frac{\zeta'_n(kl_s)}{\zeta'_n(ka_s)}, \tag{7.4.12}$$

$$\varphi^s_{mn} = -\frac{k\sin\gamma_s}{l_s}\frac{(2n+1)}{n(n+1)} \times$$
$$\times\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2} L^{*n}_{m1}(\psi_3, \theta_3, \varphi_3) \tag{7.4.13}$$

in system (7.3.10). Here $\psi_3 = \pi/2$, $\theta_3 = \alpha_s$, $\phi_3 = 3\pi/2$; $s = \pm 1$. It is assumed that $f^{s2}_{mn} \equiv 0$, if $n = 0$. Using the asymptotic formula (3.5.5) for the spherical Bessel functions and the properties of the generalized spherical functions T^n_{mk}, which appear in the expressions for the quantities L^n_{m1}, L^{*n}_{m1}, L_{mn}, we can establish that the right-hand sides of the systems [i.e., (7.4.10) and (7.4.13)] satisfy the conditions $\sum_n^\infty |f^s_{nm}|^2 < \infty$ and $\sum_n^\infty |\varphi^s_{mn}|^2 < \infty$ for each m, if the dipole does not lie on one of the spheres ($l_s > a_s$, where $s = +1$ or $s = -1$). Therefore systems (7.3.9), (7.3.10) with right hand sides (7.4.10), (7.4.13) are uniquely solvable (for each m) and their solutions will satisfy conditions (7.3.35), (7.3.36). The approximate values of the diffraction coefficients A^s_{mn} and B^s_{mn} are found by the method of truncation.

By determining A^s_{mn} and B^s_{mn} the problem for the potential u^1, v^1 is completely solved; these potentials may, on the basis of (7.3.8), be written in the form, /366

$$u^1 = \frac{ip}{k}\sum_{s=\pm1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} \times \tag{7.4.14}$$
$$\times A^s_{mn}\frac{\zeta_n(kr_s)}{r_s}P^m_n(\cos\theta_s)e^{im\varphi_s}\psi'_n(ka_s),$$

$$v^1 = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{p}{k}\sum_{s=\pm 1}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}\times \tag{7.4.15}$$

$$\times B^s_{mn}\frac{\zeta_n(kr_s)}{r_s}P_n^m\cos\theta_s)e^{im\varphi_s}\psi_n(ka_s).$$

In the case when the dipole moment p is perpendicular to the plane P at point P, the potentials of the primary field are given by the formulas in (7.4.5). Therefore, if we seek the potentials of the field scattered on the spheres in the form of series (7.4.7) - (7.4.9) in which the coefficients a^s_{mn}, b^s_{mn} are coupled with the quantities A^s_{mn} and B^s_{mn} by formula (7.3.8), then the boundary conditions (1.7.13) on the surface of each sphere will be satisfied only in the case if A^s_{mn} and B^s_{mn} are solutions to systems (7.3.9), (7.3.10) with the matrix elements (7.3.11), (7.3.12) and with right-hand sides such as

$$f^s_{mn} = -\frac{k}{l_s}\cdot\frac{2n+1}{n(n+1)}L^n_{m1}(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2}\frac{\zeta_n'(kl_s)}{\zeta_n'(ka_s)}, \tag{7.4.16}$$

$$\varphi^s_{mn} = -\frac{k}{l_s}\cdot\frac{2n+1}{n(n+1)}L^{*n}_{m1}(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2}\frac{\zeta_n(kl_s)}{\zeta_n(ka_s)}, \tag{7.4.17}$$

$$\psi_3 = 0,\ \theta_3 = \alpha_s,\ \varphi_3 = 3\pi/2.$$

Just as in the previous case, the infinite systems of linear equations obtained here differ from systems (7.3.9), (7.3.10) only in the expressions of the right-hand sides, which however does not lead to any change in their properties: when $l_s > a_s$ ($s = +1$ or $s = -1$, the dipole does not lie on the surface of one of the spheres) /367 the systems are uniquely solvable by the method of truncation. All other properties of the systems described in §3 of this chapter are also retained.

After computing A^s_{mn} and B^s_{mn} the potentials of the secondary field are found from formulas (7.4.14), (7.4.15).

In both of the particular cases studied above, the components of the electromagnetic field are found through the Debye potentials by the usual manner from equations (1.6.15), written in one of the spherical coordinate systems. In particular, in the coordinates

r, θ, ϕ with the origin at point O the components of the resulting electromagnetic field in the vicinity of the wave band will be given by the expressions [in (7.4.19) - (7.4.22), $l = \overline{OP}$ rather than $\overline{O_{-1}O_{+1}}$]

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi = k^2 p \frac{e^{ikR}}{R} W_1(\theta, \varphi),$$

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\theta = k^2 p \frac{e^{ikR}}{R} W_2, \qquad (7.4.18)$$

where W_1 and W_2, in the case when $p \in$ pl.P and $p \perp$ pl.P, are equal respectively to:

$$W_1 = \frac{ie^{ikl\cos(\alpha-\theta)}}{k^2 l^2}\left[\left\{\cos\gamma \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(2n+1)\times\right.\right.$$

$$\times L_{mn}(\tilde{\psi}_3, \theta_3, \varphi_3)(-i)^n \psi_n(kl)\frac{dP_n^m(\cos\theta)}{d\theta}e^{im\varphi} +$$

$$+ (kl)\sin\gamma \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)} L_{m1}^n(\tilde{\psi}_3, \tilde{\theta}_3, \tilde{\varphi}_3)\times$$

$$\times(-i)^n \psi_n(kl)\frac{dP_n^m(\cos\theta)}{d\theta}e^{im\varphi} +$$

$$\left.\left. + l^2 \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n e^{im\varphi}\frac{d}{d\theta}\left[P_n^m(\cos\theta)\sum_{s=\pm1} a_{mn}^s e^{iskl_0\cos\theta}\right]\right\} + \right. \qquad (7.4.19)$$

$$+ \frac{(kl)\sin\gamma}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m\frac{2n+1}{n(n+1)}(-i)^n\psi_n(kl)\times$$

$$\times L_{m1}^{*n}(\psi_3, \theta_3, \varphi_3) P_n^m(\cos\theta) e^{im\varphi} + \frac{l^2}{\sin\theta}\times$$

$$\left.\times \sum_{n=0}^{\infty}\sum_{m=-n}^{\infty}(-i)^n P_n^m(\cos\theta)\left(\sum_{s=\pm1} e^{iskl_0\cos\theta} b_{mn}^s\right)e^{im\varphi} m\right],$$

$$W_2 = -\frac{e^{ikl\cos(\alpha-\theta)}}{k^2 l^2}\left[\left\{\frac{\cos\gamma}{\sin\theta}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} m(2n+1)\times\right.\right.$$

$$\times L_{mn}(\psi_3, \theta_3, \varphi_3)(-i)^n \psi_n(kl) P_n^m(\cos\theta) e^{im\varphi} +$$

$$\left.\left. + \frac{kl\sin\gamma}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m(-i)^n \frac{2n+1}{n(n+1)}\times\right.\right.$$

$$\times L_{m1}^{n}(\psi_3,\ \theta_3,\ \varphi_3)\,\psi_n(kl)\,P_n^m(\cos\theta)\,e^{im\varphi}+\frac{l^2}{\sin\theta}\times$$

$$\times\sum_{n=1}^{\infty}\sum_{m=-n}^{n}m(-i)^n\left(\sum_{s=\pm1}a_{mn}^{s}e^{iskl_0\cos\theta}\right)P_n^m(\cos\theta)\,e^{im\varphi}\Bigg\}+$$

$$+(kl)\sin\gamma\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-i)^n\frac{2n+1}{n(n+1)}\times$$

$$\times L_{m1}^{*n}(\psi_3,\ \theta_3,\ \varphi_3)\,\psi_n(kl)\frac{dP_n^m(\cos\theta)}{d\theta}e^{im\varphi}+ \qquad (7.4.20)$$

$$+l^2\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n e^{im\varphi}\times$$

$$\times\frac{d}{d\theta}\left[P_n^m(\cos\theta)\sum_{s=\pm1}b_{mn}^{s}e^{iskl_0\cos\theta}\right]\Bigg],$$

$$\psi_3=\pi/2,\ \theta_3=\alpha_s,\ \varphi_3=3\pi/2,$$

and

$$W_1=\frac{ie^{ikl\cos(\alpha-\theta)}}{k^2l^2}\Bigg[(kl)\sin\gamma\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\frac{2n+1}{n(n+1)}\times$$

$$\times L_{m1}^{n}(\psi_3,\ \theta_3,\ \varphi_3)(-i)^n\psi_n(kl)\frac{dP_n^m(\cos\theta)}{d\theta}e^{im\varphi}+$$

$$+l^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-i)^n\frac{d}{d\theta}\Bigg[P_n^m(\cos\theta)\times$$

$$\times\sum_{s=\pm1}a_{mn}^{s}e^{iskl_0\cos\theta}\Bigg]e^{im\varphi}+\frac{kl\sin\gamma}{\sin\theta}\times \qquad (7.4.21) \quad /369$$

$$\times\sum_{n=1}^{\infty}\sum_{m=-n}^{n}m\frac{2n+1}{n(n+1)}(-i)^n\psi_n(kl)\,L_{m1}^{*n}(\psi_3,\ \theta_3,\ \varphi_3)\times$$

$$\times P_n^m(\cos\theta)\,e^{im\varphi}+\frac{l^2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-i)^n m\,P_n^m(\cos\theta)\times$$

$$\times\left(\sum_{s=\pm1}b_{mn}^{s}e^{iskl_0\cos\theta}\right)e^{im\varphi}\Bigg],$$

$$W_s = -\frac{e^{ikl\cos(\alpha-\theta)}}{k^2 l^2}\left[\frac{kl\sin\gamma}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m(-i)^n \times\right.$$

$$\times \frac{2n+1}{n(n+1)} L_{ml}^{n}(\psi_3, \theta_3, \varphi_3)\psi_n'(kl) P_n^m(\cos\theta) e^{im\varphi} +$$

$$+ \frac{l^2}{\sin\theta}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m(-i)^n \left(\sum_{s=\pm 1} a_{mn}^{s} e^{iskl_s\cos\theta}\right) \times$$

$$\times P_n^m(\cos\theta) e^{im\varphi} + (kl)\sin\gamma\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-i)^n \frac{2n+1}{n(n+1)} \times \qquad (7.4.22)$$

$$\times L_{ml}^{*n}(\psi_3, \theta_3, \varphi_3)\psi_n(kl)\frac{dP_n^m(\cos\theta)}{d\theta} e^{im\varphi} +$$

$$+ l^2\sum_{n=1}^{\infty}\sum_{m=-n}^{n}(-i)^n e^{im\varphi} \times$$

$$\left.\times \frac{d}{d\theta}\left[P_n^m(\cos\theta)\sum_{s=\pm 1} b_{mn}^{s} e^{iskl_s\cos\theta}\right]\right],$$

$$\psi_3 = 0,\ \theta_3 = \alpha_s,\ \varphi_3 = 3\pi/2.$$

The rule governing the density distribution of the surface currents /370 induced on the surface of the s-th sphere will be given by formulas (7.3.39) in which in this case,

$$j_{\varphi}^{ms} = -\frac{p\omega\varepsilon}{4\pi ka_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \times$$

$$\times B_{mn}^{s}\frac{d}{d\theta_s} P_n^m(\cos\theta_s) e^{im\varphi_s},$$

$$j_{\varphi}^{es} = \frac{p\omega\varepsilon}{4\pi ka_s}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} m\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \times$$

$$\times A_{mn}^{s}\frac{P_n^m(\cos\theta_s)}{\sin\theta_s} e^{im\varphi_s}, \qquad (7.4.23)$$

$$j_{\theta}^{ms} = \frac{ip\omega\varepsilon}{4\pi ka_s}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} m\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2} \times$$

$$\times B_{mn}^{s}\frac{P_n^m(\cos\theta_s)}{\sin\theta_s} e^{im\varphi_s},$$

$$f_\theta^{es} = -\frac{ip\omega\varepsilon}{4\pi k a_s^2}\sum_{n=1}^{\infty}\sum_{m=-n}^{n}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2}\times$$

$$\times A_{mn}^{s}\frac{d}{d\theta_s}P_n^m(\cos\theta_s)e^{im\varphi_s}$$

and the values of the coefficients A_{mn}^{s} and B_{mn}^{s} must be found from systems (7.3.9), (7.3.10) with right-hand sides (7.4.10)-(7.4.13), if the moment of the electric dipole lies in the plane P, and with right-hand sides (7.4.16), (6.4.17), in the case when the moment is perpendicular to it.

The existence in the problem of a large number of parameters and the possibility of varying them in the respective regions necessitate studying the particular cases which follow from the general formulas. In particular, if the dipole is found to lie on the polar axis Oz at point P "over" the spheres (then $\alpha_s = 0$, $s = \pm 1$), "under" the spheres (then $\alpha_s = \pi$, $s = \pm 1$) or between them (then $\alpha_{-1} = 0$, $\alpha_{+1} = \pi$), then for study of the dipole field in the presence of spheres it is sufficient to look only at the first particular problem, which will also be the general one, since then we can always construct a plane which passes through the Oz axis in such a way that the dipole moment is found to lie in it (the plane P, $\phi = 0$). These cases lead to simpler expressions which determine the diffracted field. For example, let the dipole be located "over" the spheres such that $\alpha_s = 0$, $s = \pm 1$. Then on the basis of (3.4.6), (3.4.10) and (3.4.24) the right-hand sides (7.4.10)-(7.4.13) of systems (7.3.9), (7.3.10) are found to equal: /371

$$f_{mn}^{s} = 0, \quad m \neq 0, \quad m \neq \pm 1; \qquad \varphi_{mn}^{s} = 0, \quad m \neq \pm 1;$$

$$f_{0n}^{s} = f_{0n}^{s1} = -\frac{\cos\gamma_s}{l_s^2}(2n+1)\frac{\zeta_n(kl_s)}{\zeta_n'(ka_s)}, \quad f_{\pm 1,n}^{s1} = 0,$$

$$f_{\pm 1,n}^{s} = f_{\pm 1,n}^{s2} = \mp\frac{k\sin\beta}{2l_s}\cdot\frac{2n+1}{[n(n+1)]^{1/2}}\cdot\frac{\zeta_n'(kl_s)}{\zeta_n'(ka_s)}, \qquad (7.4.24)$$

$$\varphi_{\pm 1,n}^{s} = \pm\frac{ik\sin\beta}{2l_s}\cdot\frac{2n+1}{[n(n+1)]^{1/2}}\cdot\frac{\zeta_n(kl_s)}{\zeta_n(ka_s)},$$

where $\gamma_s = \beta - \alpha_s = \beta$, as a result of which all A_{mn}^{s} and B_{mn}^{s} with the exception of A_{0n}^{s}, $A_{\pm 1,n}^{s}$ and $B_{\pm 1,n}^{s}$ will be equal to zero. These latter are found from systems (7.3.9), (7.3.10) with right-hand sides (7.4.24). Moreover if the dipole is vertical (p is directed along the Oz axis) such that $\beta = 0$, then also $f_{\pm 1,n}^{s} = 0$, $\phi_{\pm 1,n}^{s} = 0$, but

$$f_{0n}^{s} = -\frac{2n+1}{l_s^2}\cdot\frac{\zeta_n(kl_s)}{\zeta_n'(ka_s)}, \qquad (7.4.25)$$

as a result of which only the coefficients A^s_{0n} are found to be non-zero, but all others will be equal to zero. In this latter case the magnetic potentials v both of the primary and of the secondary fields are identically equal to zero, and only an electric type field will exist that is determined by the electric potential u^1, equal to

$$u^1 = ip \sum_{s=\pm 1} \sum_{n=0}^{\infty} \frac{A^s_{0n}}{kr_s} \psi'_n(ka_s)\, \zeta_n(kr_s)\, P_n(\cos\theta_s). \qquad (7.4.26)$$

If $\beta = \pi/2$ (the case of a horizontal dipole, when the dipole moment is normal to the Oz axis), then $f^s_{0n} = 0$. In this case the right-hand sides of the systems $f^s_{\pm 1, n}$ and $\phi^s_{\pm 1, n}$ will be mutually related by the equations

$$f^s_{1,n} = -f^s_{-1,n}, \quad \varphi^s_{1,n} = \varphi^s_{-1,n}, \quad s = \pm 1. \qquad (7.4.27)$$

From (7.4.27) and the property of parity of the matrix elements of systems (7.3.9), (7.3.10) relative to the index m it follows that /372

$$A^s_{1n} = -A^s_{-1,n}, \quad B^s_{1n} = B^s_{-1,n}, \quad s = \pm 1. \qquad (7.4.28)$$

Therefore, the potentials of the stray field may be written in the form,

$$u^1 = 2ip \sum_{s=\pm 1} \sum_{n=1}^{\infty} \left\{ \frac{(n-1)!}{(n+1)!} \right\}^{1/2} \times$$
$$\times A^s_{1n} \frac{\psi'_n(ka_s)}{kr_s} \zeta_n(kr_s)\, P^1_n(\cos\theta_s) \cos\varphi,$$
$$v^1 = 2ip \sqrt{\frac{\varepsilon}{\mu}} \sum_{s=\pm 1} \sum_{n=1}^{\infty} \left\{ \frac{(n-1)!}{(n+1)!} \right\}^{1/2} \times \qquad (7.4.29)$$
$$\times B^s_{1n} \frac{\psi_n(ka_s)}{kr_s} \zeta_n(kr_s)\, P^1_n(\cos\theta_s) \sin\varphi.$$

The potentials of the primary dipole field, located on the polar axis are given (in local coordinates of the s-th sphere) by expressions (7.2.18), (7.2.21) in which it follows to substitute θ_s for θ and l_s for b.

In the case of a vertical dipole located on the line of the centers of the spheres of the same radius, midway between them we

find the equation $f_{0n}^{+1} = f_{0n}^{-1}$, and in the case of a horizontal dipole located at the same point, the right-hand sides of the systems are related by the formulas,

$$f_{\pm 1,n}^{+1} = -(-1)^n f_{\pm 1,n}^{-1}, \quad \varphi_{\pm 1,n}^{+1} = (-1)^n \varphi_{\pm 1,n}^{-1}, \tag{7.4.30}$$

on the basis of which, for the spheres of equal radii from systems (7.3.9), (7.3.10) due to (7.3.65), we obtain the equations,

$$A_{\pm 1,n}^{+1} = -(-1)^n A_{\pm 1,n}^{-1}, \quad B_{\pm 1,n}^{+1} = (-1)^n B_{\pm 1,n}^{-1}. \tag{7.4.31}$$

Therefore for solving the diffraction problem in this case it is sufficient to subtract from the systems only the unknowns A_{1n}^{+1}, B_{1n}^{+1} (or A_{1n}^{-1} B_{-1n}^{-1}). The case of widely spaced spheres ($kl >> 1$) may be studied according to the same scheme as in the problem on diffraction of a plane wave on two spheres (§3 of this chapter). In the same way as in §3 of this chapter we may study also the problem of defining secondary waves of any order of scattering, produced by the s-th sphere due to its interaction with the $-s$-th sphere (at any point of observation outside the spheres the total effect of all subsequent secondary diffractions is taken into account through the coefficients A_{mn}^{s}, B_{mn}^{s}). /373

It has previously been assumed that the dipole does not lie on the surface of one of the spheres ($l_s > a_s$). However, the scheme for solving the problem remains as before also in the case when the dipole lies on the surface of one them ($l_{-1} = a_{-1}$ or $l_{+1} = a_{+1}$). Only for justifying the solvability of the infinite system obtained for the coefficients A_{mn}^{s} and B_{mn}^{s} is it necessary, rather than substituting (7.3.8), to take for a_{mn}^{s} and b_{mn}^{s} from (7.4.7) - (7.4.9) the substitution,

$$\begin{aligned} a_{mn}^{s} &= \alpha_n \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \psi_n(ka_s) A_{mn}^{s}, \\ b_{mn}^{s} &= \alpha_n \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \psi_n(ka_s) B_{mn}^{s}, \end{aligned} \tag{7.4.32}$$

where $\alpha_0 = 1$, and $\alpha_n = n$, when $n \neq 0$. Since in the electrodynamic sense an ideally conducting sphere with a radial electric dipole on its surface is equivalent to a unit annular slot, then it is obvious that when $l_{-1} = a_{-1}$ ($l_{+1} = a_{+1}$) we obtain the solution to the problem of the characteristics of emission of a unit annular slot located on one of the spheres, in the presence of another sphere located at a distance l from the first.

Magnetic Dipole. If at point P there is a magnetic dipole with the same orientation of the dipole moment as for p in the previous problem, then seeking the diffraction field is reduced to solving two particular problems in one of which $\mathrm{m} \in$ pl. P, and in the other $\mathrm{m} \perp$ pl. P. If we seek the potentials u^1 and v^1 of the secondary field in the form of series (7.4.7) - (7.4.9) in which $\sqrt{\frac{\mu}{\varepsilon}}|\mathrm{m}|$ is substituted for ip and $-i|m|$ for $\sqrt{\frac{\varepsilon}{\mu}}p$, respectively, then on the basis of (7.4.4), (7.4.6) and formulas (7.3.8) for the diffraction coefficients A^s_{mn} and B^s_{mn} we obtain systems of equations (7.3.9), (7.3.10) with matrix elements (7.3.11), (7.3.12) whose right-hand sides in the first of the particular problems will be determined by the expressions,

$$f^s_{mn} = \frac{k \sin \gamma_s}{l_s} \cdot \frac{2n+1}{n(n+1)} L^{*n}_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\zeta_n(kl_s)}{\zeta'_n(ka_s)} \qquad (7.4.33)$$

and respectively,

$$\varphi^s_{mn} = \varphi^{s1}_{mn} + \varphi^{s2}_{mn}, \qquad (7.4.34)$$

where

$$\varphi^{s1}_{mn} = \frac{\cos \gamma_s}{l_s^2}(2n+1) L_{mn}(\psi_3, \theta_3, \varphi_3) \frac{\zeta_n(kl_s)}{\zeta_n(ka_s)},$$
$$\varphi^{s2}_{mn} = \frac{k \sin \gamma_s}{l_s} \cdot \frac{(2n+1)}{n(n+1)} L^n_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\zeta'_n(kl_s)}{\zeta_n(ka_s)}, \qquad (7.4.35)$$

and in the second of the particular problems, by the expressions, /374

$$f^s_{mn} = \frac{k}{l_s} \cdot \frac{2n+1}{n(n+1)} L^{*n}_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\zeta_n(kl_s)}{\zeta'_n(ka_s)} \qquad (7.4.36)$$

and respectively,

$$\varphi^s_{mn} = \frac{k}{l_s} \cdot \frac{2n+1}{n(n+1)} L^n_{m1}(\psi_3, \theta_3, \varphi_3) \frac{\zeta'_n(kl_s)}{\zeta_n(ka_s)}. \qquad (7.4.37)$$

Here in (7.4.33) - (7.4.35) the Euler angles are equal to: $\psi_3 = \pi/2$, $\theta_3 = \alpha_s$, $\phi_3 = 3\pi/2$, and in formula (7.4.36), (7.4.37), $\psi_3 = 0$, $\theta_3 = \alpha_s$, $\phi_3 = 3\pi/2$, $s = \pm 1$.

The properties of the systems remain as before. Their unique solutions may be found by the method of truncation.

The components of the electromagnetic field are found through the Debye potentials by the usual method.

If the dipole is located on the surface of one of the spheres, then for justification of the solvability of the infinite systems it will follow to make the substitution of unknowns according to the formulas in (7.4.32) rather than (7.3.8).

If the magnetic dipole with respect to one of the spheres is horizontal and $l_{-1} = a_{-1}$ (or $l_{+1} = a_{+1}$), then the problem in the electrodynamic sense is equivalent to the problem on the field of the emitting unit "dumb-bell" slot, cut on the surface of one of the spheres, in the presence of another sphere located at a distance l from the first.

Sphere of Arbitrary Conductivity With a Non-Concentric Spherical Inclusion in the Field of the Dipole Emitter. Now let us look at the problem of the field of an electric or magnetic dipole emitter in the presence of a sphere with non-concentric spherical inclusion of a different material, on the assumption that the radius of the large sphere is equal to $r_1 = a$ and the smaller to $r_2 = b$ and that their electromagnetic properties are characterized by the constants ε_2, μ_2, σ_2 and ε_3, μ_3, σ_3, respectively. The dipole is located at point P at a distance l from the center of the sphere $r_1 = a$. Its moment forms an angle β with the plane P, passing through the line of the centers O_1O_2 of the spheres and through point P. In the general case l forms an angle α with the line of the centers of the spheres O_1O_2 (Fig. 43). The problem consists of seeking the secondary electromagnetic fields which occur as a result of excitation of the spheres by the dipole field in the region outside the larger sphere (medium I), in the region between the surfaces of the spheres (medium II) and inside the sphere $r_2 = b$ (medium III). In the spherical system of coordinates the components of the field in each of the media, just as in the previous cases, may be found through the Debye potentials according to the formulas (1.6.15). Therefore, in the mathematical formulation, the problem is reduced to solving the boundary problems for the potential functions u^i, v^i ($i = 1,2,3$), which satisfy, in each region, the Helmholtz equation, /375

$$\begin{gathered}\Delta u^i + k_i^2 u^i = 0, \quad i = 1, 2, 3,\\ k_1^2 = \omega^2 \varepsilon\mu/c^2 \text{ in medium } I,\\ k_2^2 = (\omega^2 \varepsilon_2\mu_2 + i4\pi\sigma_2\mu_2\omega)/c^2 \text{ in medium } II,\\ k_3^2 = (\omega^2 \varepsilon_3\mu_3 + i4\pi\sigma_3\mu_3\omega)/c^2 \text{ in medium } III,\end{gathered} \tag{7.4.38}$$

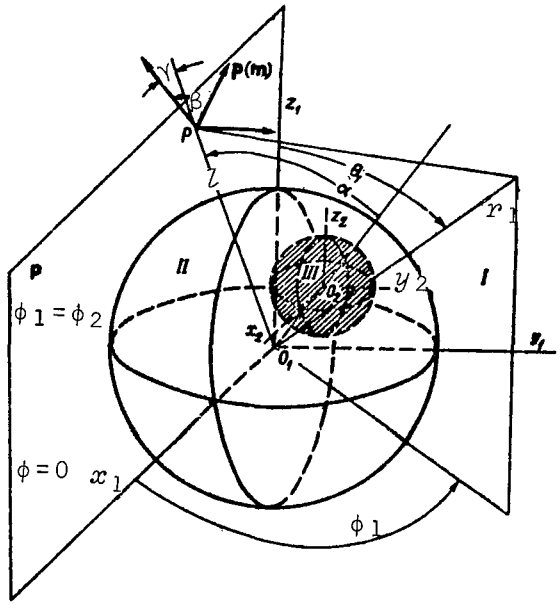
and on their interface, the conjugate conditions (1.7.10), (1.7.11), which are written in the form,

$$\left.\begin{aligned} &\frac{\partial}{\partial r_1}(r_1 u) = \frac{\partial}{\partial r_1}(r_1 u^2), \\ &\frac{k^2}{\mu} u = \frac{k_2^2}{\mu_2} u^2, \end{aligned}\right| \quad \left|\begin{aligned} &\frac{\partial (r_1 v)}{\partial r_1} = \frac{\partial (r_1 v^2)}{\partial r_1}, \\ &\mu v = \mu_2 v^2, \end{aligned}\right| \; r_1 = a,$$

$$\left.\begin{aligned} &\frac{\partial}{\partial r_2}(r_2 u^2) = \frac{\partial}{\partial r_2}(r_2 u^3), \\ &\frac{k_2^2}{\mu_2} u^2 = \frac{k_3^2}{\mu_3} u^3, \end{aligned}\right| \quad \left|\begin{aligned} &\frac{\partial}{\partial r_2}(r_2 v^2) = \frac{\partial}{\partial r_2}(r_2 v^3), \\ &\mu_2 v^2 = \mu_3 v^3, \end{aligned}\right| \; r_2 = b, \qquad (7.4.39)$$

where $u = u^0 + u^1$, $v = v^0 + v^1$. As usual, it is required of u^1 and v^1 that they satisfy the condition of radiation at infinity. To solve the problem let us introduce into study two systems of local Cartesian and spherical coordinates r_1, θ_1, ϕ_1; r_2, θ_2, ϕ_2 with origins at points O_1 and O_2 with a common polar axis Oz, passing through the centers of the spheres and with coincident planes $O_1x_1z_1$ and $O_2x_2z_2$ (then $\phi_1 = \phi_2 = \phi$). From the discussions cited at the beginning of this section, it follows that if the projection of the dipole moment on the plane P forms an angle γ with the line O_1P, then the potentials of the primary dipole field will be given by formulas (7.4.3) or (7.4.4) as a function of the type of dipole in the case when the dipole moment belongs to the plane P; and by formulas (7.4.5) or (7.4.6) in the case when it is perpendicular to it. Here just as before, in (7.4.3), (7.4.4) as the Euler angles we must take the values $\psi_3 = \pi/2$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$, and in formulas (7.4.5), (7.4.6) the values $\psi_3 = 0$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$. Furthermore in these formulas we must substitute γ for γ_s and l and θ_1 for l_s and θ_s, respectively. /37

Since the potentials of the primary field of the electric dipole are coupled with the potentials of the primary field of the magnetic dipole by the equations,



$$u_m^0 = -\frac{|\mathbf{m}|\mu}{|\mathbf{p}|\varepsilon} v_e^0, \quad v_m^0 = \frac{|\mathbf{m}|}{|\mathbf{p}|} u_e^0, \qquad (7.4.40)$$

then in the future it is sufficient to find the solution to the problem, for example, only for the case of the electric dipole. The solution to the respective problem for the magnetic dipole is obtained then from the solution for the electric dipole on the basis of (7.4.40)

Fig. 43. A Sphere With a Non-Concentric Spherical Inclusion.

(in (7.4.40) the index e indicates the potentials of the electric dipole field and the index m, the potentials of the magnetic dipole field). Let the moment p of the electric dipole lie in the plane P and for the potentials of the primary field let formulas (7.4.3) be valid (taking into account the comments made above relative to the definitions). Then if we seek the potentials of the secondary fields in region I in the form,

$$u^1 = \frac{1}{kr_1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^1_{mn}\zeta_n(kr_1)P_n^m(\cos\theta_1)e^{im\varphi} \quad v^1 = \frac{1}{kr_1}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=1}^{\infty}\sum_{m=-n}^{n} b^1_{mn}\zeta_n(kr_1)P_n^m(\cos\theta_1)e^{im\varphi}, \quad (7.4.41)$$

in region 2, /377

$$u^2 = \frac{1}{k_2r_1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^2_{mn}\psi_n(k_2r_1)P_n^m(\cos\theta_1)e^{im\varphi} + \frac{1}{k_2r_2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} c^2_{mn}\zeta_n(k_2r_2)P_n^m(\cos\theta_2)e^{im\varphi},$$

$$v^2 = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k_2r_1}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b^2_{mn}\psi_n(k_2r_1)P_n^m(\cos\theta_1)e^{im\varphi} + \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k_2r_2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} d^2_{mn}\zeta_n(k_2r_2)P_n^m(\cos\theta_2)e^{im\varphi}, \quad (7.4.42)$$

and in region 3,

$$u^3 = \frac{1}{k_3r_2}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a^3_{mn}\psi_n(k_3r_2)e^{im\varphi},$$

$$v^3 = \sqrt{\frac{\varepsilon}{\mu}}\frac{1}{k_3r_2}\times \quad (7.4.43)$$

$$\times\sum_{n=1}^{\infty}\sum_{m=-n}^{n} b^3_{mn}\psi_n(k_3r_2)P_n^m(\cos\theta_2)e^{im\varphi},$$

then after substitution of the unknown expansion coefficients a^1_{mn}, a^2_{mn}, a^3_{mn}, b^1_{mn}, b^2_{mn}, b^3_{mn}, c_{mn}, d_{mn} with new ones according to the formulas,

$$a^1_{mn} = \psi_n(ka)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}A^1_{mn},$$

$$c_{mn} = \psi_n(k_2b)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}C_{mn}, \quad (7.4.44)$$

$$b^1_{mn} = \psi_n(ka)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}B^1_{mn},$$

$$d_{mn} = \psi_n(k_2a)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2}D_{mn},$$

$$a_{nm}^2 = \zeta_n(k_2a)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} A_{mn}^2,$$
$$a_{nm}^3 = \zeta_n(k_3b)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} A_{mn}^3,$$
$$b_{mn}^2 = \zeta_n(k_2a)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} B_{mn}^2, \qquad (7.4.44)$$
$$b_{mn}^3 = \zeta_n(k_3b)\left\{\frac{(n-m)!}{(n+m)!}\right\}^{1/2} B_{mn}^3$$

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from boundary conditions (7.4.39) using the addition theorem (3.7.10) and (3.7.16) we find the systems of equations for the unknowns A_{mn}^1, A_{mn}^2, A_{mn}^3, B_{mn}^1, B_{mn}^2, B_{mn}^3, C_{mn}, D_{mn}:

$$\begin{cases} A_{mn}^2 + \sum\limits_{q=|m|}^{\infty} \alpha_{mnq}^{(21)} C_{mq} = F_{mn}, \\ C_{mn} + \sum\limits_{q=|m|}^{\infty} \alpha_{mnq}^{(12)} A_{mq}^2 = 0, \end{cases} \qquad (7.4.45)$$

$$\frac{\Delta_n^1 \zeta_n(k_2a)}{\Delta_n^2 \psi_n(ka)} A_{mn}^2 + \frac{\Delta_{mn}^3}{\Delta_n^2 \psi_n(ka)} = A_{mn}^1, \qquad (7.4.46)$$

$$\frac{\Delta_n^6 \psi_n(k_2b)}{\Delta_n^7 \zeta_n(k_3b)} C_{mn} = A_{mn}^3,$$

$$\begin{cases} B_{mn}^2 + \sum\limits_{q=|m|}^{\infty} \beta_{mnq}^{(21)} D_{mq} = \Phi_{mn}, \\ D_{mn} + \sum\limits_{q=|m|}^{\infty} \beta_{mnq}^{(12)} B_{mq}^2 = 0, \end{cases} \qquad (7.4.47)$$

$$\frac{\Delta_n^{*1} \zeta_n(k_2a)}{\Delta_n^{*2} \psi_n(ka)} B_{mn}^2 + \frac{\Delta_{mn}^{*3}}{\Delta_n^{*2} \psi_n(ka)} = B_{mn}^1, \qquad (7.4.48)$$

$$\frac{\Delta_n^{*6} \psi_n(k_2b)}{\Delta_n^7 \zeta_n(k_3b)} D_{mn} = B_{mn}^3,$$

in which we denote

$$\alpha_{mnq}^{(21)} = \frac{\Delta_n^2 \psi_q(k_2b)}{\Delta_n^4 \zeta_n(k_2a)} \left[\frac{(q-m)!}{(q+m)!} \cdot \frac{(n+m)!}{(n-m)!}\right]^{1/2} Q_{mnmq}^{(21)},$$

$$\alpha_{mnq}^{(12)} = \frac{\Delta_n^7 \zeta_q(k_2a)}{\Delta_n^8 \psi_n(k_2b)} \left[\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right]^{1/2} Q_{mnmq}^{(12)},$$

$$\beta_{mnq}^{(21)} = \frac{\Delta_n^{*2}\psi_q(k_2b)}{\Delta_n^4\zeta_n(k_2a)}\left[\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right]^{1/2} Q_{mnmq}^{(21)},$$

$$\beta_{mnq}^{(12)} = \frac{\Delta_n^{*7}\zeta_q(k_2a)}{\Delta_n^{*8}\psi_n(k_2b)}\left[\frac{(q-m)!\,(n+m)!}{(q+m)!\,(n-m)!}\right]^{1/2} Q_{mnmq}^{(12)},$$

$$F_{mn} = \frac{\Delta_{mn}^5}{\Delta_n^4\zeta_n(k_2a)}, \quad \Phi_{mn} = \frac{\Delta_{mn}^{*5}}{\Delta_n^{*4}\zeta_n(k_2a)},$$

$$\Delta_n^1 = -i\delta \quad (\delta = k_2\mu/k\mu_2), \quad \Delta_n^2 = \begin{vmatrix} \delta\zeta_n(k_2a) & \zeta_n(ka) \\ \zeta_n'(k_2a) & \zeta_n'(ka) \end{vmatrix},$$

$$\Delta_{mn}^3 = f_{mn}\begin{vmatrix} \psi_n(ka) & \zeta_n(k_2a) \\ \psi_n'(ka) & \zeta_n'(k_2a) \end{vmatrix}, \qquad (7.4.49)$$

$$\Delta_n^4 = \begin{vmatrix} \delta\psi_n(k_2a) & \zeta_n(ka) \\ \psi_n'(k_2a) & \zeta_n'(ka) \end{vmatrix},$$

$$\Delta_{mn}^5 = if_{mn}, \quad \Delta_n^6 = i\delta_2 \quad (\delta_2 = \mu_3 k_2/\mu_2 k_3),$$

$$\Delta_n^7 = \begin{vmatrix} \delta_2\psi_n(k_2b) & \psi_n(k_3b) \\ \psi_n'(k_2b) & \psi_n'(k_3b) \end{vmatrix},$$

$$\Delta_n^8 = \begin{vmatrix} \delta_2\zeta_n(k_2b) & \zeta_n'(k_2b) \\ \psi_n(k_3b) & \psi_n'(k_3b) \end{vmatrix}$$

and where

$$Q_{mnmq}^{(ij)} = \frac{2i^{n-q}}{N_{mn}}\sum_{\sigma=|n-q|}^{n+q} i^\sigma b_\sigma^{(nmqm)} j_\sigma(kl_0) P_\sigma(\cos\theta_{ij}), \qquad (7.4.50)$$

$$i, j = 1, 2, \; j \neq i.$$

In (7.4.48) expressions Δ_n^* differ from Δ_n only in the fact that in them δ and δ_2 are substituted by $k\mu_2/k_2\mu$ and $k_3\mu_2/k_2\mu_3$, respectively, and f_{mn} by ϕ_{mn}, where

$$f_{mn} = f_{mn}^1 + f_{mn}^2,$$

$$f_{mn}^1 = -\frac{ip}{l^2}\cos\gamma(2n+1)\times$$

$$\times L_{mn}(\psi_3, \theta_3, \varphi_3)\zeta_n(kl)\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2}, \qquad (7.4.51)$$

$$f_{mn}^2 = -\frac{ikp}{l}\sin\gamma\,\frac{2n+1}{n(n+1)}\times$$

$$\times L_{m1}^n(\psi_3, \theta_3, \varphi_3)\zeta_n'(kl)\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2},$$

$$\varphi_{mn} = -\frac{kp}{l}\sin\gamma\sqrt{\frac{\varepsilon}{\mu}}\frac{2n+1}{n(n+1)}\times$$
$$\times L^{*n}_{ml}(\psi_3,\ \theta_3,\ \varphi_3)\,\zeta_n(kl)\left[\frac{(n+m)!}{(n-m)!}\right]^{1/2}.$$

The matrix elements of systems (7.4.45), (7.4.47) possess the property of parity relative to the index m. For them

$$\alpha^{(ij)}_{mnq} = \alpha^{(ij)}_{-mnq},\ \beta^{(ij)}_{mnq} = \beta^{(ij)}_{-mnq},\ i,\ j = 1,\ 2,\ i \neq j. \tag{7.4.52}$$

Furthermore they satisfy condition (1.9.19) and their right-hand sides satisfy the condition $\sum_n |F_{mn}|^2 < \infty$ and $\sum_n |\Phi_{mn}|^2 < \infty$, by virtue of which the Gilbert alternative on solvability is applicable to them. The assumption that homogeneous systems corresponding to the heterogeneous systems (7.4.45), (7.4.47) may have non-trivial solutions would mean, as is obvious from (7.4.46) and (7.4.48) an ambigous determination of the expansion coefficients of the Debye potentials (7.4.41) of the secondary field outside the sphere $r_1 = b$ and in the same manner ambiguous determination of the potentials themselves and through them the secondary electromagnetic fields as well. However, this latter will contradict the theorem of uniqueness of the solution to the system of Maxwell equations in the presence of conditions of radiation at infinity and the conjugate conditions of tangential components at the interface of the media. Therefore from the Gilbert alternative there follows the uniqueness of solutions tc systems (7.4.45) and (7.4.47) for each m, satisfying the conditions $\sum_n^\infty |A^i_{mn}|^2 < \infty$, $\sum_n^\infty |B^i_{mn}|^2 < \infty$, $\sum_n^\infty |C_{mn}|^2 < \infty$, $\sum_n^\infty |D_{mn}|^2 < \infty$ $i = 1, 2, 3$. These solutions may be found by the method of truncation.

By virtue of the relationships,

$$L_{-m,n} = (-1)^m \frac{(n+m)!}{(n-m)!} L_{mn},$$
$$L^n_{-m,1} = -\frac{(n+m)!}{(n-m)!} L^n_{m1}, \tag{7.4.53}$$
$$L^{*n}_{-m,1} = \frac{(n+m)!}{(n-m)!} L^{*n}_{m1},$$

which are satisfied for the Euler angles $\psi_3 = \pi/2$, $\phi_3 = 3\pi/2$, it follows that the right-hand sides of the systems will satisfy the equations /381

$$F_{-m,n} = -F_{mn}$$

(m is odd; with an even m in the expression $F_{-m,n}$

$$f_{-m,n} = -f^1_{mn} + f^2_{mn}),$$
$$\Phi_{-m,n} = \Phi_{mn} \qquad (7.4.54)$$
(for all m)

Therefore on the basis of (7.4.52) we always have

$$B^i_{-m,n} = B^i_{mn},\ D_{-m,n} = D_{mn} \quad (i = 1, 2, 3),$$
$$A^i_{-m,n} = -A^i_{mn},\ C_{-m,n} = -C_{mn} \qquad (7.4.55)$$
($i = 1, 2, 3$, m is odd).

From (7.4.51) it is obvious that if $\gamma = \pi/2$ (the dipole is horizontal with respect to the sphere with radius $r = a$), then $f^1_{mn} = 0$, and for all m

$$A^i_{-m,n} = A^i_{mn},\ C_{-m,n} = C_{mn} \quad (i = 1, 2, 3). \qquad (7.4.56)$$

If $\gamma = 0$ (the dipole is vertical with respect to the sphere with radius $r = a$) then $f^2_{mn} = 0$ and for all m

$$A_{-m,n} = (-1)^m A_{mn},\ C_{-m,n} = (-1)^m C_{mn}$$
$$(i = 1, 2, 3). \qquad (7.4.57)$$

Let us look at several particular cases which follow from the general formulas.

1. The vertical electric dipole, $\gamma = 0$. Then $\Phi_{mn} = 0$ and

$$f_{mn} = f^1_{mn} = -\frac{ip(2n+1)}{l^2} \times$$
$$\times L_{mn}(\psi_3, \theta_3, \varphi_3)\,\zeta_n(kl)\left\{\frac{(n+m)!}{(n-m)!}\right\}^{1/2}$$

Therefore all $B^i_{mn} = 0$ ($i = 1, 2, 3$) and $D_{mn} = 0$ such that also $v^i = 0$ ($i = 1, 2, 3$). The coefficients A^i_{mn}, C_{mn} are found from (7.4.45) and (7.4.46) in which

$$f^1_{-m\,n} = (-1)^m f^1_{mn}$$

and (7.4.57) is valid. In this case from (7.4.41) we find for the potential u^1 of the secondary field, scattered on the sphere with radius $r = a$, the expression /382

$$u^1 = \frac{1}{kr_1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \times$$

$$A^{(1)}_{mn} \psi_n(ka)\, \zeta_n(kr_1)\, P_n^m(\cos\theta_1) \cos m\varphi.$$

Hence in the vicinity of the wave band

$$u^1 = \frac{e^{ikr_1}}{ikr_1} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \times$$

$$\times (-i)^n A^1_{mn} \psi_n(ka)\, P_n^m(\cos\theta_1) \cos m\varphi$$

and in this vicinity the components of the secondary electromagnetic field will be equal to

$$E^1_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H^1_\varphi = k^2 p\, \frac{e^{ikR}}{R}\, W_1(\theta_1, \varphi),$$

$$E^1_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\, H^1_\theta = -k^2 p\, \frac{e^{ikR}}{R}\, W_2(\theta_1, \varphi),$$

where

$$W_1 = \frac{e^{ikl\cos(\alpha-\theta_1)}}{k^2 p} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \times$$

$$\times (-i)^n \psi_n(ka) \frac{d}{d\theta_1} [P_n^m(\cos\theta_1)] \cos m\varphi,$$

$$W_2 = \frac{e^{ikl\cos(\alpha-\theta_1)}}{k^2 p} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \varepsilon_m \left\{ \frac{(n-m)!}{(n+m)!} \right\}^{1/2} \times$$

$$\times (-i)^n \psi_n(ka) \frac{P_n^m(\cos\theta_1)}{\sin\theta_1} \sin m\varphi.$$

If $\alpha = 0$ here (the dipole is located on the polar axis), then $A^i_{mn} = C_{mn} = 0$, if $m \neq 0$, but A^i_{0n}, C_{0n} are found from (7.4.45), (7.4.46) with $m = 0$. Here

$$f_{0n} = f^1_{0n} = -\frac{ip}{l^2}\,(2n+1)\,\zeta_n(kl)$$

and for the stray electromagnetic field in the wave band /383

$$E^1_\theta = \sqrt{\frac{\mu}{\varepsilon}}\,H^1_\varphi = k^2 p\,\frac{e^{ikR}}{R}\,W(\theta_1),$$

where

$$W(\theta_1) = -\frac{e^{ikl\cos\theta_1}}{k^2 p}\sum_{n=0}^{\infty} A^1_{0n}(-i)^n\,\psi_n(ka)\,P^1_n(\cos\theta_1).$$

This case ($\gamma = 0, \alpha = 0$) was studied previously in [83].

The components of the primary dipole field in the wave band, when $\gamma = 0$, is determined in the coordinates of the system r_1, θ_1, ϕ_1 by the expressions

$$E^0_\theta = \sqrt{\frac{\mu}{\varepsilon}}\,H^0_\varphi = k^2 p\,\frac{e^{ikR}}{R}\,W_{01}(\theta_1, \varphi),$$

$$E^0_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\,H^0_\theta = -k^2 p\,\frac{e^{ikR}}{R}\,W_{02}(\theta_1, \varphi),$$

where

$$W_{01} = \frac{ie^{ikl\cos(\alpha-\theta_1)}}{k^2 l^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varepsilon_m(2n+1)(-i)^n \times$$

$$\times\,\psi_n(kl)\,L_{mn}(\psi_3, \theta_3, \varphi_3)\,\frac{d}{d\theta_1}\,[P^m_n(\cos\theta_1)]\cos m\varphi,$$

$$W_{02} = \frac{e^{ikl\cos(\alpha-\theta_1)}}{k^2 l^2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\varepsilon_m(2n+1)\,m\,(-i)^n \times$$

$$\times\,\psi_n(kl)\,L_{mn}(\psi_3, \theta_3, \varphi_3)\,\frac{P^m_n(\cos\theta_1)}{\sin\theta_1}\cos m\varphi,$$

and in the case when $\alpha = 0$, by the expressions

$$E^0_\theta = \sqrt{\frac{\mu}{\varepsilon}}\,H^0_\varphi = k^2 p\,\frac{e^{ikR}}{R}\,W_0(\theta_1),$$

where $W_0(\theta_1) = \sin\theta_1$.

Figure 44 gives the directional diagrams constructed for the secondary electromagnetic field H_φ^1 in the wave band and in the case when $\gamma = \alpha = 0$, medium I is a vacuum, medium II is a dielectric ($\sigma_2 = 0$), medium III is an ideal conductor ($\sigma_3 = \infty$), and $k_2 b = 1$, $k_2 a = 4$, $k_2 l_0 = 2$, $k_1 l = {}_1 4$, $k_1 a = 2$. The graphs of the functions $|W(\theta_1)|$, through which H_φ is determined in the wave band, are constructed for the various increasing orders of truncation of the system (7.4.45) ($m = 0$, $N = 2$, 3, 4 and 5; correspondingly in (7.4.45) and in the expression of the function $W(\theta_1)$ we assumed n, $q = 1, \ldots, N$). For these parameters a fully satisfactory coincidence of the results is observed when $N \blacksquare 4$. /384

2. The horizontal electric dipole, $\gamma = \pi/2$. In this case $f_{mn} = f^2_{mn}$, and $f^2_{-mn} = -f^2_{mn}$, so that also $F_{-mn} = -F_{mn}$, and in systems (7.4.45) - (7.4.48)

$$A^i_{-m,n} = -A^i_{mn},\ B^i_{-m,n} = B^i_{mn},\ C_{-m,n} = -C_{mn},$$
$$D_{-m,n} = D_{mn}\ (i = 1, 2, 3).$$

In particular if $\alpha = 0$. then A^i_{mn}, B^i_{mn}, C_{mn}, $D_{mn} = 0$ ($i = 1, 2, 3$), if $m \neq \pm 1$, but $A^i_{\pm 1 n}$, $B^i_{\pm 1 n}$ and $C_{\pm 1 n}$, $D_{\pm 1 n}$ ($i = 1, 2, 3$), are found from equations (7.4.45) - (7.4.48) when $m = \pm 1$. Here for the potentials of the stray field we obtain the expressions,

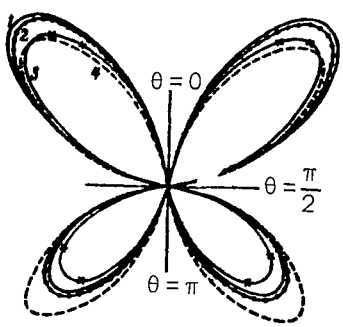

Fig. 44. The Function $|W(\theta_1)|$ for Various Values of N. 1. $N = 5$; 2. 4; 3. 3; 4. 2.

$$u^1 = \frac{2}{kr_1} \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^{1/2}} \times$$
$$\times A^1_{1n} \psi_n(ka)\, \zeta_n(kr_1)\, P^1_n(\cos\theta_1) \cos\varphi,$$
$$v^1 = \frac{2i}{kr_1} \sum_{n=1}^{\infty} \frac{1}{[n(n+1)]^{1/2}} \times$$
$$\times B^1_{1n} \psi_n(ka)\, \zeta_n(kr_1)\, P^1_n(\cos\theta_1) \sin\varphi,$$

on the basis of which in the wave band the components of the stray electromagnetic field are equal to

$$E^1_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H^1_\varphi = k^2 p\, \frac{e^{ikR}}{R} \cos\varphi\, W_1(\theta_1),$$

$$E^1_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\, H^1_\theta = -k^2 p\, \frac{e^{ikR}}{R} \sin\varphi\, W_2(\theta_1),$$

where

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$$W_1(\theta) = \frac{-2e^{ikl\cos\theta_1}}{k^2 p} \times$$

$$\times \left\{ \sum_{n=1}^{\infty} (-i)^n A_{1n}^1 \frac{\psi_n(ka)}{[n(n+1)]^{1/2}} \tau_n(\cos\theta_1) + \right.$$

$$\left. + \sqrt{\frac{\mu}{\varepsilon}}\, i \sum_{n=1}^{\infty} (-i)^n \frac{\psi_n(ka)}{[n(n+1)]^{1/2}} B_{1n}^1 \pi_n(\cos\theta_1) \right\},$$

$$W_2(\theta) = \frac{-2e^{ikl\cos\theta_1}}{k^2 p} \times$$

$$\times \left\{ \sum_{n=1}^{\infty} \frac{(-i)^n}{[n(n+1)]^{1/2}} \psi_n(ka) A_{1n}^1 \pi_n(\cos\theta_1) + i \times \right.$$

$$\left. \times \sqrt{\frac{\mu}{\varepsilon}} \sum_{n=1}^{\infty} \frac{(-i)^n}{[n(n+1)]^{1/2}} \psi_n(ka) B_{1n}^1 \tau_n(\cos\theta_1) \right\}.$$

For the primary dipole field in this case when $\alpha \neq 0$ we will have

$$E_\theta^0 = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^0 = k^2 p \frac{e^{ikR}}{R} W_1(\theta_1, \varphi),$$

$$E_\varphi^0 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\theta^0 = -k^2 p \frac{e^{ikR}}{R} W_2(\theta_1, \varphi),$$

where

$$W_1 = \frac{ie^{ikl\cos(\alpha-\theta_1)}}{kl} \left[\sum_{n=1}^{\infty} \sum_{m=-n}^{n} (-i)^n \frac{2n+1}{n(n+1)} \times \right.$$

$$\times \left\{ L_{m1}^n \psi_n(kl) \frac{dP_n^m(\cos\theta_1)}{d\theta_1} + \right.$$

$$\left. \left. + \frac{im}{\sin\theta_1} L_{m1}^{*n} \psi_n(kl) P_n^m(\cos\theta_1) \right\} \right] e^{im\varphi},$$

$$W_2 = \frac{e^{ikl\cos(\alpha-\theta_1)}}{kl} \left[\sum_{n=1}^{\infty} \sum_{m=-n}^{n} (-i)^n \frac{2n+1}{n(n+1)} \times \right.$$

$$\times \left\{ L_{m1} \psi_n'(kl) \frac{P_n^m(\cos\theta_1)}{\sin\theta_1} m + \right.$$

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$$\left. \left. + iL_{m1}^{*n} \psi_n(kl) \frac{dP_n^m(\cos\theta_1)}{d\theta_1} \right\} \right] e^{im\varphi},$$

and when $\alpha = 0$,

$$W_1 = -\frac{e^{ikl\cos\theta_1}}{kl}\left\{\sum_{n=1}^{\infty}(-i)^n\frac{2n+1}{n(n+1)}\times\right.$$

$$\left.\times[\psi_n'(kl)\tau_n(\cos\theta_1) + i\psi_n(kl)\pi_n(\cos\theta_1)]\right\}\cos\varphi,$$

$$W_2 = -\frac{ie^{ikl\cos\theta_1}}{kl}\left\{\sum_{n=1}^{\infty}(-i)^n\frac{2n+1}{n(n+1)}\times\right.$$

$$\left.\times[\psi_n(kl)\tau_n(\cos\theta_1) + \psi_n(kl)\pi_n(\cos\theta_1)]\right\}\sin\varphi.$$

3. When $l_0 = 0$ (the case of two concentric spheres) from the general formulas cited above we obtain the solution to the problem on the diffraction of the dipole field with arbitrary orientation of the moment p on two concentric spheres which, when $b = 0$, convert to the solution of the problem of diffraction on one absorbing sphere.

4. In the case when the sphere with a radius $r_2 = b$ is an ideally conducting one, we find $A^3_{mn} = B^3_{mn} = 0$ (then the potential $u^3 = v^3 \equiv 0$). In regions *I*, *II* the potentials are determined by expressions (7.4.41), (7.4.42), in which the unknown expansion coefficients are found from (7.4.44) where in turn A^i_{mn}, B^i_{mn}, C_{mn}, D_{mn}, $i = 1, 2$, are determined from (7.4.45) - (7.4.48) without the fourth and later equations under the condition that there we have

$$\Delta^7_n = \Delta^{*7}_n = \psi_n(k_2b), \text{ a } \Delta^8_n = \Delta^{*8}_n = \zeta_n(k_2b).$$

Let us note that relationships (7.4.45) - (7.4.48) and (7.4.49) may be to a significant degree simplified also in a number of other cases, depending on the choice of values of the constants ε_i, μ_i, σ_i in the adjacent regions and values of the parameters k_ia, k_ib, etc.

Now let the moment of the electric dipole be directed perpendicular to the plane P. Then the scheme for solving the problem remains as before: if we seek the potentials of the secondary field in the form of series (7.4.41) - (7.4.43) and make the substitution of the unknown coefficients of the series from the formulas in (7.4.44), then from the conjugate conditions on the surface of the spheres we again obtain the systems of equations (7.4.45) - (7.4.48) with the same matrix elements whose right-hand sides will be given /387 respectively by the expressions

$$f_{mn} = -\frac{ikp}{l} \cdot \frac{2n+1}{n(n+1)} \times$$
$$\times \left\{ \frac{(n+m)!}{(n-m)!} \right\}^{1/2} L^{n}_{m1}(\psi_3, \theta_3, \varphi_3)\, \zeta'_n(kl)$$

and

$$\varphi_{mn} = -\sqrt{\frac{\varepsilon}{\mu}}\, \frac{ikp}{l} \cdot \frac{2n+1}{n(n+1)} \times$$
$$\times \left\{ \frac{(n+m)!}{(n-m)!} \right\}^{1/2} L^{*n}_{m1}(\psi_3, \theta_3, \varphi_3)\, \zeta_n(kl).$$

($f^{1}_{mn} \equiv 0$), in which the Euler angles are equal to $\psi_3 = 0$, $\theta_3 = \alpha$, $\phi_3 = 3\pi/2$.

Similarly we solve the problem also in the case of the magnetic dipole.

Let us note also that the scheme for solving the problem does not change in the case when the dipole is located on the surface of a sphere with the radius $r = a$ or when it is found inside one of the spheres.

§5. Problem of the Field of Two Spherical Emitters.

Let us look at the problem of the electromagnetic field of a system, composed of two coaxial spherical emitters (antennas) with surfaces, separated by a small gap to which is supplied from without (for example, using a coaxial line) an exciting voltage which is symmetrical with respect to the axis, on the assumption that the surfaces of the emitters are ideally conducting ones and that the medium surrounding them is a uniform and isotropic unbounded space with electric and magnetic permeabilities ε, μ and a conductivity $\sigma = 0$ and that the influence of the supply lines on the field of the system can be ignored. As usual, the problem consists of seeking a solution to the system of Maxwell equations (1.1.4) outside the emitters ($\rho^{(ext)} = 0$, $J^{(ext)} = 0$), which on the surface of each emitter satisfies several of the boundary conditions and the conditions of radiation at infinity and also possesses (on the strength of the above assumptions relative to the location of the emitters and their property of excitation) axial symmetry ($\partial/\partial\phi \equiv 0$).

In local spherical coordinates r_s, θ_s, ϕ_s, associated with the /388
s-th emitter (the geometry of the problem is shown in Fig. 45), $s = \pm 1$, the solution to the system of Maxwell equations for the field of the uniform s-th emitter under the condition that $\partial/\partial\phi_s = 0$, may be represented on the basis of (1.6.27) in the form,

$$E^s_{r_s} = \frac{i}{k_0 \varepsilon} \cdot \frac{1}{r_s^2 \sin\theta_s} \cdot \frac{\partial u^s}{\partial \theta_s},$$

$$E^s_{\theta_s} = -\frac{i}{k_0 \varepsilon} \cdot \frac{1}{r_s \sin\theta_s} \cdot \frac{\partial u^s}{\partial r_s}, \qquad (7.5.1)$$

$$H^s_{\varphi_s} = \frac{1}{r_s \sin\theta_s} u^s,,$$

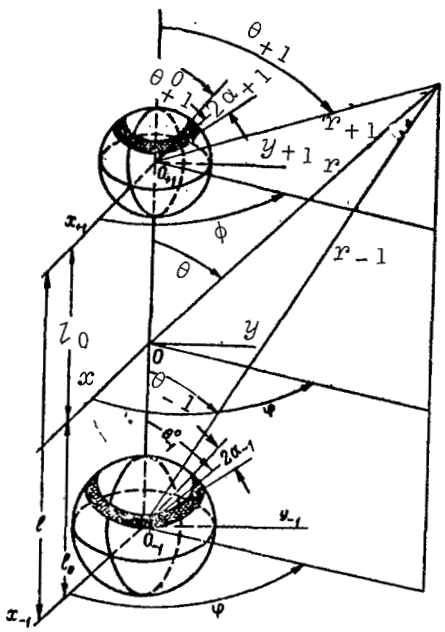

Fig. 45. Geometry of the Problem of the Field of Two Spherical Emitters.

where u^s is the Abraham scalar potential, which satisfies the equation (1.6.28),

$$\frac{\partial^2 u^s}{\partial r_s^2} + \frac{\sin\theta_s}{r_s^2} \cdot \frac{\partial}{\partial \theta_s}\left(\frac{1}{\sin\theta_s} \cdot \frac{\partial u^s}{\partial \theta_s}\right) + k^2 u^s = 0. \qquad (7.5.2)$$

If we substitute into equation (7.5.2) the function $r_s \sin\theta_s H^s_{\phi_s}$ from (7.5.1) rather than u^s then for the component $H^s_{\phi_s}$, we find the equation,

$$\frac{\partial}{\partial r_s^2}(r_s H^s_{\varphi_s}) + \frac{1}{r_s^2} \times$$

$$\times \frac{\partial}{\partial \theta_s}\left[\frac{1}{\sin\theta_s} \cdot \frac{\partial}{\partial \theta_s}(r_s \sin\theta_s H^s_{\varphi_s})\right] + k^2 H^s_{\varphi_s} r_s = 0, \qquad (7.5.3)$$

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which appears as the Helmholtz equation $\Delta H^s_{\phi_s} + k^2 H^s_{\phi_s} = 0$, written in spherical coordinates and possessing an azimuth number $m = 1$. Its particular solutions which satisfy the condition of radiation at infinity and which are bounded at the terminals $\theta_s = 0$, $\theta_s = \pi$, will be the wave functions $h_n^{(1)}(kr_s)P_n^1(\cos\theta_s)$, where $n = 1, 2, \ldots$ Therefore, the component $H^s_{\phi_s}$ of the field of the uniform emitter may be represented in the form of a series,

$$H^s_{\varphi_s} = \frac{\sqrt{\frac{\varepsilon}{\mu}}}{kr_s} \sum_{n=1}^{\infty} a_n^s \zeta_n(kr_s) P_n^1(\cos\theta_s). \qquad (7.5.4)$$

As is obvious from (7.5.1) the components E^s_r, E^s_θ of the electric field are expressed through H^s_ϕ by the formulas,

$$E_r^s = \frac{i}{\varepsilon k_0 r_s \sin\theta_s} \cdot \frac{\partial}{\partial\theta_s}(\sin\theta_s H_\varphi^s),$$
$$E_\theta^s = \frac{-i}{k_0 \varepsilon r_s} \cdot \frac{\partial}{\partial r_s}(r_s H_\varphi^s). \qquad (7.5.5)$$

(these relationships may be obtained directly from the Maxwell system).

We shall seek the resulting electromagnetic field of two emitters in the form of the sum $\mathbf{E} = \sum_{s=\pm 1} \mathbf{E}^s$, $\mathbf{H} = \sum_{s=\pm 1} \mathbf{H}^s$, where $E^s_\phi = H^s_r = H^s_\theta = 0$, and E^s_r, E^s_θ and H^s_ϕ, $s = \pm 1$, are determined by formulas (7.5.4), (7.5.5). The interaction of the emitters is taken into account through the expansion coefficients a_n^s, which are subject to determination. For this we use the boundary conditions

$$E_\theta|_{r_s = a_s} = \begin{cases} (1)\ 0 \text{ when } r_s = a_s \text{ outside the gap;} \\ (2)\ \text{the field applied to the gap of the } s\text{-th emitter when } r_s = a_s \text{ and } \theta_s^0 - \alpha_s \leq \theta_s \leq \theta_s^0 + \alpha_s, \text{ where } \alpha_s \text{ is the angular width of the gap of the } s\text{-th emitter, } s = \pm 1, \end{cases}$$

which must be satisfied simultaneously on the surface of each emitter of the system for the adjacent component of the vector of the electric intensity of the complete field. After repeating the arguments similar to those given, for example in [8, 7], where a discussion /390 was given of the method of solving a similar problem for the uniform spherical emitter obtained by Stratton and Chu, we find that

condition (7.5.6) may be rewritten in the form,

$$E_\theta|_{r_s=a_s} = \frac{V_s}{a_s} \times$$

$$\times \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} P_n^1(\cos\theta_s^0)\, P_n^1(\cos\theta_s),\ s = \pm 1,$$

if we assume that α_s are sufficiently small values, $s = \pm 1$. Here through V_s we denote the symmetrically exciting voltage applied to the gap of the s-th emitter which is assumed to be given. On the basis of (7.5.5) this boundary condition assumes the form,

$$\frac{\partial}{\partial r_s}(r_s H_\varphi)|_{r_s=a_s} = ik_0 \varepsilon V_s \times$$

$$\times \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} P_n^1(\cos\theta_s^0)\, P_n^1(\cos\theta_s),\ s = \pm 1. \qquad (7.5.7)$$

If instead of H_ϕ^s we substitute here the series,

$$H_\varphi^s = \sum_{s=\pm 1} \left\{ \frac{\sqrt{\frac{\varepsilon}{\mu}}}{kr_s} \sum_{n=1}^{\infty} a_n^s \zeta_n(kr_s) P_n^1(\cos\theta_s) \right\}, \qquad (7.5.8)$$

we find, after using the addition theorem (3.7.10) and substituting the new unknowns A_n^s for the unknowns a_n^s according to the formula,

$$a_n^s = \psi_n(ka_s) A_n^s, \quad s = \pm 1, \qquad (7.5.9)$$

and infinite system of linear equations for determining A_n:

$$A_n^s + \sum_{m=1}^{\infty} \alpha_{nm}^{(-s,s)} A_m^{-s} = f_n^s \qquad (7.5.10)$$

with the matrix elements

$$\alpha_{nm}^{(-s,s)} = \frac{\psi_m'(ka_{-s})}{\zeta_n'(ka_s)} Q_{1n1m}^{(1)}(r_{-s,s},\ \theta_{-s,s}) \qquad (7.5.11)$$

and with the right-hand side,

$$f_n^s = \frac{ik^2(2n+1) V_s P_n^1(\cos\theta_s^0)}{2n(n+1)\psi_n'(ka_s)\zeta_n'(ka_s)}, \qquad (7.5.12)$$

where, just as in the previous problems $\theta_{-1,1} = 0$, $\theta_{1,-1} = \pi$, and /391 $r_{-s,s} = l$ is the distance between centers of the spheres (it is assumed that the spheres do not touch one another). System (7.5.10) differs from the infinite system (7.3.9), if this latter system is assumed, with a value of $m = 1$, [taking into account (7.3.8)] to be the expression of the right-hand side. However, it is easy to see that (7.5.12) satisfies condition (1.9.20). Therefore, system (7.5.10) will possess all properties of system (7.3.9). Its solution (which is unique on the strength of the uniqueness of the solution to this problem) may be found by the method of truncation. The determinant of system (7.5.10) may be written in the form of (7.3.14), and the system itself after the introduction of the definitions (7.3.18), (7.3.19) may be written in the normal form of (7.3.17). Solution to the system satisfies the condition $\sum_{n=1}^{\infty} |A_n^s|^2 < \infty$.

The approximate numerical values of the coefficients A_n^s may be found from the finite (truncated) systems which have the form of (7.3.61) or (7.3.63) with matrix elements (7.5.11) and with right-hand sides (7.5.12) according to the same scheme as in all the previous problems.

By discovering the coefficients A_n^s the problem is completely solved. The components of the electromagnetic field emitted by the system of two coaxial spherical emitters, are found according to formulas (7.5.8), (7.5.5) ($H_\varphi = \sum_{s=\pm 1} H_\varphi^s$, $E_\theta = \sum_{s=\pm 1} E_\theta^s$, $E_r = \sum_{s=\pm 1} E_r^s$). In the vicinity of the wave band in spherical coordinates r, θ, ϕ with the origin at point O (see Fig. 45), when approximately $\theta_s = \theta$, $r_s = r - sl_0\cos\theta$ and $r_s = r$ ($r \gg l$), $l_0 = l/2$, we will have

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi = \frac{e^{ikr}}{ikr}\, W(\theta), \qquad (7.5.13)$$

where

$$W(\theta) = \sum_{s=\pm 1} e^{-iskl_0\cos\theta} \sum_{n=1}^{\infty} (-i)^n A_n^s \psi_n'(ka_s) P_n^1(\cos\theta). \qquad (7.5.14)$$

On the basis of (7.5.13), (7.5.14) for the function $\Psi(\theta,\phi)$ which appears in the expression of the coefficient of the directed effect (1.4.18) of the emitting system, we find

$$\Psi(\theta, \varphi) = \frac{1}{k^2} |W(\theta)|^2,$$

and then for the emitting system which we are studying

$$g(\theta, \varphi) = \frac{|W(\theta)|^2}{W},$$

where on the basis of (7.5.14), (7.5.9) /392

$$W = \int_0^{\pi} |W(\theta)|^2 \sin\theta \, d\theta =$$

$$= \sum_{s=\pm 1} \sum_{n=1}^{\infty} \frac{|\tilde{a}_n^s|^2 n(n+1)}{2n+1} + \sum_{n,p=1}^{\infty} \tilde{a}_n^{+1} \tilde{a}_p^{-1*} a_{np} +$$

$$+ \sum_{n,p=1}^{\infty} \tilde{a}_n^{-1} \tilde{a}_p^{+1*} b_{np},$$

and $$a_{np} = \sum_{\sigma=|n-p|}^{n+p} (-i)^{\sigma} b_{\sigma}^{(1n1m)} j_{\sigma}(kl), \quad b_{np} = \sum_{\sigma=|n-p|}^{n+p'} i^{\sigma} b_{\sigma}^{(1n1m)} j_{\sigma}(kl),$$

$$\tilde{a}_n^s = (-i)^n \psi_n'(ka_s) A_n^s.$$

Let us look at several particular cases which follow from the formulas obtained.

1. For example, let $a_s = a$, $s = \pm 1$ and $\theta_s^0 = \pi/2$, $s = \pm 1$. Then the relationship $\alpha_{nm}^{(-1,+1)} = (-1)^{n+m} \times \alpha_{nm}^{(1,-1)} = \alpha_{nm}$ and from (7.5.10) (7.5.12) we can establish that with a cophasal supply of the emitters, when $V = V_{+1} = V_{-1}$,

$$A_{1,3,\ldots}^{+1} = A_{1,3,\ldots}^{-1}, \quad A_{2,4,\ldots}^{+1} = -A_{2,4,\ldots}^{-1}, \tag{7.5.15}$$

and with antiphase supply, when $V = V_{+1} = -V_{-1}$,

$$A_{1,3,\ldots}^{+1} = -A_{1,3,\ldots}^{-1*}, \quad A_{2,4,\ldots}^{+1} = A_{2,4,\ldots}^{-1}. \tag{7.5.16}$$

As a result in the vicinity of the wave band for the function $W(\theta)$ we find the expressions

$$W(\theta) = 2\cos(kl_0 \cos\theta) \times$$
$$\times \sum_{n=1,3,\ldots} A_n^{+1} \psi_n'(ka)(-i)^n P_n^1(\cos\theta) - 2i \sin(kl_0 \cos\theta) \times \tag{7.5.17}$$
$$\times \sum_{n=2,4,\ldots} A_n^{+1} \psi_n'(ka)(-i)^n P_n^1(\cos\theta)$$

and

$$W(\theta) = 2i \sin(kl_0 \cos\theta) \times \times \sum_{n=1,3,\ldots} A_n^{-1} (-i)^n \psi_n'(ka) P_n^1(\cos\theta) + 2\cos(kl_0 \cos\theta) \times \times \sum_{n=2,4,\ldots} A_n^{-1} (-i)^n \psi_n'(ka) P_n^1(\cos\theta), \quad (7.5.18)$$

from which it follows that in the case when H_ϕ attains a dominant value only because of the waves of first order ($n = 1$) the field H_ϕ will have the form, /393

$$H_\varphi = A \frac{e^{ikr}}{r} \cos(kl_0 \cos\theta) \sin\theta \quad (7.5.19)$$

and

$$H_\psi = B \frac{e^{ikr}}{r} \sin(kl_0 \cos\theta) \sin\theta \quad (7.5.20)$$

respectively, where A and B are certain constants. In particular, (7.5.19) and (7.5.20) will be valid for sufficiently small values of the parameter ka ($ka \ll 1$, as the computations show, for example when $ka = 1$ and $kl \geq 3$, approximately $A_n^s = f_n^s$ and in (7.5.8) it is sufficient to confine outselves only to terms with $n = 1$). Expressions (7.5.19), (7.5.20) show that for small ka the field of the system of two spherical emitters is found to be identical to the field of the system of two unit electric dipoles with the respective values of their moments, located at a distance l from one another.

In this case the approximate values of the quantities A_n^s, $s = \pm 1$ may be found by solving two mutually independent systems of N-th order,

$$x_n + \sum_{m=1}^{N} (-1)^m \alpha_{nm} x_m = F_n, \quad (7.5.21)$$

$$y_n - \sum_{m=1}^{N} (-1)^m \alpha_{nm} y_m = \Phi_n \quad (7.5.22)$$

with the unknowns

$$x_n = A_n^{+1} + (-1)^n A_n^{-1}, \; y_n = A_n^{+1} - (-1)^n A_n^{-1}$$

and with right-hand sides

$$F_n = f_n^{+1} + (-1)^n f_n^{-1}, \; \Phi_n = f_n^{+1} - (-1)^n f_n^{-1},$$

which are obtained in the same way as in the previous problems, on the basis of the relationship established above between the matrix elements of the system (7.5.10) when $a_s = a$, $s = \pm 1$.

2. We shall assume now that the distance between emitters is much greater than the wave length ($kl >> 1$) and is high in comparison with the radii of the spheres. Then on the assumption that $kl >> n + m$, where $n, m = 1, 2, \ldots, N$, on the basis of the formula which is asymptotic with respect to thr argument for the spherical Bessel function $h_\sigma^{(1)}(kl)$, applied to (7.5.11) it is found that in this

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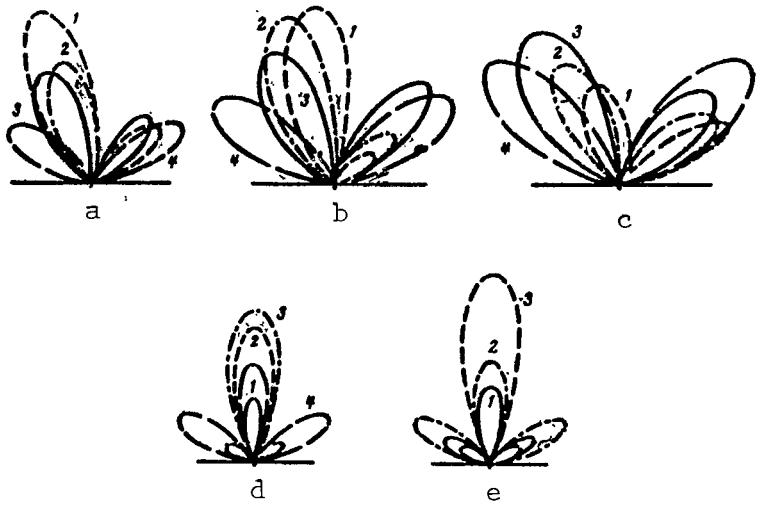

Fig. 46. The Function $|H_\phi|^2$ as a Function of the Angle θ. a - $kl = 5$, $ka = 1$; b - $kl = 5$, $ka = \pi/2$; c - $kl = 5$, $ka = 2$ when $V_{+1} = e^{i\pi/4} V_{-1}$ (1), $V_{+1} = e^{i\pi/2} V_{-1}$ (2), $V_{+1} = e^{i3\pi/4} V_{-1}$ (3), $V_{+1} = e^{i\pi} V_{-1}$ (4); d - $kl = 5$, $V_{+1} = V_{-1}$ when $ka = \pi/4$ (1), 1 (2), $\pi/2$ (3) and 2 (4); e - $ka = 2$, $V_{+1} = V_{-1}$ when $kl = 4.5$ (1), 5 (2) and 6 (3).

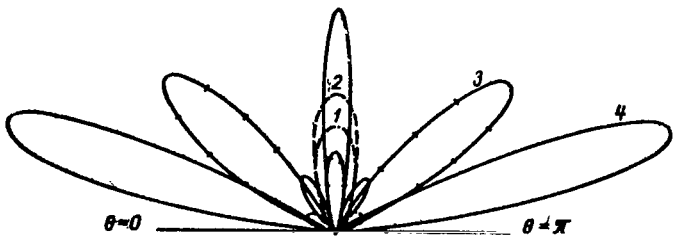

Fig. 47. The Function $H|\phi|^2$ as a Function of the Angle θ.($l = 3a$, $V_{+1} = V_{-1}$) when $ka = 1.5$ (1), 2 (2), π (3) and 4 (4).

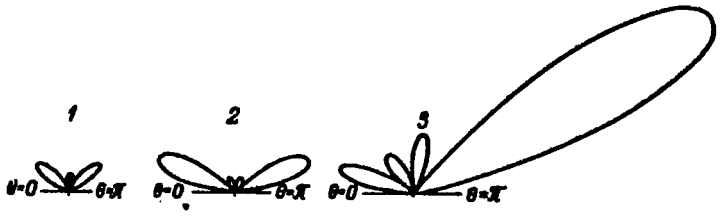

Fig. 48. The Function $|H_\phi|^2$ as a Function of the Angle θ when ka = π, l = 3a and $V_{+1} = V_{-1}$ (1), $V_{+1} = -V_{-1}$ (2), $V_{+1} = -iV_{-1}$ (3).

approximation $\alpha_{nm}^{(-s,s)} = 0$ for all n and m, such that /395

$$A_n^s = f_n^s. \tag{7.5.23}$$

This result means that in the case of widely separated spherical symmetrically excited emitters, when the distance between them is sufficiently great in comparison with the radii of the spheres, the secondary field which arises during the subsequent reciprocal diffractions on the emitters, can be ignored in comparison with the primary fields of the emitters. For such a case the directional properties of the emitters when $a_s = a$, $\theta_s^0 = \pi/2$, $s = \pm 1$, will be described for a cophasal supply of the emitters by the formula,

$$W(\theta) = -ik^2V\cos(kl_0\cos\theta)\times$$
$$\times\sum_{n=1,3,\ldots} i^n\frac{2n+1}{n(n+1)}\cdot\frac{P_n^1(0)}{\zeta_n'(ka)}P_n^1(\cos\theta), \tag{7.5.24}$$

and with antiphase supply, by the formula,

$$W(\theta) = k^2V\sin(kl_0\cos\theta)\times$$
$$\times\sum_{n=1,3,\ldots} i^n\frac{2n+1}{n(n+1)}\cdot\frac{P_n^1(0)}{\zeta_n'(ka)}P_n^1(\cos\theta). \tag{7.5.25}$$

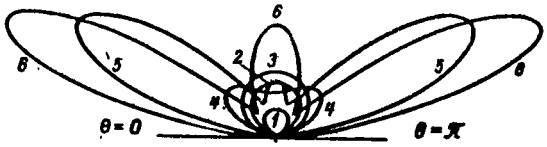

Fig. 49. The Function $|H_\phi|^2$ as a Function of the Angle θ For a Single Spherical Emitter When ka = π/4 (1); 1 (2); 1.5 (3); 2 (4); π (5); and 4 (6).

In the case of arbitrary values of kl ($l > a_{-1} + a_{+1}$) the values of the secondary fields of all subsequent orders of scattering may be found according to the same scheme as, for example, in §3 of this chapter, by applying to the system (7.5.10) the method successive approximations.

In the general case the directional properties of the system of two symmetrical spherical emitters will depend on the values of the parameters kl, ka_s, θ_s^0, the phase difference of the supply currents, the exciting emitters and their amplitudes, etc. In particular, in the case when $kl \gg 1$, $\theta_s^0 = \pi/2$, $a_s = a$, $V_s = e^{is\alpha}V$, $s = \pm 1$, the directional properties of the system in the wave band are determined by the formula, /396

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}} H_\varphi = V \frac{e^{ikr}}{r} W(\theta),$$

where

$$W(\theta) = k_0 a \cos(\alpha - kl_0 \cos\theta) \times$$

$$\times \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} \cdot \frac{P_n^1(0)}{\zeta_n'(ka)} P_n^1(\cos\theta). \qquad (7.5.26)$$

As examples illustrating the method, let us cite the directional diagrams constructed for $|H_\phi|^2$ as a function of the angle θ for various values of the parameters kl, $ka_{+1} = ka_{-1} = ka$, V_s and $\theta_s^0 = \pi/2$, $s = \pm 1$ (Figs. 46-48). All the diagrams have been computed according to formulas (7.5.21), (7.5.22), and (7.5.13), (7.5.14). They show that by varying these parameters with one and the same dimensions of the emitters we can in the necessary manner vary the directionality of the radiation from the system.

Formula (7.5.26) can be written in the form,

$$H_\varphi = f_1(\theta) f_2(\theta), \qquad (7.5.27)$$

where

$$f_1(\theta) = k_0 a V \frac{e^{ikr}}{r} \sum_{n=1}^{\infty} \frac{2n+1}{2n(n+1)} \cdot \frac{P_n^1(0)}{\zeta_n'(ka)} P_n^1(\cos\theta),$$

$$f_2(\theta) = 2\cos(\alpha - kl_0 \cos\theta).$$

In such a form the function H_ϕ is the usual function of the directionality of the system for linear systems of two identical emitters.

Here $f_1(\theta)$ is the function of directionality of the self-contained emitter of the system, and $f_2(\theta)$ is the function of directionality of the two undirected symmetrical vibrators located at a distance l from one another with a phase difference α in the vibrators. The directional properties of the self-contained emitter depends substantially on the value of ka (Fig. 49); with an increase in ka, the diagrams become multi-lobed, and the intensity of emission in the direction $\theta = \pi/2$ is generally not maximal for all ka. If, for example, the maximal intensity of emission of the self-contained emitter is reached when $\theta = \theta^0$, then for the value $\alpha = kl \cos \theta_0$ $H_\phi(\theta_0) = 2f_1(\theta_0)$.

The method given here for solving the problem of a field of two symmetrically excited spherical emitters may be extended also to the linear system of any finite number of spherical emitters with a common axis of rotation [93].

CHAPTER 8

DIFFRACTION ON ELLIPTICAL CYLINDERS

§1. Diffraction of a Plane Electromagnetic Wave on an Elliptical Cylinder

On an infinitely long ideally conducting elliptical cylinder, /397
occupying the region $\xi \leq \xi^0$ an unbounded homogeneous and isotropic space with electric and magnetic permeabilities ε, μ and a conductivity $\sigma = 0$, let there be (in the elliptical system of coordinates ξ, η, z with the axis Oz, directed along the axis of the cylinder) incident a plane electromagnetic wave (1.3.4) propagating in the direction of the unit vector $\mathbf{n}$ which forms a certain angle θ with the positive direction of the Oz axis and let there also be an angle α between its projection on the plane Oxy and the positive direction of the Ox axis which coincides with the large diameter of the ellipse of the cross section of the cylinder with the plane Oxy (fig. 50). In the general case the wave is polarized such that the vector of the electric intensity $\mathbf{E}^0$ forms an angle β with the plane P, which passes through the direction $\mathbf{n}$ and the axis Oz. Let us formulate the problem of seeking the secondary field scattered on the cylinder. Since in the coordinates of the elliptical cylinder the electromagnetic field permits representation in the form of a superposition of fields of electric and magnetic types, then the solution to this problem may be obtained from the solution to two particular problems, corresponding to the cases:

(1) the vector $\mathbf{E}^0 \parallel$ pl. P (TM type field).

(2) the vector $\mathbf{E}^0 \perp$ pl. P (TE type field).

If we introduce into study the scalar potential functions u, v, associated with the components of the electromagnetic field by formulas (1.6.20), (1.6.19), then in mathematical formulation the solution to the particular problems consists of finding the functions u^1, v^1 which satisfy the Helmholtz equation, the boundary conditions (1.7.7), (1.7.8) [in (1.7.7) $u = u^0 + u^1$ and in (1.7.8) $v = v^0 + v^1$, where u^0 and v^0 are potential functions of the primary field of TM type or TE type, respectively] on the surface of the cylinder and the supplementary condition of radiation at infinity. Since in the case of a TM type field the longitudinal component of /398
the vector $\mathbf{E}^0$ is equal to

$$E_z = E \sin\theta\, e^{ihz + ik \sin\theta (\mathbf{nr})}$$

and in the case of a TE type field the longitudinal component of the vector $\mathbf{H}^0$ is equal to

$$H_z = E \sin\theta\, e^{ihz + ik \sin\theta (\mathbf{nr})},$$

where in elliptical coordinates $(nr) = f(\operatorname{ch}\xi\cos\eta\cos\alpha + \operatorname{sh}\xi \times \sin\eta\sin\alpha)$, then in the same way as in §1, Chapter 6, in a similar problem for the circular cylinder, on the basis of (4.4.10) we find that the functions u^0 and v^0 can be represented by the expansions,

$$u^0 = \frac{2E\sin\theta}{\delta^2}\, e^{ihz} \sum_{n=0}^{\infty} [c_n \mathrm{Ce}_n(\xi, q)\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\alpha, q) + d_n \mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\alpha, q)] \tag{8.1.1}$$

and

$$v^0 = \frac{2E\sin\theta}{\delta^2}\, e^{ihz} \sum_{n=0}^{\infty} [c_n \mathrm{Ce}_n(\xi, q)\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\alpha, q) + d_n \mathrm{Se}_n(\xi, q)\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\alpha, q)],$$

where $\delta^2 = k^2 - h^2$, $q = \frac{(fk\sin\theta)^2}{4}$ и $h = k\cos\theta$. (8.1.2)

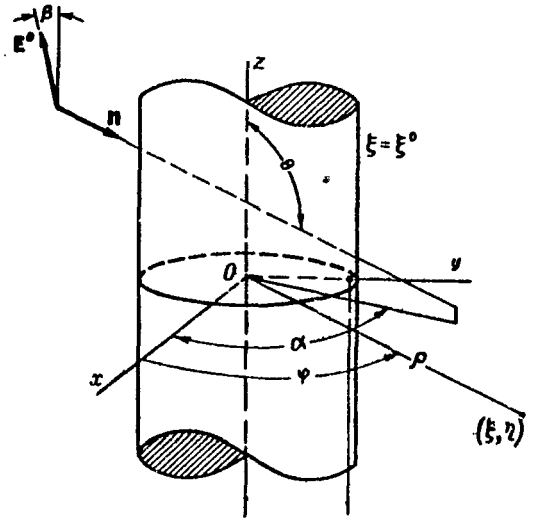

Fig. 50. Diffraction of a Plane Wave on an Elliptical Cylinder.

where $\delta^2 = k^2 - h^2$, $q = \frac{(fk\ \sin\ \theta)^2}{4}$ and $h = k\ \cos\ \theta$.

Case of a TM Type Field. If we seek the potential u^1 of the field scattered on the cylinder in the form of a series of elliptical wave functions (4.1.9), (4.1.10), /399

$$u^1 = \frac{2E\sin\theta}{\delta^2}\, e^{ihz} \sum_{n=0}^{\infty} [a_n c_n \mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{ce}_n(\eta, q) + b_n d_n \mathrm{Ne}_n^{(1)}(\xi, q)\,\mathrm{se}_n(\eta, q)] \tag{8.1.3}$$

with undetermined expansion coefficients a_n and b_n, then from the boundary condition (1.7.7) on the surface of the cylinder ($\xi = \xi^0$) we find that this latter will be satisfied if

$$a_n = -\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{ce}_n(\alpha, q), n = 0, 1, \ldots,$$

$$b_n = -\frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{se}_n(\alpha, q), n = 1, 2, \ldots \qquad (8.1.4)$$

Then

$$u^1 = -\frac{2E\sin\theta}{\delta^2}\,e^{ihz}\sum_{n=0}^{\infty}\left\{c_n\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{ce}_n(\alpha, q)\times\right.$$

$$\left.\times\mathrm{ce}_n(\eta, q) + d_n\frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi, q)\,\mathrm{se}_n(\alpha, q)\,\mathrm{se}_n(\eta, q)\right\}, \qquad (8.1.5)$$

and the potential of the complete field ($u = u^0 + u^1$)

$$u = \frac{2E\sin\theta}{\delta^2}\,e^{ihz}\sum_{n=0}^{\infty}\left\{c_n\left[\mathrm{Ce}_n(\xi, q) - \frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi, q)\right]\times\right.$$

$$\times\,\mathrm{ce}_n(\alpha, q)\,\mathrm{ce}_n(\eta, q) + d_n\left[\mathrm{Se}_n(\xi, q) - \right. \qquad (8.1.6)$$

$$\left.\left.-\frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi, q)\right]\,\mathrm{se}_n(\alpha, q)\,\mathrm{se}_n(\eta, q)\right\}.$$

In the vicinity of the wave band for the stray field

$$E_z^1 = -\sqrt{\frac{\mu}{\varepsilon}}\,H_\eta^1 = \frac{e^{i\delta\rho}}{\sqrt{\rho}}\,A(\varphi)\sin\theta, \qquad (8.1.7)$$

where the amplitude function

$$A(\varphi) = -2E\sqrt{\frac{2}{\delta\pi}}\,e^{ihz-\frac{i\pi}{4}}\sum_{n=0}^{\infty}\left\{\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{ce}_n(\alpha, q)\,\mathrm{ce}_n(\varphi, q) + \right.$$

$$\left. + \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{se}_n(\alpha, q)\,\mathrm{se}_n(\varphi, q)\right\}. \qquad (8.1.8)$$

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With degeneration of the elliptical cylinder into a band with a width of $2f(\xi^0 = 0)$,

$$A(\varphi) = -2E\sqrt{\frac{2}{\delta\pi}}\, e^{ihz-\frac{i\pi}{4}} \sum_{n=0}^{\infty} \frac{\mathrm{Ce}_n(0,\ q)}{\mathrm{Me}_n^{(1)}(0,\ q)} \times \\ \times \mathrm{ce}_n(\alpha,\ q)\,\mathrm{ce}_n(\varphi,\ q). \tag{8.1.9}$$

The law governing the density distribution of surface currents, induced on the elliptical cylinder, is given [as follows from (1.1.7) and (1.6.19), (8.1.6)] by the formulas ($\theta = \pi/2$)

$$j_z = -\sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{cE}{\pi^2 k h_\xi} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha,\ q)\,\mathrm{ce}_n(\eta,\ q)}{c_n \mathrm{Me}_n^{(1)}(\xi^0,\ q)} + \right. \\ \left. + \frac{\mathrm{se}_n(\alpha,\ q)\,\mathrm{se}_n(\eta,\ q)}{d_n \mathrm{Ne}_n^{(1)}(\xi^0,\ q)} \right\} (-1)^n, \tag{8.1.10}$$
$$j_\xi = j_\eta = 0.$$

In the case of band with a width $2f(\xi^0 = 0)$,

$$j_z = -\sqrt{\frac{\varepsilon}{\mu}} \cdot \frac{Ec}{\pi^2 k h_\xi} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha,\ q)\,\mathrm{ce}_n(\eta,\ q)}{c_n \mathrm{Me}_n^{(1)}(0,\ q)} + \right. \\ \left. + \frac{\mathrm{se}_n(\alpha,\ q)\,\mathrm{se}_n(\eta,\ q)}{d_n \mathrm{Ne}_n^{(1)}(0,\ q)} \right\} (-1)^n \tag{8.1.11}$$

(the equating to zero of the component j_η, normal to the edge of the band, means automatic satisfaction of the Meixner condition on the edge of the band when $\xi = \eta = 0$).

With normal incidence of the plane wave on the cylinder (with respect to the generatrices of the cylinder), when $\theta = \pi/2$, the value of the back scattering cross section (radar cross section) for the elliptical cylinder is computed from the formula,

$$\sigma_B = \frac{16}{k} \left| \sum_{n=0}^{\infty} (-1)^n \left\{ \frac{\mathrm{Ce}_n(\xi^0,\ q)}{\mathrm{Me}_n^{(1)}(\xi^0,\ q)}\ \mathrm{ce}_n^2(\alpha,\ q) + \right. \right.$$

$$\left. \left. + \frac{\mathrm{Se}_n(\xi^0,\ q)}{\mathrm{Ne}_n^{(1)}(\xi^0,\ q)}\ \mathrm{se}_n^2(\alpha,\ q) \right\} \right|^2. \tag{8.1.12}$$

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In the case when the wave is propagated along the Ox axis ($\alpha = 0$),

$$\sigma_B = \frac{16}{k} \left| \sum_{n=0}^{\infty} (-1)^n \frac{\mathrm{Ce}_n(\xi^0,\ q)}{\mathrm{Me}_n^{(1)}(\xi^0,\ q)}\ \mathrm{ce}_n^2(0,\ q) \right|^2, \tag{8.1.13}$$

and when $\alpha = \pi/2$,

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0,2,\ldots}\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{ce}_n^2\left(\frac{\pi}{2}, q\right) - \right.$$

$$\left. - \sum_{n=1,3,\ldots}\frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{se}_n^2\left(\frac{\pi}{2}, q\right)\right|^2. \qquad (8.1.14)$$

With degeneration of the elliptical cylinder into a band with a width of $2f(\xi^0 = 0)$,

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0}^{\infty}(-1)^n\frac{\mathrm{Ce}_n(0, q)}{\mathrm{Me}_n^{(1)}(0, q)}\,\mathrm{ce}_n^2(\alpha, q)\right|^2. \qquad (8.1.15)$$

The complete scattering cross section is equal to

$$\sigma_s = \frac{8}{k}\sum_{n=0}^{\infty}\{|a_n|^2 + |b_n|^2\}, \qquad (8.1.16)$$

where a_n and b_n are determined from formula (8.1.14). When $\alpha = 0$,

$$\sigma_s = \frac{8}{k}\sum_{n=0}^{\infty}\left|\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{ce}_n(0, q)\right|^2, \qquad (8.1.17)$$

when $\alpha = \pi/2$,

$$\sigma_s = \frac{8}{k}\left\{\sum_{n=0,2,\ldots}^{\infty}\left|\frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{ce}_n\left(\frac{\pi}{2}\,q\right)\right|^2 + \right.$$

$$\left. + \sum_{n=1,3,\ldots}^{\infty}\left|\frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{se}_n\left(\frac{\pi}{2}\,q\right)\right|^2\right\} \qquad (8.1.18)$$

and finally in the case of a band with a width $2f(\xi^0 = 0)$,

$$\sigma_s = \frac{8}{k}\sum_{n=0}^{\infty}\left|\frac{\mathrm{Ce}_n(0, q)}{\mathrm{Me}_n^{(1)}(0, q)}\,\mathrm{ce}_n(\alpha, q)\right|^2. \qquad (8.1.19)$$

Case of a TE Type Field. For a stray field the potential v^1 /402 is sought in the form of a series similar to (8.1.3),

$$v^1 = \frac{2E\sin\theta}{\delta^2}\,e^{ihz}\sum_{n=0}^{\infty}\{a_n c_n\,\mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{ce}_n(\eta, q) + $$

$$+ b_n d_n\,\mathrm{Ne}_n^{(1)}(\xi, q)\,\mathrm{se}_n(\eta, q)\}, \qquad (8.1.20)$$

whose coefficients a_n and b_n are found from (1.7.8), taking into account (8.1.12), and are equal to[1] (the terms with d_0 must be dropped everywhere)

$$a_n = -\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{ce}_n(\alpha, q),\quad b_n = -\frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{se}_n(\alpha, q). \tag{8.1.21}$$

Therefore,

$$v^1 = \frac{2E\sin\theta}{\delta^2}\times$$
$$\times e^{ihz}\sum_{n=0}^{\infty}\left\{c_n\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{ce}_n(\alpha, q)\,\mathrm{ce}_n(\eta, q) + \right. \tag{8.1.22}$$
$$\left. + d_n\frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi, q)\,\mathrm{se}_n(\alpha, q)\,\mathrm{se}_n(\eta, q)\right\},$$

and the potential of the complete field $v = v^0 + v^1$

$$v = \frac{2E\sin\theta}{\delta^2}\,e^{ihz}\sum_{n=0}^{\infty}\left\{c_n\left[\mathrm{Ce}_n(\xi, q) - \right.\right.$$
$$\left.-\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi, q)\right]\mathrm{ce}_n(\alpha, q)\,\mathrm{ce}_n(\eta, q) +$$
$$+ d_n\left[\mathrm{Se}_n(\xi, q) - \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi, q)\right]\times \tag{8.1.23}$$
$$\left.\times\,\mathrm{se}_n(\alpha, q)\,\mathrm{se}_n(\eta, q)\right\}.$$

In the vicinity of the wave band for the stray field

$$E_\eta^1 = \sqrt{\frac{\mu}{\varepsilon}}\,H_z^1 = \frac{e^{i\delta\rho}}{\sqrt{\rho}}\,A^*(\varphi)\sin\theta, \tag{8.1.24}$$

where the amplitude function

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$$A^*(\varphi) = -\sqrt{\frac{\mu}{\varepsilon}}\,2E\sqrt{\frac{2}{\delta\pi}}\,e^{ihz-\frac{i\pi}{4}}\times$$

[1] The prime in the ordinary and in the modified Mathieu functions means differentiation over the argument. In particular, $\mathrm{Ce}_n'(\xi^0, q) = \frac{d}{d\xi}\,\mathrm{Ce}_n(\xi, q)|_{\xi=\xi^0}$ etc.

$$\times \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)} \mathrm{ce}_n(\alpha, q)\mathrm{ce}_n(\varphi, q) + \right.$$
$$\left. + \frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)'}(\xi^0, q)} \mathrm{se}_n(\alpha, q)\mathrm{se}_n(\varphi, q) \right\}. \qquad (8.1.25)$$

For a band with the width $2f(\xi^0 = 0)$

$$A^*(\varphi) = -\sqrt{\frac{\mu}{\varepsilon}}\, 2E \sqrt{\frac{2}{\delta\pi}}\, e^{ihz - \frac{i\pi}{4}} \times$$
$$\times \sum_{n=0}^{\infty} \frac{\mathrm{Se}_n'(0, q)}{\mathrm{Ne}_n^{(1)'}(0, q)} \mathrm{se}_n(\alpha, q)\mathrm{se}_n(\varphi, q). \qquad (8.1.26)$$

The law governing the density distribution of the surface currents is given by the formulas

$$j_\xi = 0,$$
$$j_\eta = -\frac{Ec\sin\theta}{i\pi^2} e^{ihz} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha, q)\mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(\xi^0, q)} + \right.$$
$$\left. + \frac{\mathrm{se}_n(\alpha, q)\mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)} \right\} (-1)^n, \qquad (8.1.27)$$
$$j_z = \frac{Ec\cos\theta}{h_\xi \pi^2 k \sin\theta} e^{ihz} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha, q)\mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(\xi^0, q)} + \right.$$
$$\left. + \frac{\mathrm{se}_n(\alpha, q)\mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)} \right\} (-1)^n.$$

It should be noted that the component j_z of the density of the surface currents along the Oz axis does not create a stray field, since it generates the potential function u, identically equal to zero for a TE type field. For a band with a width $2f(\xi^0 = 0)$

$$j_\eta = -\frac{Ec\sin\theta}{i\pi^2} e^{ihz} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha, q)\mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(0, q)} + \right.$$
$$\left. + \frac{\mathrm{se}_n(\alpha, q)\mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(0, q)} \right\} (-1)^n, \qquad (8.1.28) \quad /404$$
$$j_z = \frac{Ec\cos\theta}{h_\xi \pi^2 k \sin\theta} e^{ihz} \sum_{n=0}^{\infty} \left\{ \frac{\mathrm{ce}_n(\alpha, q)\mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(0, q)} + \right.$$
$$\left. + \frac{\mathrm{se}_n(\alpha, q)\mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(0, q)} \right\} (-1)^n.$$

These expressions satisfy the Meixner conditions at the edge of the band. With normal incidence of the wave on the cylinder (with respect to the generatrices of the cylinder), when $\theta = \pi/2$ the back scattering cross section (radar cross section)

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0}^{\infty}(-1)^n\left\{\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{ce}_n^2(\alpha, q)+\right.\right.$$
$$\left.\left.+\frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)'}(\xi^0, q)}\,\mathrm{se}_n^2(\alpha, q)\right\}\right|^2. \qquad (8.1.29)$$

When $\alpha = 0$

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0}^{\infty}(-1)^n\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{ce}_n^2(0, q)\right|^2, \qquad (8.1.30)$$

and when $\alpha = \pi/2$

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0,2,\ldots}\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{ce}_n^2\left(\frac{\pi}{2}, q\right)-\right.$$
$$\left.-\sum_{n=1,3,\ldots}\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)'}(\xi^0, q)}\,\mathrm{se}_n^2\left(\frac{\pi}{2}, q\right)\right|^2 \qquad (8.1.31)$$

and finally in the case of band with a width of $2f(\xi^0 = 0)$

$$\sigma_B = \frac{16}{k}\left|\sum_{n=0}^{\infty}(-1)^n\frac{\mathrm{Se}_n'(0, q)}{\mathrm{Ne}_n^{(1)'}(0, q)}\,\mathrm{se}_n^2(\alpha, q)\right|^2. \qquad (8.1.32)$$

The total cross (linear) section of scattering is equal to

$$\sigma_s = \frac{8}{k}\sum_{n=0}^{\infty}\{|a_n|^2 + |b_n|^2\}, \qquad (8.1.33)$$

where a_n and b_n are determined from the formulas in (8.1.21). When $\alpha = 0$,

$$\sigma_s = \frac{8}{k}\sum_{n=0}^{\infty}\left|\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{ce}_n(0, q)\right|^2, \qquad (8.1.34)$$

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when $\alpha = \pi/2$,

$$\sigma_s = \frac{8}{k}\left\{\sum_{n=0,2,\ldots}\left|\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{ce}_n\left(\frac{\pi}{2}\, q\right)\right|^2+\right.$$
$$\left.+\sum_{n=1,3,\ldots}\left|\frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)'}(\xi^0, q)}\,\mathrm{se}_n\left(\frac{\pi}{2}, q\right)\right|^2\right\}$$

and finally in the case of the band

$$\sigma_s = \frac{8}{k} \sum_{n=0}^{\infty} \left| \frac{\mathrm{Se}_n'(0, q)}{\mathrm{Ne}_n^{(1)'}(0, q)} \, \mathrm{se}_n(\alpha, q) \right|^2. \qquad (8.1.35)$$

Using the representations of the Mathieu functions (4.2.29), (4.3.7) and (2.2.26) - (2.2.28) which are asymptotic relative to the index n, it is easy to establish the absolute and uniform convergence of the series[2] which are encountered in the solutions obtained above for the problems for both cases of polarization of the field. These series are suitable for numerical computation (without resorting to a computer) for relatively small values of the parameter q, whose value depends on the dimensions of the cross section of the cylinder - ellipse (on the focal distance f) and on the wave length of excitation. In the general case the numerical computation of the obtained expressions is made more difficult since the theory of elliptical wave functions is more complicated and their tabulation is more difficult than, for example, wave functions of a circular cylinder. In this respect, the Tables of Mathieu Functions are valid only for comparatively small ratio between the focal distance of the ellipse of the cross section of the cylinder to the wave length [43, 183-185, 188]. The boundedness of the extent of the available Tables of Mathieu Functions and the lack of data on their asymptotics for large values of q leads to the solutions of (8.1.5) - (8.1.35) being of very little practical feasibility in the optical wave band, when because of the small wave length the value of q is quite large. However, even in the centimeter wave band with the proper value of the interfocal distance of the ellipse of the cross section of the cylinder all solutions become feasible for numerical computations of the problem. The theory of approximate methods for computing infinite series of Mathieu wave functions in the case of large q, on the basis of which they could be converted into expressions that are more suitable for computation and analysis of the field, for example, as is done in the case of a circular cylinder, has been developed insufficiently. More difficulties arise /406
also in solving the problem on the diffraction of electromagnetic waves on an elliptical cylinder of a material which possesses a finite conductivity ($\sigma_2 \neq 0$), when q is (in the general case) a complex value inside the cylinder and even when $\sigma_2 = 0$ inside the cylinder but the magnetic and electric permeabilities of the cylinder differ from the permeabilities of the surrounding atmosphere. The singularity of the solution to the diffraction problem for such an elliptical cylinder is that the desired potential functions inside and outside the cylinder are suitable for expansion of the Mathieu wave functions with different values of the parameter q, for which the angular Mathieu functions become nonorthogonal on the surface of the cylinder. As a result of the reexpansion of the functions for the coefficients of the series, an infinite system of linear

[2] The series converge non-uniformly relative to q [171-174].

equations is obtained with an infinite number of unknowns [123].

§2. Field of a Longitudinal Emitter in the Presence of an Elliptical Cylinder

Let us look at the problem of a unit electric or magnetic dipole field in the presence of an ideally conducting infinitely long elliptical cylinder, occupying in an unbounded uniform isotropic space with constant ε, μ and $\sigma = 0$ the region $\xi \leq \xi^0$ (the Oz axis of the elliptical system of coordinates ξ, η, z is directed along

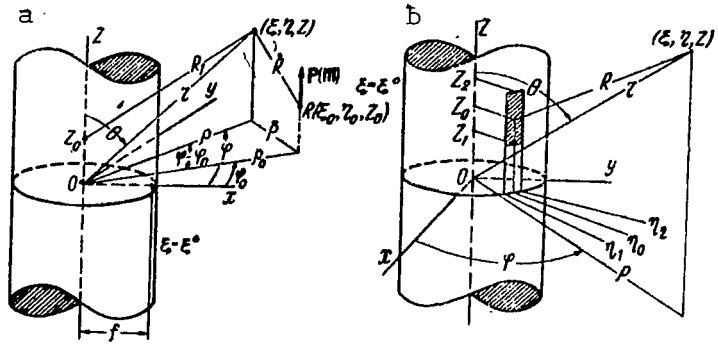

Fig. 51. Excitation of an Elliptical Cylinder of a Plane Wave (a) and Generalization of the Problem to the Case of a Longitudinal Slot on the Cylinder (b).

the axis of the cylinder) on the assumption that the dipole moment located at a given point $R(\xi_0, \eta_0, z_0)$ of space, oriented parallel to the generatrix of the cylinder (Fig. 51, a). The problem is solved according to the same scheme as in §2, Chapter 6 for the case of a circular cylinder. Just as in Chapter 6, the solution obtained for the dipole emitter is generalized then to the case of a linear /407 source of finite or infinite length. In the extreme case when the magnetic dipole is located on the surface of the cylinder, a solution is obtained that is equivalent in the electrodynamic sense to the solution of the problem of the field of a unit emitting slot located on the surface of the cylinder of elliptical cross section (band) which may then be generalized to the case of a longitudinal slot of infinite width.

According to the same scheme as in §2, Chapter 6, we find that in the case of an electric dipole with the moment $\mathbf{p} = \{0, 0, p\}$ the potential Hertz function u^0 of the primary field in free space is given by a series of Fourier-type integrals,

$$u^0 = ip\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}(-1)^n\left\{\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\left[\begin{matrix} c_n^2\,\mathrm{Me}_n^{(1)}(\xi, q)\,\mathrm{Ce}_n(\xi_0, q) \\ c_n^2\,\mathrm{Me}_n^{(1)}(\xi_0, q)\,\mathrm{Ce}_n(\xi, q)\end{matrix}\right] + \right.$$

$$+ \operatorname{se}_n(\eta, q)\operatorname{se}_n(\eta_0, q)\begin{bmatrix} d_n^2 \operatorname{Ne}_n^{(1)}(\xi, q)\operatorname{Se}_n(\xi_0, q) \\ d_n^2 \operatorname{Ne}_n^{(1)}(\xi_0, q)\operatorname{Se}_n(\xi, q) \end{bmatrix}\Bigg\} e^{ih(z-z_0)}\,dh, \quad \begin{matrix}\xi > \xi_0 \\ \xi < \xi_0,\end{matrix} \tag{8.2.1}$$

where $q = f^2(k^2 - h^2)/4$, $p = I(z_0)dz_0/-i4\pi\omega\varepsilon$, and $I(z_0)$ is the given electric current on the dipole, and in the case of a magnetic dipole with the moment $\mathfrak{m} = \{0, 0, m\}$ the Hertz potential is equal to

$$v^0 = im\sum_{n=0}^{\infty}{}'\int_{-\infty}^{\infty}(-1)^n\Bigg\{\operatorname{ce}_n(\eta, q)\operatorname{ce}_n(\eta_0, q)\begin{bmatrix} c_n^2 \operatorname{Me}_n^{(1)}(\xi, q)\operatorname{Ce}_n(\xi_0, q) \\ c_n^2 \operatorname{Me}_n^{(1)}(\xi_0, q)\operatorname{Ce}_n(\xi, q) \end{bmatrix} +$$
$$+ \operatorname{se}_n(\eta, q)\operatorname{se}_n(\eta_0, q)\begin{bmatrix} d_n^2 \operatorname{Ne}_n^{(1)}(\xi, q)\operatorname{Se}_n(\xi_0, q) \\ d_n^2 \operatorname{Ne}_n^{(1)}(\xi_0, q)\operatorname{Se}_n(\xi, q) \end{bmatrix}\Bigg\} e^{ih(z-z_0)}\,dh, \quad \begin{matrix}\xi > \xi_0 \\ \xi < \xi_0,\end{matrix} \tag{8.2.2}$$

where $m = K(z_0)dz_0/-i4\pi\omega\mu$, and $K(z_0)$ is the given magnetic current in the dipole.

In both cases in the elliptical system of coordinates ξ, η, z (the positon of the dipole is determined by the values $\xi = \xi_0$, $\eta = \eta_0$, $z = z_0$) the components of the desired electromagnetic field at any point outside the cylinder are found through u^0, v^0 and the potentials u^1, v^1 of the stray field from (1.6.19), (1.6.20) (for the field of longitudinal electric dipole the magnetic Hertz potential $v \equiv 0$, and for the field of the longitudinal dipole the electric Hertz potential $u \equiv 0$). /408

Electric Dipole. If we seek the Hertz potential of the secondary field u^1 in the form of a series of Fourier integrals

$$u^1 = ip\sum_{n=0}^{\infty}{}'\int_{-\infty}^{\infty}(-1)^n\left[a_n(h)\varsigma_n \operatorname{Me}_n^{(1)}(\xi, q)\operatorname{ce}_n(\eta, q) + b_n(h)d_n \operatorname{Ne}_n^{(1)}(\xi, q)\operatorname{se}_n(\eta, q)\right] e^{ih(z-z_0)}\,dh, \tag{8.2.3}$$

Then from boundary condition (1.7.7), where $u = u^0 + u^1$, we find that this latter will be satisfied if

$$a_n(h) = -\frac{\operatorname{Ce}_n(\xi^0, q)}{\operatorname{Me}_n^{(1)}(\xi^0, q)}\, c_n \operatorname{Me}_n^{(1)}(\xi_0, q)\operatorname{ce}_n(\eta_0, q),$$
$$b_n(h) = -\frac{\operatorname{Se}_n(\xi^0, q)}{\operatorname{Ne}_n^{(1)}(\xi^0, q)}\, d_n \operatorname{Ne}_n^{(1)}(\xi_0, q)\operatorname{se}_n(\eta_0, q). \tag{8.2.4}$$

As a result the potential u of the resulting field in the region $\xi_0 < \xi$ is determined by the expression

$$u = ip \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} (-1)^n e^{ih(z-z_0)} \left\{ c_n^2 \mathrm{ce}_n(\eta, q) \mathrm{ce}_n(\eta_0, q) \left[\mathrm{Ce}_n(\xi_0, q) - \right.\right.$$
$$\left. - \frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)} \mathrm{Me}_n^{(1)}(\xi_0, q) \right] \mathrm{Me}_n^{(1)}(\xi, q) +$$
$$+ d_n^2 \mathrm{se}_n(\eta, q) \mathrm{se}_n(\eta_0, q) \left[\mathrm{Se}_n(\xi_0, q) - \right. \qquad (8.2.5)$$
$$\left.\left. - \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)} \mathrm{Ne}_n^{(1)}(\xi_0, q) \right] \mathrm{Ne}_n^{(1)}(\xi, q) \right\} dh.$$

The formal solution to the problem is then essentially finished since the desired electromagnetic field is now found from (1.6.19) by substituting (8.2.5) into it.

It is obvious that to carry out integration over the variable h in (8.2.5) (or in the corresponding expressions for the components of the sought field) is more difficult than in the analogous problem for the circular cylinder. However, in the case when the field is studied in the wave band for which, as usual, we assume $\xi \to \infty$ (kf ch $\xi \to \infty$), series (8.2.5) may be converted into a simpler form that does not contain the integral expressions.

Thus, if in the integrals,

$$J_1 = \int_{-\infty}^{\infty} c_n \mathrm{Me}_n^{(1)}(\xi, q) f_n(\xi_0, \xi^0; h) e^{ih(z-z_0)} dh,$$

$$J_2 = \int_{-\infty}^{\infty} d_n \mathrm{Ne}_n^{(1)}(\xi, q) g_n(\xi_0, \xi^0; h) e^{ih(z-z_0)} dh \qquad /409$$

we substitute the radial Mathieu functions by their asymptotic approximations (4.3.9) and then apply them to the integrals obtained by the method of steepest descents then on the basis of the discussions analogous to those given in [52] for the circular cylinder, we find the asymptotic expressions,

$$J_1 = -2i \frac{e^{ikR}}{R} f_n(\xi^0, \xi_0; k\cos\theta), \qquad (8.2.6)$$

$$J_2 = -2i \frac{e^{ikR}}{R} g_n(\xi^0, \xi_0; k\cos\theta), \qquad (8.2.7)$$

in which

$$f_n(\xi^0, \xi_0; h) = c_n \operatorname{ce}_n(\eta, q) \operatorname{ce}_n(\eta_0, q) \times$$
$$\times \left[\operatorname{Ce}_n(\xi_0, q) - \frac{\operatorname{Ce}_n(\xi^0, q)}{\operatorname{Me}_n^{(1)}(\xi^0, q)} \operatorname{Me}_n^{(1)}(\xi_0, q) \right]$$
$$g_n(\xi^0, \xi_0; h) = d_n \operatorname{se}_n(\eta, q) \operatorname{se}_n(\eta_0, q) \times \qquad (8.2.8)$$
$$\times \left[\operatorname{Se}_n(\xi_0, q) - \frac{\operatorname{Se}_n(\xi^0, q)}{\operatorname{Ne}_n^{(1)}(\xi^0, q)} \operatorname{Ne}_n^{(1)}(\xi_0, q) \right]$$

[in (8.2.6), (8.2.7) we have dropped terms which contain higher powers of the value R^{-1}, $R = \sqrt{\rho^2 + (z - z_0)^2}$; $\theta = \arctan \rho/z$, $\rho \approx f \operatorname{ch} \xi$). If we convert from elliptical coordinates to spherical coordinates R, θ, ϕ, in which in this approximation,

$$\begin{aligned} E_\theta &= E_\xi \cos\theta - E_z \sin\theta, & H_\theta &= H_\xi \cos\theta - H_z \sin\theta, \\ E_R &= E_\xi \sin\theta + E_z \cos\theta, & H_R &= H_\xi \sin\theta + H_z \cos\theta, \\ E_\varphi &= E_\eta, & H_\varphi &= H_\eta, \end{aligned} \qquad (8.2.9)$$

then we find that in the wave band, approximately,

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}} H_\varphi = k^2 p \frac{e^{ikR}}{R} \sin\theta \, e^{-ikz_0\cos\theta} W(\theta, \varphi), \qquad (8.2.10)$$

where the damping coefficient $W(\theta, \phi)$ is equal to

$$W(\theta, \varphi) = 2 \sum_{n=0}^{\infty} [f_n(\xi^0, \xi_0; k\cos\theta, \varphi) + g_n(\xi^0, \xi_0; k\cos\theta, \varphi)]. \qquad (8.2.11)$$

Generalization to the case of a linear antenna, extending from $z = z_1$ to $z = z_2$ with a given current $I(z_0)$, is found by integration over z_0 from $z_0 = z_1$ to $z_0 = z_2$. For its field,

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}} \frac{ike^{ikR}}{4\pi cR} \sin\theta \left\{ \int_{z_1}^{z_2} I(z_0) e^{ikz_0\cos\theta} \, dz_0 \right\} W(\theta, \varphi). \qquad (8.2.12)$$

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Of definite interest is the particular case when the source of the primary field is an infinitely long filament with an electric current. In this case, after carrying out integration over z_0 from $z = -\infty$ to $z = +\infty$ we find (in elliptical coordinates) $E_\xi = E_\eta = 0$, and (in the wave band),

$$E_z = -\sqrt{\frac{\mu}{\varepsilon}} H_\eta = \frac{e^{ikp}}{\sqrt{p}} W(\varphi), \qquad (8.2.13)$$

where

$$W(\varphi) = -I\sqrt{\frac{\mu}{\varepsilon}}\sqrt{\frac{k}{2\pi}}\,ce^{-\frac{i\pi}{4}}\sum_{n=0}^{\infty}(-i)^n\left\{\left[\mathrm{Ce}_n(\xi_0, q) - \right.\right.$$
$$\left. - \frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi_0, q)\right] c_n\,\mathrm{ce}_n(\varphi, q)\,\mathrm{ce}_n(\eta_0, q) +$$
$$\left. + \left[\mathrm{Se}_n(\xi_0, q) - \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n(\xi^0, q)}\,\mathrm{Ne}_n(\xi_0, q)\right] d_n\,\mathrm{se}_n(\varphi, q)\,\mathrm{se}_n(\eta_0, q)\right\},$$
$$q^{1/2} = f\frac{k}{2}.$$

In this case of an infinitely long filament with a current, for the potential of the complete field we find the expression (when $\xi > \xi_0$; $q = \left(f\frac{k}{2}\right)^2$),

$$u = -\frac{I}{2\omega\varepsilon}\sum_{n=0}^{\infty}(-1)^n\left\{\left[\mathrm{Ce}_n(\xi_0, q) - \right.\right.$$
$$\left. - \frac{\mathrm{Ce}_n(\xi^0, q)}{\mathrm{Me}_n^{(1)}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi_0, q)\right] c_n^2\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\,\mathrm{Me}_n^{(1)}(\xi, q) +$$
$$+ \left[\mathrm{Se}_n(\xi_0, q) - \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi_0, q)\right] \times \qquad (8.2.14)$$
$$\left. \times\, d_n^2\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)\,\mathrm{Ne}_n^{(1)}(\xi, q)\right\},$$

which gives a strict solution to the problem on the diffration of a cylindrical wave $\frac{I}{4k}\sqrt{\frac{\mu}{\varepsilon}}\,H_0^{(1)}(k\rho)$ on an ideally conducting cylinder of elliptical cross section.

Magnetic Dipole. If we seek the potential of the stray field in the form of series (8.2.3) by substituting m for p in it, then from boundary condition (1.7.8), where $v = v^0 + v^1$, we find that /411

$$a_n(h) = -\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,c_n\,\mathrm{Me}_n^{(1)}(\xi_0, q)\,\mathrm{ce}_n(\eta_0, q),$$
$$b_n(h) = -\frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,d_n\,\mathrm{Ne}_n^{(1)}(\xi_0, q)\,\mathrm{se}_n(\eta_0, q). \qquad (8.2.15)$$

Therefore, the potential of the resulting field will be equal to

$$v = im\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}(-1)^n\,e^{ih(z-z_0)}\left\{c_n^2\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\left[\mathrm{Ce}_n(\xi_0, q) - \right.\right.$$

$$-\frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi_0, q)\Big]\,\mathrm{Me}_n^{(1)}(\xi, q) + d_n^2\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)\times$$

$$\times\Big\{\Big[\mathrm{Se}_n(\xi_0, q) - \frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi_0, q)\Big]\,\mathrm{Ne}_n^{(1)}(\xi, q)\Big\}\,dh,$$

if $\xi > \xi_0$, and

$$v = im\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}(-1)^n e^{ih(z-z_*)}\Big\{c_n^2\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\times$$

$$\times\Big[\mathrm{Ce}_n(\xi, q) - \frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi, q)\Big]\times$$

$$\times\,\mathrm{Me}_n^{(1)}(\xi_0, q) + d_n^2\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)\times$$

$$\times\Big[\mathrm{Se}_n(\xi, q) - \frac{\mathrm{Se}_n'(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi, q)\Big]\,\mathrm{Ne}_n^{(1)}(\xi_0, q)\Big\}\,dh,$$

if $\xi_0 > \xi$, $q = f^2(k^2 - h^2)/4$.

In the vicinity of the wave band ($\eta \approx \phi$) by substituting the integral expressions by their asymptotic approximations (8.2.6) - (8.2.7) and denoting

$$f_n = c_n\,\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)\times$$

$$\times\Big[\mathrm{Ce}_n(\xi_0, q) - \frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi_0, q)\Big],$$

$$g_n = d_n\,\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)\times \qquad (8.2.16)$$

$$\times\Big[\mathrm{Se}_n(\xi_0, q) - \frac{\mathrm{Se}_n(\xi^0, q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0, q)}\,\mathrm{Ne}_n^{(1)}(\xi_0, q)\Big],$$

we find from (1.6.20) and (8.2.9) (in spherical coordinates R, θ, ϕ), that /41

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}\,H_\theta = i\sqrt{\frac{\mu}{\varepsilon}}\,k^2 m\,\frac{e^{ikR}}{R}\,\sin\theta\,W^*(\theta, \varphi), \qquad (8.2.17)$$

where the damping coefficient

$$W^*(\theta, \varphi) = 2\sum_{n=0}^{\infty}\left[f_n(\xi^0, \xi_0; k\cos\theta\,\varphi) + g_n(\xi^0, \xi_0; k\cos\theta, \varphi)\right]. \qquad (8.2.18)$$

By integrating over z_0 from $z_0 = z_1$ to $z_0 = z_2$ we find the generalization of the solution of the problem to the case of a linear antenna with a given magnetic current $K = K(z_0)$. Here we will have

$$E_\varphi = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{ik^2 e^{ikR}}{4\pi\omega\mu R}\sin\theta\left\{\int_{z_1}^{z_2} K(z_0)\,e^{ikz_0\cos\theta}\,dz_0\right\}W^*(\theta,\varphi). \qquad (8.2.19)$$

By integration in (8.2.15) over z_0 from $z_0 = -\infty$ to $z_0 = \infty$, we find the generalization of the solution to the case of an infinitely long filament with a magnetic current parallel to the axis of the cylinder. For its field the Hertz potential will be equal to ($\xi > \xi_0$)

$$\begin{aligned} v \doteq -\frac{K}{2\omega\mu}\sum_{n=0}^{\infty}(-1)^n\Bigg\{c_n^2\,\mathrm{ce}_n(\eta,q)\,\mathrm{ce}_n(\eta_0,q)\times \\ \times\left[\mathrm{Ce}_n(\xi_0,q)-\frac{\mathrm{Ce}_n'(\xi^0,q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0,q)}\mathrm{Me}_n^{(1)}(\xi_0,q)\right]\mathrm{Me}_n^{(1)}(\xi,q)+ \\ +d_n^2\,\mathrm{se}_n(\eta,q)\,\mathrm{se}_n(\eta_0,q)\left[\mathrm{Se}_n(\xi_0,q)-\right. \\ \left.-\frac{\mathrm{Se}_n(\xi^0,q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0,q)}\mathrm{Ne}_n^{(1)}(\xi_0,q)\right]\mathrm{Ne}_n^{(1)}(\xi,q)\Bigg\}. \end{aligned} \qquad (8.2.20)$$

Such a result is found also with a strict solution to the problem on the diffraction of a cylindrical wave $-\frac{K}{4k}\sqrt{\frac{\mu}{\varepsilon}}\,H_0^{(1)}(k\rho)$ on an ideally conducting elliptical cylinder. Using (1.1.7), (8.2.20) and (1.6.20) we find the law for the density distribution of the surface current induced on the surface of the elliptical cylinder by the field of the filament with the current. We have $j_\xi = j_z = 0$, and /413

$$\begin{aligned} j_\eta = -\frac{iKk}{4\pi^2}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=0}^{\infty}\Bigg\{\frac{\mathrm{Me}_n^{(1)}(\xi_0,q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0,q)}\,\mathrm{ce}_n(\eta,q)\,\mathrm{ce}_n(\eta_0,q)+ \\ +\frac{\mathrm{Ne}_n^{(1)}(\xi_0,q)}{\mathrm{Ne}_n^{(1)\prime}(\xi^0,q)}\,\mathrm{se}_n(\eta,q)\,\mathrm{se}_n(\eta_0,q)\Bigg\}. \end{aligned} \qquad (8.2.21)$$

In particular, if $\eta_0 = 0$ (the dipole is located in the plane Oxz), then

$$j_\eta = -\frac{iKk}{4\pi^2}\sqrt{\frac{\varepsilon}{\mu}}\sum_{n=0}^{\infty}\frac{\mathrm{Me}_n^{(1)}(\xi_0,q)}{\mathrm{Me}_n^{(1)\prime}(\xi^0,q)}\,\mathrm{ce}_n(0,q)\,\mathrm{ce}_n(\eta,q). \qquad (8.2.22)$$

Let us note that in the case of a dipole emitter or a linear antenna extending from $z_0 = z_1$ to $z_0 = z_2$, the previously obtained expressions for the function E_ϕ are substantially simplified, if the dipole is located in the plane Oxz, when $\eta_0 = 0$. Then instead of (8.2.18) we find

$$W^*(\theta, \varphi) = 2\sum_{n=0}^{\infty}\left[\mathrm{Ce}_n(\xi_0, q) - \right.$$
$$\left. - \frac{\mathrm{Ce}_n'(\xi^0, q)}{\mathrm{Me}_n^{(1)'}(\xi^0, q)}\,\mathrm{Me}_n^{(1)}(\xi_0, q)\right] c_n \mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(0, q),\ \eta \approx \varphi. \tag{8.2.23}$$

If here the elliptical cylinder is degenerated into a band with a width of $2f$, when $\xi^0 = 0$, then

$$W^*(\theta, \varphi) = 2\sum_{n=0}^{\infty}\mathrm{Ce}_n(\xi_0, q) c_n \mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(0, q), \tag{8.2.24}$$

since $\mathrm{Ce}_n'(0, q) = 0$. Hence it follows that the characteristics of the emission from a longitudinal magnetic emitter do not depend on the presence in its field of an infinitely long ideally conducting band, in whose plane the emitter is located.

In the limiting case, when $\xi_0 \to \xi^0$ (the magnetic dipole approaches the surface of the cylinder)

$$W^*(\theta, \varphi) = \frac{4}{i\pi}\sum_{n=0}^{\infty}(-1)^n\left\{\frac{\mathrm{ce}_n(\eta, q)\,\mathrm{ce}_n(\eta_0, q)}{c_n \mathrm{Me}_n^{(1)'}(\xi^0, q)} + \right.$$
$$\left. + \frac{\mathrm{se}_n(\eta, q)\,\mathrm{se}_n(\eta_0, q)}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)}\right\}. \tag{8.2.25}$$

Instead of the dipole if we look at an antenna of finite length extending from $z_0 = z_1$ to $z_0 = z_2$, then at the limit when $\xi_0 \to \xi^0$ we obtain (8.2.19) with the damping coefficient determined by formulas (8.2.25). In the electrodynamic sense this case is equivalent to the case of a narrow longitudinal emitting slot (Fig. 51, b) with a width $h_{\eta_0}\,d\,\eta_0$ ($h_\eta = f\sqrt{\mathrm{ch}^2\xi^0 - \cos^2\eta}$, extending in length from $z_0 = z_1$ to $z_0 = z_2$ and located on the surface of the cylinder. /414

If we substitute in (8.2.19) the distribution of the current $K(z_0)$ by the distribution of the voltage on the slot $E_\eta(\xi^0, \eta^0, z^0)\,h_{\eta_0}\,d\eta_0$ and then integrate in (8.2.19) over η_0 from $\eta_0 = \eta_1$ to $\eta_0 = \eta_2$, we then find the generalization to the solution of the problem to the case of a rectangular slot of finite width, extending in length from $z_0 = z_1$ to $z_0 = z_2$ and located between $\eta_0 = \eta_1$ and $\eta_0 = \eta_2$ with a given distribution of the field $E_\eta(\xi^0, \eta^0, z^0)$ in the opening of the slot. If we assume that the voltage on the slot

$V(z_0) = E_\eta(\xi^0, \eta^0, z^0)h_\eta(\eta_2 - \eta_1)$ we find, for the field emitted by the slot in the vicinity of the wave band in spherical coordinates R, θ, ϕ, the representation,

$$E_\varphi = \frac{e^{ikR}}{R} S(\theta) M(\xi^0, \eta), \tag{8.2.26}$$

in which

$$S(\theta) = \frac{cik \sin\theta}{2\pi} \int_{z_1}^{z_2} V(z_0) e^{-ikz_0\cos\theta} dz_0. \tag{8.2.27}$$

$$M(\xi^0, \eta) = \frac{4}{i\pi} \sum_{n=0}^{\infty} \left\{ \frac{G_n \mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(\xi^0, q)} + \right.$$

$$\left. + \frac{F_n \mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)} \right\}, \tag{8.2.28}$$

where

$$G_n = \sum_{r=0,1}' A_r^{(n)} \frac{\cos r\delta \sin r\delta_1}{r\delta_1},$$

$$F_n = \sum_{r=1,2} B_r^{(n)} \frac{\sin r\delta \sin r\delta_1}{r\delta_1},$$

$$\delta = \frac{\eta_2 + \eta_1}{2}, \quad \delta_1 = \frac{\eta_2 - \eta_1}{2}.$$

If the voltage distribution in the slot is almost sinusoidal and

$$V(z_0) = V_0 \frac{\sin k(l - |z_0|)}{\sin kl}, \quad z_1 = -l, \quad z_2 = l, \tag{8.2.29}$$

where V_0 is the voltage in the middle of the slot, then after computing the integral in (8.2.27) we obtain /415

$$S(\theta) = icV_0 \frac{\cos(kl\cos\theta) - \cos kl}{\pi \sin kl \sin\theta},$$

whence when $l = \lambda_0/4$ (the length of the slot is equal to half the wave length),

$$S(\theta) = icV_0 \frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta}.$$

For a slot that is symmetrically located relative to the plane Oxz, when $\eta_1 = -\eta_0$, and $\eta_2 = \eta_0$, we will have $F_n \equiv 0$, but

$$M(\xi^0, \eta) = \frac{4}{i\pi} \sum_{n=0}^{\infty} \frac{G_n \operatorname{ce}_n(\eta, q)}{c_n \operatorname{Me}_n^{(1)'}(\xi^0, q)},$$
$$G_n = \sum_{r=0,1}^{\infty}{}' A_r^{(n)} \frac{\sin r\eta_0}{r\eta_0}. \qquad (8.2.30)$$

Since the function $S(\theta)$, determined by formula (8.2.27) is a characteristic of the directionality of the slot, located on an infinite ideally conducting plane and extending in the axial direction from $z_1 = -l$ to $z_2 = l$, then the function $M(\xi^0, \eta)$ is the damping coefficient which takes into account the finiteness of the radius of curvature of the elliptical cylinder (or the factor of directionality of the narrow longitudinal slot, located on the surface of the elliptical ideally conducting cylinder). With a fixed value of $h = k \cos\theta$ the quantity $M(\xi^0, \eta)$ is a function of the angle ϕ (in the wave band $\eta \approx \phi$), which characterizes the azimuth directionality of the slot.

The factor of directionality for a thin elliptical cylinder (band) is equal to

$$M(0, \eta) = \frac{4}{i\pi} \sum_{n=0}^{\infty} \left\{ \frac{G_n \operatorname{ce}_n(\eta, q)}{c_n \operatorname{Me}_n^{(1)'}(0, q)} + \right.$$
$$\left. + \frac{F_n \operatorname{se}_n(\eta, q)}{d_n \operatorname{Ne}_n^{(1)'}(0, q)} \right\}. \qquad (8.2.31)$$

Using formula (1.4.17) we can compute the conductivity of the emission from the longitudinal slot, located on the surface of the elliptical cylinder, which is determined from the computation of the power emitted by the slot, transmitted through an infinite sphere with the center at the origin. In the case of a half-wave slot /416 ($l = \lambda_0/4$), with a sinusoidal distribution of the voltage at the slot, we find

$$2\bar{P} = \sqrt{\frac{\varepsilon}{\mu}} \int_0^{2\pi} \int_0^{\pi} |E_\varphi|^2 R^2 \sin\theta \, d\theta \, d\varphi, \qquad (8.2.32)$$

where

$$E_\varphi = \frac{e^{ikR}}{R} \cdot \frac{4cV_0}{\pi} \cdot \frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta} \times$$

$$\times \sum_{n=0}^{\infty} \left\{ \frac{G_n \mathrm{ce}_n(\eta, q)}{c_n \mathrm{Me}_n^{(1)'}(\xi^0, q)} + \frac{F_n \mathrm{se}_n(\eta, q)}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)} \right\}, \qquad (8.2.33)$$

$$q = (fk \sin\theta)^2/4.$$

By virtue of the relationships of the orthogonality of the angular Mathieu functions, from (1.4.17) we obtain

$$G = \frac{16c^2}{\pi} \sqrt{\frac{\varepsilon}{\mu}} \int_0^{\pi} \frac{\cos^2\left(\frac{\pi}{2}\cos\theta\right)}{\sin\theta} \sum_{n=0}^{\infty} \left\{ \left| \frac{G_n}{c_1 \mathrm{Me}_n^{(1)'}(\xi^0, q)} \right|^2 + \right. \qquad (8.2.34)$$

$$\left. + \left| \frac{F_n}{d_n \mathrm{Ne}_n^{(1)'}(\xi^0, q)} \right|^2 \right\}.$$

The conductivity of the emission G may be found by numerical integration [52].

§3. Diffraction of a Plane Wave on Two Parallel Elliptical Cylinders

In an unbounded uniform and isotropic space with electromagnetic constants ε, μ, σ ($\sigma = 0$) let there be found two mutually unconnected and nontangent infinitely long cylinders of elliptical cross section with parallel longitudinal axes whose surfaces in local elliptical coordinate systems ξ_s, η_s, z_s ($s = \pm 1$) are given by the equations $\xi_{+1} = \xi_{+1}^0$, $\xi_{-1} = \xi_{-1}^0$ (the Oz axes of the systems coincide with the longitudinal axes of the cylinders). The cylinders are ideally conducting, their focal distances f_{-1} and f_{+1} in the general case are different. The large parameters of the ellipses of the cross section along which the axes $O_{-1}x_{-1}$ and $O_{+1}x_{+1}$ of the local coordinate systems are directed, from between themselves the angle β. The distance between the axes of the cylinders are equal to l. /417
The plane electromagnetic wave (1.3.4) is incident on the cylinders and is propagated in the direction of the unit vector $\mathbf{n}$ which forms the angle $\pi/2$ with the axes of the cylinders and the angle α with the positive direction of the Ox axis (O is the origin of the system x, y, z, superposed with the middle of the line of the centers of the cross section ellipses of the cylinders, along which the axis Ox is directed). With the axes $O_s x_s$, $s = \pm 1$, the vector $\mathbf{n}$ forms the angles $\alpha_{-1} = \alpha + \gamma$ and $\alpha_{+1} = \alpha + \gamma + \beta$, respectively. By $\mathbf{R}$ in (1.3.4) we mean the radius-vector which joins the point of observation with the point O (Fig. 52, a). The incident wave is a wave of only one type: type TM, when the vector $\mathbf{E}^0$ is parallel to the generatrices of the cylinders and type TE, when the vector $\mathbf{H}^0$ is parallel to them.

We pose the problem of seeking the secondary electromagnetic field scattered on the cylinders. In elliptical coordinates this problem in mathematical formulation is reduced to solving the bound-

ary problems for the Helmholtz equation relative to the potential u^1 or v^1 of the secondary field, which satisfies on the surface of the cylinders the boundary condition (1.7.7) or (1.7.8), respectively, and the supplementary condition of radiation at infinity. By virtue of the uniformity of the electromagnetic and geometric properties of the cylinders and the medium surrounding them in the direction of the longitudinal axes of the cylinders, the diffraction problems will be two-dimensional and independent of the spatial coordinate z.

The plane wave (1.3.4) which has, in the coordinates x, y, z with the origin at point O, the form,

$$E^0 = E_0 e^{ik(\alpha x + \beta y)}, \tag{8.3.1}$$

where $|E_0| = E = \text{const}$; in local coordinates of the s-th cylinder it is written in the form,

$$E^0 = E_0 e^{ik(nR_{0s}) + ik(nR_s)}, \tag{8.3.2}$$

since $R = R_s + R_{0_s}$, where R_{0_s} is the radius-vector which joins the point O with the origin O_s of the system $O_s x_s y_s z_s$, $s = \pm 1$. Here $(nR_{0_s}) = sl_0 \cos\alpha$, $l_0 = 1/2$. Therefore in local coordinates of the s-th cylinder the electric Hertz potential of the primary field u^0 (the TM case) is written on the basis of (8.1.1) in the form,

$$u^0 = \frac{2Ee^{iskl_0\cos\alpha}}{k^2} \sum_{n=0}^{\infty} [c_n \mathrm{Ce}_n(\xi_s, q_s)\mathrm{ce}_n(\eta_s, q_s)\mathrm{ce}_n(\alpha_s, q_s) + d_n \mathrm{Se}_n(\xi_s, q_s)\mathrm{se}_n(\eta_s, q_s)\mathrm{se}_n(\alpha_s, q_s)], \tag{8.3.3}$$

and the Hertz potential of the primary field of magnetic type (the TE case on the basis of (8.1.2)) is written in the form,

$$v^0 = \frac{2Ee^{iskl_0\cos\alpha}}{k^2} \sum_{n=0}^{\infty} [c_n \mathrm{Ce}_n(\xi_s, q_s)\mathrm{ce}_n(\eta_s, q_s)\mathrm{ce}_n(\alpha_s, q_s) + d_n \mathrm{Se}_n(\xi_s, q_s)\mathrm{se}_n(\eta_s, q_s)\mathrm{se}_n(\alpha_s, q_s)], \tag{8.3.4}$$

where $c_n = c_n(q_s)$, $d_n = d_n(q_s)$, $q_s = (f_s k/2)^2$.

TM Type Field. Let us first look at the case of a TM-polarized plane wave, whose Hertz potential is given by formula (8.3.3). We shall seek the potential of the complete field in the form of a sum

$$u = u^0 + u^1, \tag{8.3.5}$$

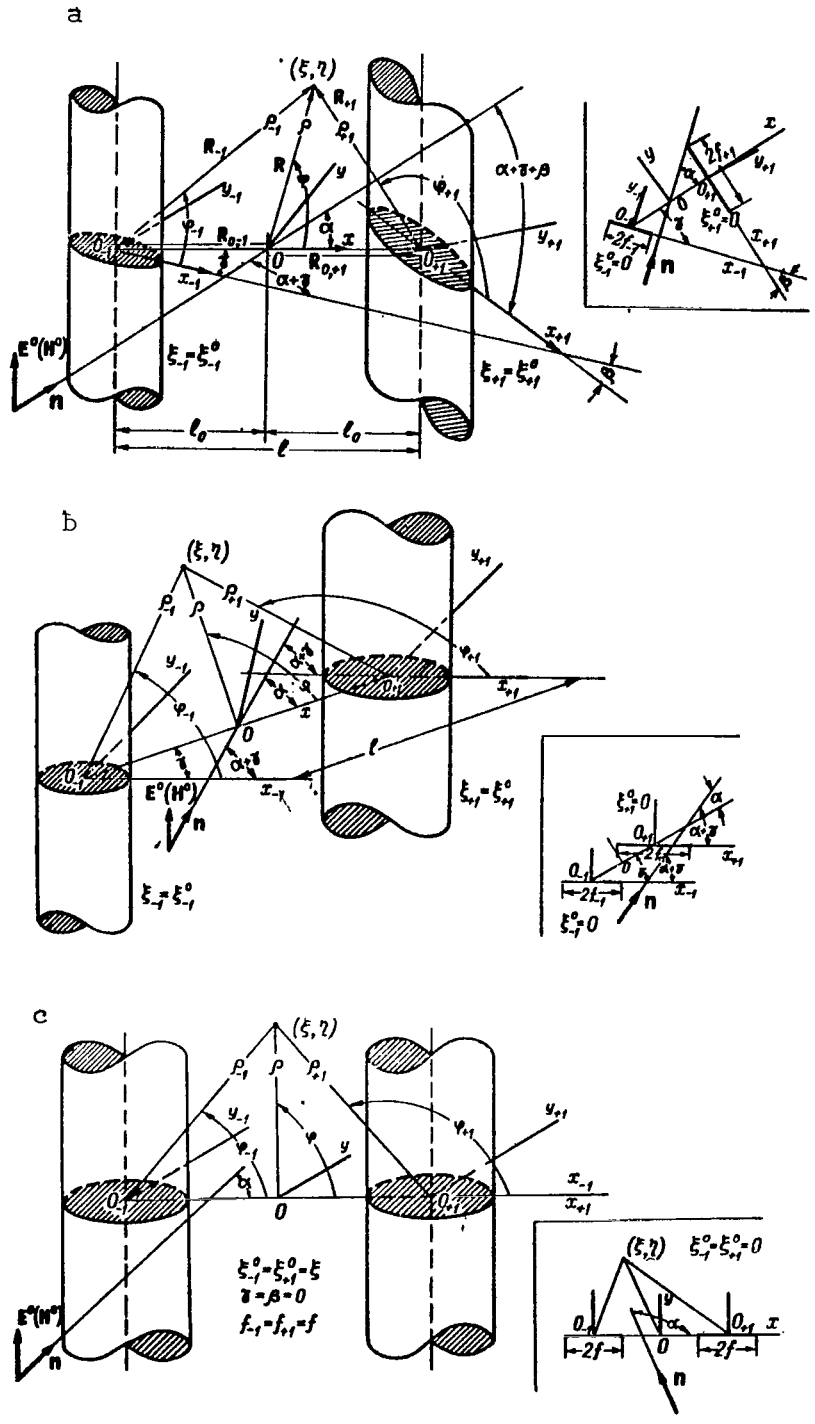

Fig. 52. Diffraction of a Plane Wave on Two Elliptical Cylinders (Bands). (a) Arbitrary Orientation of the Axes of the Ellipses; (b) Major Axes of the Ellipses are Parallel; (c) Major Axes of the Ellipses Lie on the *Ox* Axis.

where $u^1 = \sum_{s=\pm 1} u_s$ (then $E_z = k^2 u$), and u_s is the potential of the secondary field, produced by the presence in space of the s-th cylinder, $s = \pm 1$.

To determine u_s, if we use the method of perturbations, then we obtain,

$$u_s = \sum_{m=1}^{\infty} {}^s u^m, \tag{8.3.6}$$

where ${}^s u^m$ is the potential of the secondary field of m-th order of scattering, satisfying on the surface of the cylinders the boundary condition,

$$\begin{gathered} {}^s u^m + {}^{-s} u^{m-1} = 0, \quad \xi_s = \xi_s^0, \\ s = \pm 1, \ m = 1, 2, \ldots \end{gathered} \tag{8.3.7}$$

Let us represent ${}^s u^m$ in the form of a series of elliptical wave functions such as (4.1.15), (4.1.16),

$$\begin{aligned} {}^s u^m = \frac{2E}{k^2} \sum_{n=0}^{\infty} \left[{}^s A_n^m c_n^2 \operatorname{Ce}_n(\xi_s^0, q_s) \operatorname{Me}_n^{(1)}(\xi_s, q_s) \operatorname{ce}_n(\eta_s, q_s) + \right. \\ \left. + {}^s B_n^m d_n^2 \operatorname{Se}_n(\xi_s^0, q_s) \operatorname{Ne}_n^{(1)}(\xi_s, q_s) \operatorname{se}_n(\eta_s, q_s) \right] \end{aligned} \tag{8.3.8}$$

($q_s = (f_s k/2)^2$). Then

$$\begin{aligned} u^1 = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \left[{}^s A_n^m c_n^2 \operatorname{Ce}_n(\xi_s^0, q_s) \operatorname{Me}_n^{(1)}(\xi_s, q_s) \operatorname{ce}_n(\eta_s, q_s) + \right. \\ \left. + {}^s B_n^m d_n^2 \operatorname{Se}_n(\xi_s^0, q_s) \operatorname{Ne}_n^{(1)}(\xi_s, q_s) \operatorname{se}_n(\eta_s, q_s) \right], \end{aligned} \tag{8.3.9}$$

where the scattering coefficients ${}^s A_n^m$ and ${}^s B_n^m$ are computed from the recurrent formulas, /420

$$\begin{aligned} {}^s A_n^m = -2 \sum_{k=0}^{\infty} \left\{ \frac{c_k(q_{-s}) \operatorname{Ce}_k(\xi_{-s}^0, q_{-s})}{c_n(q_s) \operatorname{Me}_n^{(1)}(\xi_s^0, q_s)} Q_{kn}^{(1)}(-s, s; \beta) {}^{-s} A_k^{m-1} + \right. \\ \left. + \frac{d_k(q_{-s}) \operatorname{Se}_k(\xi_{-s}^0, q_{-s})}{c_n(q_s) \operatorname{Me}_n^{(1)}(\xi_s^0, q_s)} \tilde{Q}_{kn}^{(1)}(-s, s; \beta) {}^{-s} B_k^{m-1} \right\}, \end{aligned} \tag{8.3.10}$$

$$ {}^{s}B_n^m = -2\sum_{k=0}^{\infty}\left\{\frac{c_k(q_{-s})\,\mathrm{Ce}_k(\xi^0_{-s},q_{-s})}{d_n(q_s)\,\mathrm{Ne}_n^{(1)}(\xi^0_s,q_s)}\,R_{kn}^{(1)}(-s,\,s;\,\beta)\,{}^{-s}A_k^{m-1}+\right. $$

$$ \left.+\frac{d_k(q_{-s})\,\mathrm{Se}_k(\xi^0_{-s},q_{-s})}{d_n(q_s)\,\mathrm{Ne}_n^{(1)}(\xi^0_s,q_s)}\,\tilde{R}_{kn}^{(1)}(-s,\,s,\,\beta)\,{}^{-s}B_k^{m-1}\right\}, \qquad (8.3.11) $$

in which

$$ {}^{s}A_n^1 = -e^{ikl_s\cos\alpha}\,\frac{\mathrm{ce}_n(\alpha_s,q_s)}{\mathrm{Me}_n^{(1)}(\xi^0_s,q_s)}, \qquad (8.3.12) $$

$$ {}^{s}B_n^1 = -e^{ikl_s\cos\alpha}\,\frac{\mathrm{se}_n(\alpha_s,q_s)}{\mathrm{Ne}_n^{(1)}(\xi^0_s,q_s)}. \qquad (8.3.13) $$

Formulas (8.3.10) - (8.3.13) are obtained from (8.3.7) using the addition theorems for wave functions of an elliptical cylinder (4.5.7), (4.5.8). The scattering coefficients (8.3.12), (8.3.13) determine the diffraction waves of first order of scattering without taking into account the mutual interaction of the cylinders.

The values $Q_{kn}^{(1)}$, $\tilde{Q}_{kn}^{(1)}$, $R_{kn}^{(1)}$ and $\tilde{R}_{kn}^{(1)}$ which appear in expressions (8.3.12), (8.3.13) are the same as in (4.5.12) - (4.5.15) if we set there $\gamma = \alpha_{-1,+1}$, and $\alpha_{+1,-1} = \gamma + \beta + \pi$.

Series (8.3.10), (8.3.11) converge well only in the case when the distance l between the cylinders is much larger than the wave length of excitation, and the geometric dimensions of the cross section ellipses of the cylinders are small in comparison with it ($kl \gg 1$, but $q \ll 1$). Then

$$ |{}^{s}u^1| \gg |{}^{s}u^2| \gg |{}^{s}u^3| \gg \ldots, \qquad (8.3.14) $$

and in this case we can confine ourselves to the secondary fields of the first order of scattering, by ignoring the secondary waves of higher orders of scattering. However, with a decrease in distance between the cylinders, in computing the diffraction effect at the observation point, it is essential in addition to the waves of first order of scattering that the secondary waves of the subsequent increasing orders of scattering be taken into account, thus leading /421 to considerable difficulty in the numerical solution to the problem.

If we seek the potentials u_s of the particular fields in the form of a series

$$ u_s = \frac{2E}{k^2}\sum_{n=0}^{\infty}\left[a_n^s c_n\,\mathrm{Me}_n^{(1)}(\xi_s,\,q_s)\,\mathrm{ce}_n(\eta_s,\,q_s)+\right. $$

$$ \left.+\,b_n^s d_n\,\mathrm{Ne}_n^{(1)}(\xi_s,\,q_s)\,\mathrm{se}_n(\eta_s,\,q_s)\right] \qquad (8.3.15) $$

($s = \pm 1$) with the still undetermined coefficients a_n^s and b_n^s, then after substituting them by new unknowns A_n^s and B_n^s according to the formulas,

$$a_n^s = c_n \operatorname{Ce}_n(\xi_s^0, q_s) A_n^s, \quad b_n^s = d_n \operatorname{Se}_n(\xi_s^0, q_s) B_n^s \tag{8.3.16}$$

and applying the addition theorems (4.5.7), (4.5.8) from the boundary condition (1.7.7) on the surface of each cylinder we obtain for A_n^s and B_n^s an infinite system of linear equations,

$$\begin{aligned} A_n^s + \sum_{m=0}^{\infty} \{ \alpha_{nm}^{(-s,s)} A_m^{-s} + \beta_{nm}^{(-s,s)} B_m^{-s} \} &= F_n^s, \\ B_n^s + \sum_{m=0}^{\infty} \{ \gamma_{nm}^{(-s,s)} A_m^{-s} + \delta_{nm}^{(-s,s)} B_m^{-s} \} &= \Phi_n^s, \\ s = \pm 1;\ n = 0,\ 1,\ 2,\ \ldots, & \end{aligned} \tag{8.3.17}$$

where

$$\begin{aligned} \alpha_{nm}^{(-s,s)} &= 2 \frac{c_m \operatorname{Ce}_m(\xi_{-s}^0, q_{-s})}{c_n \operatorname{Me}_n^{(1)}(\xi_s^0, q_s)} Q_{mn}^{(1)}(-s, s; \beta), \\ \beta_{nm}^{(-s,s)} &= 2 \frac{d_m \operatorname{Se}_m(\xi_{-s}^0, q_{-s})}{c_n \operatorname{Me}_n^{(1)}(\xi_s^0, q_s)} \tilde{Q}_{mn}^{(1)}(-s, s; \beta), \\ \gamma_{nm}^{(-s,s)} &= 2 \frac{c_m \operatorname{Ce}_m(\xi_{-s}^0, q_{-s})}{d_n \operatorname{Ne}_n^{(1)}(\xi_s, q_s)} R_{mn}^{(1)}(-s, s; \beta), \\ \delta_{nm}^{(-s,s)} &= 2 \frac{d_m \operatorname{Se}_m(\xi_{-s}^0, q_{-s})}{d_n \operatorname{Ne}_n^{(1)}(\xi_s^0, q_s)} \tilde{R}_{mn}^{(1)}(-s, s; \beta), \end{aligned} \tag{8.3.18}$$

$$\begin{aligned} F_n^s &= -\frac{e^{iksl_0 \cos\alpha} \operatorname{ce}_n(\alpha_s, q_s)}{c_n \operatorname{Me}_n^{(1)}(\xi_s^0, q_s)}, \\ \Phi_n^s &= -\frac{e^{iksl_0 \cos\alpha} \operatorname{se}_n(\alpha_s, q_s)}{d_n \operatorname{Ne}_n^{(1)}(\xi_s^0, q_s)} \end{aligned} \tag{8.3.19}$$

($\alpha_{-1,+1} = \gamma$, $\alpha_{+1,-1} = \gamma + \pi + \beta$, $\alpha_{-1} = \gamma + \alpha$, $\alpha_{+1} = \alpha + \gamma + \beta$). After introduction of new definitions for the unknowns A_n^s and B_n^s,

$$\begin{aligned} z_{|4i-1|} &= A_i^{+1}, \ z_{|4i-1|+1} = A_i^{-1}, \ i = 0,\ 1,\ \ldots, \\ z_{4i+1} &= B_i^{+1}, \ z_{4i+2} = B_i^{-1}, \ i = 1,\ 2,\ \ldots, \end{aligned} \tag{8.3.20}$$

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the system (8.3.17) may be written in normal form,

$$z_i + \sum_{j=1}^{\infty} c_{ij} z_j = f_i, \quad i = 1, 2, \ldots, \tag{8.3.21}$$

in which

$$\begin{aligned} f_{|4l-1|} &= F_l^{+1}, \; f_{|4l-1|+1} = F_l^{-1}, \quad i = 0, 1, \ldots, \\ f_{4l+1} &= \Phi_l^{+1}, \; f_{4l+2} = \Phi_l^{-1}, \quad i = 0, 1, \ldots, \end{aligned} \tag{8.3.22}$$

and

$$\begin{gathered} c_{|4l-1|,|4j-1|+1} = \alpha_{lj}^{(-1,+1)}, \; c_{|4l-1|+1,|4j-1|} = \alpha_{lj}^{(+1,-1)}, \; i, j = 0, 1, \ldots, \\ c_{|4l-1|,4j+2} = \beta_{lj}^{(-1,+1)}, \; c_{|4l-1|+1,4j+1} = \beta_{lj}^{(+1,-1)}, \\ i = 0, 1, \ldots; \; j = 1, 2, \ldots, \\ c_{4l+1,|4j-1|+1} = \gamma_{lj}^{(-1,+1)}, \; c_{4l+2,|4j-1|} = \gamma_{lj}^{(+1,-1)}, \\ i = 1, 2, \ldots; \; j = 0, 1, \ldots, \\ c_{4l+1,4j+2} = \delta_{lj}^{(-1,+1)}, \; c_{4l+2,4j+1} = \delta_{lj}^{(+1,-1)}, \; i, j = 1, 2, \ldots, \\ c_{ij} \equiv 0, \text{ if } |i-j| = 2m, \; m \text{ is a whole number.} \end{gathered} \tag{8.3.23}$$

The determinant of the system (8.3.21) has the form,

$$\Delta = \begin{vmatrix} 1 & \alpha_{00}^{-}0 & \alpha_{01}^{-}0 & \beta_{01}^{-}0 & \alpha_{02}^{-}0 & \beta_{02}^{-}0 & \alpha_{03}^{-}0 & \beta_{03}^{-}0 & \alpha_{04}^{-} \cdots \\ \alpha_{00}^{+}1 & \alpha_{01}^{+}0 & \beta_{01}^{+}0 & \alpha_{02}^{+}0 & \beta_{02}^{+}0 & \alpha_{03}^{+}0 & \beta_{03}^{+}0 & \alpha_{04}^{+}0 & \cdots \\ 0 & \alpha_{10}^{-}1 & \alpha_{11}^{-}0 & \beta_{11}^{-}0 & \alpha_{12}^{-}0 & \beta_{12}^{-}0 & \alpha_{13}^{-}0 & \beta_{13}^{-}0 & \alpha_{14}^{-} \cdots \\ \alpha_{10}^{+}0 & \alpha_{11}^{+}1 & \beta_{11}^{+}0 & \alpha_{12}^{+}0 & \beta_{12}^{+}0 & \alpha_{13}^{+}0 & \beta_{13}^{+}0 & \alpha_{14}^{+}0 & \cdots \\ 0 & \gamma_{10}^{-}0 & \gamma_{11}^{-}1 & \delta_{11}^{-}0 & \gamma_{12}^{-}0 & \delta_{12}^{-}0 & \gamma_{13}^{-}0 & \delta_{13}^{-}0 & \gamma_{14}^{-} \cdots \\ \gamma_{10}^{+}0 & \gamma_{11}^{+}0 & \delta_{11}^{+}1 & \gamma_{12}^{+}0 & \delta_{12}^{+}0 & \gamma_{13}^{+}0 & \delta_{13}^{+}0 & \gamma_{14}^{+}0 & \cdots \\ 0 & \alpha_{20}^{-}0 & \alpha_{21}^{-}0 & \beta_{21}^{-}1 & \alpha_{22}^{-}0 & \beta_{22}^{-}0 & \alpha_{23}^{-}0 & \beta_{23}^{-}0 & \alpha_{24}^{-} \cdots \\ \alpha_{20}^{+}0 & \alpha_{21}^{+}0 & \beta_{21}^{+}0 & \alpha_{22}^{+}1 & \beta_{22}^{+}0 & \alpha_{23}^{+}0 & \beta_{23}^{+}0 & \alpha_{24}^{+}0 & \cdots \\ 0 & \gamma_{20}^{-}0 & \gamma_{21}^{-}0 & \delta_{21}^{-}0 & \gamma_{22}^{-}1 & \delta_{22}^{-}0 & \gamma_{23}^{-}0 & \delta_{23}^{-}0 & \gamma_{24}^{-} \cdots \\ \gamma_{20}^{+}0 & \gamma_{21}^{+}0 & \delta_{21}^{+}0 & \gamma_{22}^{+}0 & \delta_{22}^{+}1 & \gamma_{23}^{+}0 & \delta_{23}^{+}0 & \gamma_{24}^{+}0 & \cdots \\ 0 & \alpha_{30}^{-}0 & \alpha_{31}^{-}0 & \beta_{31}^{-}0 & \alpha_{32}^{-}0 & \beta_{32}^{-}1 & \alpha_{33}^{-}0 & \beta_{33}^{-}0 & \alpha_{34}^{-} \cdots \\ \alpha_{30}^{+}0 & \alpha_{31}^{+}0 & \beta_{31}^{+}0 & \alpha_{32}^{+}0 & \beta_{32}^{+}0 & \alpha_{33}^{+}1 & \beta_{33}^{+}0 & \alpha_{34}^{+}0 & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \\ \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdot\cdot & \cdots \end{vmatrix}$$

$\alpha_{ij}^{-} = \alpha_{ij}^{(-1,+1)}$, $\alpha_{ij}^{+} = \alpha_{ij}^{(+1,-1)}$, $\beta_{ij}^{-} = \beta_{ij}^{(-1,+1)}$, $\beta_{ij}^{+} = \beta_{ij}^{(+1,-1)}$, $\gamma_{ij}^{-} = \gamma_{ij}^{(-1,+1)}$, $\gamma_{ij}^{+} = \gamma_{ij}^{(+1,-1)}$, $\delta_{ij}^{-} = \delta_{ij}^{(-1,+1)}$, $\delta_{ij}^{+} = \delta_{ij}^{(+1,-1)}$.

With increase in kl all c_{ij} tend to zero, and $\Delta \to 1$. /43

Using formulas (4.3.7), (4.2.30), (4.2.31) and (2.2.26) we can establish that the matrix elements of the system (8.3.21) [or equally that systems (8.3.17)] satisfy condition (1.9.19), and its right-hand sides satisfy condition (1.9.20) (if the cylinders do not touch one another). Therefore, on the basis of the Gilbert alternative, system (8.3.21) is uniquely solvable on the strength of the uniqueness of the solution to this problem and[3]

$$\sum_{l=1}^{\infty} |z_l|^2 < \infty. \tag{8.3.24}$$

Using (8.3.24) and (8.3.16), (8.3.20), (4.3.7), (4.2.29) we can be sure of the absolute and uniform convergence of series (8.3.14) (for each q_s) which give the solution to the mathematical problem for the potential u^1. The function u^1 on the basis of (8.3.16) is written in the form,

$$u^1 = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \{A_n^s c_n^2 \mathrm{Ce}_n(\xi_s^0, q_s) \mathrm{Me}_n^{(1)}(\xi_s, q_s) \mathrm{ce}_n(\eta_s, q_s) + B_n^s d_n^2 \mathrm{Se}_n(\xi_s^0, q_s) \mathrm{Ne}_n^{(1)}(\xi_s, q_s) \mathrm{se}_n(\eta_s, q_s)\}. \tag{8.3.25}$$

In the vicinity of the wave band in polar coordinates ρ, ϕ with the origin at point O, the components of the field scattered on the diffracting system are determined by the expressions

$$E_z^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\eta^1 = \frac{e^{ik\rho}}{\sqrt{\rho}} A(\varphi), \tag{8.3.26}$$

where the amplitude function

$$A(\varphi) = 2E \sqrt{\frac{2}{\pi k}}\, e^{-\frac{i\pi}{4}} \sum_{s=\pm 1} e^{-iskl_0 \cos\varphi} \sum_{n=0}^{\infty} \{A_n^s c_n \times \mathrm{Ce}_n(\xi_s^0, q_s) \mathrm{ce}_n(\eta_s, q_s) + B_n^s d_n \mathrm{Se}_n(\xi_s^0, q_s) \mathrm{se}_n(\eta_s, q_s)\} \tag{8.3.27}$$

(here $\eta_{-1} = \phi + \gamma$, and $\eta_{+1} = \phi + \gamma + \beta$ and $\rho = fe^{\xi}/2$). The value of the linear cross section of back scattering (radar cross section) for the two ideally conducting parallel elliptical cylinders is computed from the formula

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[3] In fact, $\sum_{l=1}^{\infty} |z_l| < \infty$, since we can show that $|z_l| < \mathrm{const}\left(\frac{a_{-s}}{l - a_s}\right)^l$, where a_s are the major semiaxes of the cross section ellipses of the cylinders, $s = \pm 1$.

$$\sigma_B = \frac{16}{k}\left| \sum_{s=\pm 1} e^{iskl_0 \cos\alpha} \sum_{n=0}^{\infty} (-1)^n \{A_n^s c_n \mathrm{Ce}_n(\xi_s^0, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + \right.$$
$$\left. + B_n^{si} d_n \mathrm{Se}_n(\xi_s^0, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}\right|^2, \qquad (8.3.28)$$

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with degeneration of the cylinders into bands with a width $2f_{-1}$ and $2f_{+1}$, respectively ($\eta_{-1} = \alpha + \gamma$, $\eta_{+1} = \alpha + \gamma + \beta$)

$$\sigma_B = \frac{16}{k}\left| \sum_{s=\pm 1} e^{iskl_0 \cos\alpha} \sum_{n=0}^{\infty} (-1)^n A_n^s c_n \times \right.$$
$$\left. \times \mathrm{Ce}_n(0, q_s)\,\mathrm{ce}_n(\eta_s, q_s)\right|^2 \qquad (8.3.29)$$

Simpler expressions for σ_B are obtained for example, with values of the angle α, equal to 0, $\pi/2$; when $q_{-1} = q_{+1}$, etc. The complete linear cross section of the scattering for the two parallel ideally conducting elliptical cylinders may be computed from the formula,

$$\sigma_s = \int_0^{2\pi} |A(\varphi)|^2 d\varphi, \qquad (8.3.30)$$

where the amplitude function $A(\phi)$ is determined by expression (8.3.27). In the particular case of two identical elliptical cylinders with parallel respective axes of their cross section ellipses ($\beta = 0$)

$$\sigma_s = \frac{8}{k} \mathrm{Re}\left\{ \sum_{n=0}^{\infty} [|a_n^{+1}|^2 + |b_n^{+1}|^2 + |a_n^{-1}|^2 + |b_n^{-1}|^2] + \right.$$
$$+ \sum_{n,m=0}^{\infty} a_n^{-1} a_m^{+1*} p_{nm} + \sum_{n,m=1}^{\infty} b_n^{-1} b_m^{+1*} r_{nm} + \qquad (8.3.31)$$
$$\left. + \sum_{n,m=0}^{\infty} a_n^{+1} a_m^{-1*} q_{nm} + \sum_{n,m=1}^{\infty} b_n^{+1} b_m^{-1*} s_{nm}\right\},$$

where

$$p_{nm} = \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} A_r^{(n)} A_t^{(m)} \{i^{r-t} J_{r-t}(kl)\cos(r-t)\gamma +$$
$$+ i^{r+t} J_{r+t}(kl)\cos(r+t)\gamma\},$$
$$r_{nm} = \sum_{r=0}^{\infty}\sum_{t=0}^{\infty} B_r^{(n)} B_t^{(m)} \{i^{r-t} J_{r-t}(kl)\cos(r-t)\gamma -$$

$$-i^{r+t}J_{r+t}(kl)\cos(r+t)\gamma\},$$

$$q_{nm}=\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}A_r^{(n)}A_t^{(m)}\{i^{t-r}J_{r-t}(kl)\cos(r-t)\gamma+$$

$$+i^{-r-t}J_{r+t}(kl)\cos(r+t)\gamma\}, \qquad (8.3.32)$$

$$s_{nm}=\sum_{r=0}^{\infty}\sum_{t=0}^{\infty}B_r^{(n)}B_t^{(m)}\{i^{t-r}J_{r-t}(kl)\cos(r-t)\gamma-$$

$$-i^{-r-t}J_{r+t}(kl)\cos(r+t)\gamma\}.$$

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Here $A_r^{(n)}$ and $B_r^{(n)}$ are the coefficients of the Mathieu functions, $q_{-1}=q_{+1}$. With degeneration of the cylinders into bands with a width $2f$ each, the formula

$$\sigma_s=\frac{8}{k}\operatorname{Re}\left\{\sum_{n=0}^{\infty}\left[|a_n^{+1}|^2+|a_n^{-1}|^2\right]+\right.$$

$$\left.+\sum_{n,m=0}^{\infty}\left[a_n^{-1}a_m^{+1*}p_{nm}+a_n^{+1}a_m^{-1*}q_{nm}\right]\right\}$$

will determine the value of the complete cross section (linear) for the two ideally conducting parallel bands ($\beta = 0$).

Using the addition theorems (4.5.7), (4.5.8) we can obtain for the components of the complete field the expressions which are suitable for analysis of the field near the cylinders (in the region for which $l > f_s \operatorname{ch}\xi_s$, where l is the distance between centers of the cylinders). Thus, by applying (4.5.7) and (4.5.8) to (8.3.25) and taking (8.3.3), (8.3.17) into account we obtain after several transformations the following representation for the potential of the complete field in local coordinates of the s-th cylinder:

$$u=\frac{E}{k^2}\sum_{n=0}^{\infty}\{c_n^2A_n^s[\operatorname{Ce}_n(\xi_s^0,q_s)\operatorname{Me}_n^{(1)}(\xi_s,q_s)-$$

$$-\operatorname{Se}_n(\xi_s,q_s)\operatorname{Me}_n^{(1)}(\xi_s^0,q_s)]\operatorname{ce}_n(\eta_s,q_s)+d_n^2B_n^s\times$$

$$\times[\operatorname{Ce}_n(\xi_s^0,q_s)\operatorname{Ne}_n^{(1)}(\xi_s,q_s)-\operatorname{Se}_n(\xi_s,q_s)\operatorname{Ne}_n^{(1)}(\xi_s^0,q_s)]\operatorname{se}_n(\eta_s,q_s)\}$$

If we substitute this expression into

$$H_\xi=-ik_0\varepsilon\frac{1}{f\sqrt{\operatorname{ch}^2\xi_s-\cos^2\eta_s}}\cdot\frac{\partial u}{\partial\eta_s},$$

$$H_\eta = ik_0\varepsilon \frac{1}{f\sqrt{\operatorname{ch}^2\xi_s - \cos^2\eta_s}} \cdot \frac{\partial u}{\partial \xi_s}, \quad s = \pm 1,$$
$$E_z = k^2 u,$$

we find formulas for computing the field in the neighboring band. Hence we also find formulas which determine the rule of density distribution of the surface currents induced on the surface of the s-th cylinder which takes into account the interaction of the cylinders. Since on the surface of the s-th cylinder, (1.1.7) is valid, we then find

$$j_\xi = j_\eta = 0,$$

but

$$j_z = -\sqrt{\frac{\varepsilon}{\mu}} \frac{cE}{2\pi^2} \cdot \frac{1}{kf\sqrt{\operatorname{ch}^2\xi_s^0 - \cos^2\eta_s}} \sum_{n=0}^{\infty} (-1)^n \{A_n^s \operatorname{ce}_n(\eta_s, q_s) + B_n^s \operatorname{se}_n(\eta_s, q_s)\}.$$

Finding the numerical values of the quantities which determine the diffracted field first involves computation of the unknowns A_n^s, B_n^s from system (8.3.21). Their approximate values may be computed with the required degree of accuracy from the truncated systems

$$z_i + \sum_{j=1}^{4N+2} c_{ij} z_j = f_i, \quad i = 1, 2, \ldots, 4N+2; \; N = 0, 1, \ldots, \qquad (8.3.33)$$

obtained from (8.3.21) for the various increasing values of N, the order of truncation. As the original value of N we can take the number $N \approx 2ka$, where $a = \max_{s=\pm 1} a_s$ (a_s is the length of the major semiaxis of the s-th ellipse). The matrix of the coefficient of the truncated system (8.3.33) has the form,

$$\begin{vmatrix}
1 & c_{12} & 0 & c_{14} & \ldots \\
c_{21} & 1 & c_{13} & 0 & \ldots \\
0 & c_{32} & 1 & c_{34} & \ldots \\
c_{41} & 0 & c_{43} & 1 & \ldots \\
\cdot & \cdot & \cdot & \cdot & \ldots \\
\cdot & \cdot & \cdot & \cdot & \ldots \\
\cdot & \cdot & \cdot & \cdot & \ldots \\
0 & c_{4N-1,2} & 0 & c_{4N-1,4} & \ldots \\
c_{4N,1} & 0 & c_{4N,3} & 0 & \ldots \\
0 & c_{4N+1,2} & 0 & c_{4N+1,4} & \ldots \\
c_{4N+2,1} & 0 & c_{4N+2,3} & 0 & \ldots
\end{vmatrix}$$

$$\begin{vmatrix}
\cdots & 0 & c_{1,4N} & 0 & c_{1,4N+2} \\
\cdots & c_{2,4N-1} & 0 & c_{2,4N+1} & 0 \\
\cdots & 0 & c_{3,4N} & 0 & c_{3,4N+2} \\
\cdots & c_{4,4N-1} & 0 & c_{4,4N+1} & 0 \\
\cdots & \cdot & \cdot & \cdot & \cdot \\
\cdots & \cdot & \cdot & \cdot & \cdot \\
\cdots & \cdot & \cdot & \cdot & \cdot \\
\cdots & 1 & c_{4N-1,4N} & 0 & c_{4N-1,4N+2} \\
\cdots & c_{4N,4N-1} & 1 & c_{4N,4N+1} & 0 \\
\cdots & 0 & c_{4N+1,4N} & 1 & c_{4N+1,4N+2} \\
\cdots & c_{4N+2,4N-1} & 0 & c_{4N+2,4N+1} & 1
\end{vmatrix} \qquad (8.3.34)$$

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where c_{ij} are expressed through (8.3.18) by formulas (8.3.23). Since when $a/l \to 0$ ($kl >> 1$, $l \to \infty$) all $a_{ij} \to 0$, then when $kl >> 1$, the determinant of system (8.3.3) differs little from unity in absolute value.

Seeking the approximate values of the diffraction coefficients A_n^s and B_n^s involves solution of $4N+2$ equations with $4N$ unknowns, whose matrix elements and right-hand sides are complex-valued. With comparatively small ratios of the diameters of the cross section ellipses of the cylinders to the wave length of excitation (q is of the order of several units and less) the numerical solution to the problem can be obtained with the aid of the simplest calculators and the available Tables of Mathieu Functions [43, 183-185, 188], Bessel Functions [179] and others. As is clear from (8.3.13)-(8.3.18), the basic problems (of a technical order) may arise in computing the matrix elements of the system, containing the coefficients $Q_{mn}^{(1)}$, $\tilde{Q}_{mn}^{(1)}$, $R_{mn}^{(1)}$ and $\tilde{R}_{mn}^{(1)}$, having a comparatively complex structure [see formulas (4.5.13) - (4.5.16)]. As noted in §2, Chapter 4, when $q \to 0$ all the Mathieu coefficients $A_r^{(n)}$ and $B_r^{(n)}$ tend to zero, with the exception only of the coefficients $A_n^{(n)}$ and $B_n^{(n)}$, which tend to unity. Hence it follows that with small values of q ($q > 0$) the coefficients $A_n^{(n)}$ and $B_n^{(n)}$ will be dominant values. Therefore in computing the coefficients (4.5.13) - (4.5.16) in the series it is feasible to begin the summation over r and p from values of $r = n$ and $p = m$. It follows to remember also that the summation in (4.5.13) - (4.5.16) is carried out over the indices r and p, having the same parity as n and m, respectively.

In a number of cases, because of the choice of special values of the parameters appearing in the problem, we may significantly simplify the matrix elements (8.3.21) of system (8.3.33) and substitute the solution to system (8.3.33) by the solution to equivalent

systems with a smaller number of equation for the same N.

1. Let $\beta = 0$ (the corresponding semiaxes of the cross section ellipses of the cylinders with the plane Oxy are parallel, Fig. 52,b). Then as is obvious from (4.5.13) - (4.5.16) which appear in (8.3.18) the expressions $Q_{mn}^{(1)}$, $\tilde{Q}_{mn}^{(1)}$, $R_{mn}^{(1)}$ and $\tilde{R}_{mn}^{(1)}$ are simplified, assuming the form /428

$$Q_{mn}^{(1)}(-s,\ s;\ 0) = \frac{1}{2}\sum_{r=0}^{\infty}\sum_{p=0}^{\infty} A_r^{(n)}(q_s)\,A_p^{(m)}(q_{-s})\times$$
$$\times\left\{i^{r-p}H_{r-p}^{(1)}(kl)\cos(r-p)\alpha_{-s,s} + i^{r+p}H_{r+p}^{(1)}(kl)\cos(r+p)\alpha_{-s,s}\right\},$$
$$\tilde{Q}_{mn}^{(1)}(-s,\ s;\ 0) = -\frac{1}{2}\sum_{r=0}^{\infty}\sum_{p=1}^{\infty} A_r^{(n)}(q_s)\,B_p^{(m)}(q_{-s})\times$$
$$\times\left\{i^{r-p}H_{r-p}^{(1)}(kl)\sin(r-p)\alpha_{-s,s} - i^{r+p}H_{r+p}^{(1)}(kl)\sin(r+p)\alpha_{-s,s}\right\}, \qquad (8.3.35)$$
$$R_{mn}^{(1)}(-s,\ s;\ 0) = \frac{1}{2}\sum_{r,p} B_r^{(n)}(q_s)\,A_p^{(m)}(q_{-s})\times$$
$$\times\left\{i^{r-p}H_{r-p}^{(1)}(kl)\sin(r-p)\alpha_{-s,s} + i^{r+p}H_{r+p}^{(1)}(kl)\sin(r+p)\alpha_{-s,s}\right\},$$
$$\tilde{R}_{mn}^{(1)}(-s,\ s;\ 0) = \frac{1}{2}\sum_{r,p} B_r^{(n)}(q_s)\,B_p^{(m)}(q_{-s})\times$$
$$\times\left\{i^{r-p}H_{r-p}^{(1)}(kl)\cos(r-p)\alpha_{-s,s} - i^{r+p}H_{r+p}^{(1)}(kl)\cos(r+p)\alpha_{-s,s}\right\},$$

where $\alpha_{-1,+1} = \gamma$, and $\alpha_{+1,-1} = \gamma + \pi$. Since $\cos n(\alpha + \pi) = (-1)^n \times \cos n\alpha$, and the parity of the indices of summation r and p in (8.3.35) agree with the parity of the indices n and m, respectively, then it is obvious that in the case $q_{-1} = q_{+1}$ (the interfocal distances of the cross section ellipses of the cylinders are equal) we find the following relationships to be satisfied:

$$\begin{aligned}
Q_{mn}^{(1)}(-s,\ s;\ 0) &= (-1)^{n+m}Q_{mn}^{(1)}(s,\ -s;\ 0),\\
\tilde{Q}_{mn}^{(1)}(-s,\ s;\ 0) &= (-1)^{m+n}\tilde{Q}_{mn}^{(1)}(s,\ -s;\ 0),\\
R_{mn}^{(1)}(-s,\ s;\ 0) &= (-1)^{n+m}R_{mn}^{(1)}(s,\ -s;\ 0),\\
\tilde{R}_{mn}^{(1)}(-s,\ s;\ 0) &= (-1)^{n+m}\tilde{R}_{mn}^{(1)}(s,\ -s;\ 0),
\end{aligned} \qquad (8.3.36)$$

on the basis of which when $\xi_{-1}^{0} = \xi_{+1}^{0}$ (the ellipses of the cross section of the cylinders are mutually equal) for the matrix elements of system (8.3.17) we find the equations

$$\alpha_{nm} = \alpha_{nm}^{(-1,+1)} = (-1)^{n+m}\alpha_{nm}^{(+1,-1)},$$

$$\beta_{nm}^{(-1,+1)} = \beta_{nm}^{(-1,+1)}(-1)^{n+m} = \beta_{nm},$$
$$\gamma_{nm} = \gamma_{nm}^{(-1,+1)} = (-1)^{n+m}\gamma_{nm}^{(+1,-1)}, \qquad (8.3.37)$$
$$\delta_{nm}^{(-1,+1)} = (-1)^{n+m}\delta_{nm}^{(+1,-1)} = \delta_{nm}.$$

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Using (8.3.37), system (8.3.17) is reduced to an equivalent system of equations:

$$\left.\begin{aligned} a_n + \sum_{m=0}^{\infty}\{(-1)^m \alpha_{nm} a_m + (-1)^m \beta_{nm} b_m\} = f_n, \\ b_n + \sum_{m=0}^{\infty}\{(-1)^m \gamma_{nm} a_m + (-1)^m \delta_{nm} b_m\} = \varphi_n, \end{aligned}\right\} \qquad (8.3.38)$$

$$\left.\begin{aligned} c_n - \sum_{m=0}^{\infty}[\alpha_{nm} c_m + \beta_{nm} d_m](-1)^m = g_n, \\ d_n - \sum_{m=0}^{\infty}[\gamma_{nm} c_m + \delta_{nm} d_m](-1)^m = e_n, \end{aligned}\right\} \qquad (8.3.39)$$

in which

$$\begin{aligned} a_n = A_n^{+1} + (-1)^n A_n^{-1},\ c_n = A_n^{+1} - (-1)^n A_n^{-1}, n = 0,\ 1,\ \ldots, \\ b_n = B_n^{+1} + (-1)^n B_n^{-1},\ d_n = B_n^{+1} - (-1)^n B_n^{-1},\quad n = 1,\ 2,\ \ldots, \\ f_n = F_n^{+1} + (-1)^n F_n^{-1},\ g_n = F_n^{+1} - (-1)^n F_n^{-1},\quad n = 0,\ 1,\ \ldots, \\ \varphi_n = \Phi_n^{+1} + (-1)^n \Phi_n^{-1},\quad e_n = \Phi_n^{+1} - (-1)^n \Phi_n^{-1},\quad n = 1,\ 2,\ \ldots \end{aligned} \qquad (8.3.40)$$

Systems (8.3.38) and (8.3.39) are solved independently of one another. If we denote

$$\begin{aligned} a_i = z_{2i},\quad c_i = u_{2i},\quad i = 1,\ 2,\ \ldots, \\ b_i = z_{2i+1},\ d_i = u_{2i+1},\quad i = 1,\ 2,\ \ldots, \end{aligned} \qquad (8.3.41)$$

$$c_0 = u_1,\ a_0 = z_1, \qquad (8.3.42)$$

then they are written in the form

$$z_i + \sum_{j=1}^{\infty} a_{ij} z_j = \tilde{f}_i, \qquad (8.3.43)$$

$$u_i - \sum_{j=1}^{\infty} b_{ij} u_j = \tilde{\varphi}_i, \qquad (8.3.44)$$

$$i = 1,\ 2,\ \ldots,$$

where /430

$$f_i = \tilde{f}_{2i},\ g_i = \varphi_{2i},$$
$$\varphi_i = \tilde{f}_{2i+1},\ e_i = \tilde{\varphi}_{2i+1},\quad i = 1,\ 2,\ \ldots,$$
$$f_0 = \tilde{f}_1,\ g_0 = \tilde{\varphi}_1, \qquad (8.3.45)$$

but

$$a_{2i,2j} = (-1)^j \alpha_{ij},\ a_{2i+1,2j} = (-1)^j \gamma_{ij},\ a_{11} = \alpha_{00},\ a_{1,2j+1} = (-1)^j \beta_{1j},$$
$$a_{2i,2j+1} = (-1)^j \beta_{ij},\ a_{2i+1,2j+1} = (-1)^j \delta_{ij},$$
$$a_{2i+1,1} = -\gamma_{i1},\ a_{2i,1} = -d_{i1},$$
$$a_{1,2j} = (-1)^j \alpha_{1j},\ i, j = 1,\ 2,\ \ldots$$

The matrix elements b_{ij} are determined in the same manner. They will be represented by the truncated systems,

$$z_i + \sum_{j=1}^{2N+1} a_{ij} z_j = \tilde{f}_i,$$
$$u_i - \sum_{j=1}^{2N+1} b_{ij} u_j = \tilde{\varphi}_i, \qquad (8.3.46)$$
$$i = 1,\ 2,\ \ldots,\ 2N+1;\ N = 1,\ 2,\ \ldots,$$

each of which is a system of the order of $2N+1$, i.e., half the order of system (8.3.33) with the same value of the parameter ka.

2. A still greater simplification can be attained in the case when the cylinders are located such that the major semiaxes of the ellipses of their cross section are found to lie on one straight line, i.e., the Ox axis (Fig. 52,c). Then $\gamma = \beta = 0$, and $\alpha_{-1,+1} = 0$, $\alpha_{+1,-1} = \pi$ and in (8.3.18) we will have

$$\tilde{Q}_{mn}^{(1)}(-s,\ s;\ 0) = R_{mn}^{(1)}(-s,\ s,\ 0) \equiv 0,$$

by virtue of which we also have

$$\beta_{nm}^{(-s,s)} = \gamma_{nm}^{-s,s} = 0. \qquad (8.3.47)$$

As a result, system (8.3.17) is divided into two independent systems of equations,

$$A_n^s + \sum_{m=0}^{\infty} \alpha_{nm}^{(-s,s)} A_m^{-s} = F_n^s \qquad (8.3.48)$$

and

$$B_n^s + \sum_{m=1}^{\infty} \delta_{nm}^{(-s,s)} B_m^{-s} = \Phi_n^s, \tag{8.3.49}$$

the first of which, with truncation to $n, m = 0, 1, \ldots, N$, is a system of $2N + 2$-nd, and the second with truncation to $n, m = \pm 1, 2, \ldots, N$ is a system of $2N$-th order relative to the unknowns $A_n^{\pm 1}$ and $B_n^{\pm 1}$. In the case, if $q_{-1} = q_{+1}$ and $\xi_{-1}^0 = \xi_{+1}^0$, systems (8.3.48) and (8.3.49) are reduced, respectively, to the form /431

$$\begin{aligned} z_n + \sum_{m=0}^{\infty} (-1)^m \alpha_{nm} z_m &= f_n, \\ u_n - \sum_{m=0}^{\infty} (-1)^m \alpha_{nm} u_m &= \tilde{f}_n, \end{aligned} \tag{8.3.50}$$

where

$$\begin{aligned} z_n = A_n^{+1} + (-1)^n A_n^{-1},\ u_n = A_n^{+1} - (-1)^n A_n^{-1}, \\ f_n = F_n^{+1} + (-1)^n F_n^{-1},\ \tilde{f}_n = F_n^{+1} - (-1)^n F_n^{-1}, \end{aligned} \tag{8.3.51}$$

and to the form

$$\begin{aligned} c_n + \sum_{m=1}^{\infty} (-1)^m \delta_{nm} c_m &= \varphi_n, \\ d_n - \sum_{m=1}^{\infty} (-1)^m \delta_{nm} d_m &= \tilde{\varphi}_n, \end{aligned} \tag{8.3.52}$$

where

$$\begin{aligned} c_n = B_n^{+1} + (-1)^n B_n^{-1},\ d_n = B_n^{+1} - (-1)^n B_n^{-1}, \\ \varphi_n = \Phi_n^{+1} + (-1)^n \Phi_n^{-1},\ \tilde{\varphi}_n = \Phi_n^{+1} - (-1)^n \Phi_n^{-1}. \end{aligned} \tag{8.3.53}$$

With truncation of the systems (8.3.50), (8.3.52) to $n, m = 0, 1, \ldots, N - 1$ for seeking the approximate values of z_n, u_n and c_n, d_n we find systems of N and $N - 1$ equations with N and $N - 1$ unknowns, respectively.

3. An important particular case is that when the diffraction system consists of two bands (elliptical cylinders are degenerated into bands with a width of $2f_{-1}$ and $2f_{+1}$, respectively), whose surfaces are determined by the equations $\xi_s^0 = 0$, $s = \pm 1$. Such a case in (8.3.17) will be represented by the values

$$\beta_{nm}^{(-s,s)} = \delta_{nm}^{(-s,s)} = 0,$$

as a result of which it is written in the form,

$$A_n^s + \sum_{m=0}^{\infty} \alpha_{nm}^{-s,s} A_m^{-s} = F_n^s, \tag{8.3.54}$$

$$B_n^s = \Phi_n^s - \sum_{m=0}^{\infty} \gamma_{nm}^{(-s,s)} A_m^{-s}. \tag{8.3.55}$$

Thus in this case for finding the approximate values of the expansion coefficients A_n^s and B_n^s we must solve only one system of $2N$ + 2-nd order, /432

$$A_n^s + \sum_{m=0}^{N} \alpha_{nm}^{(-s,s)} A_m^{-s} = F_n^s, \quad s = \pm 1, \; n = 0, \; 1, \; \ldots, \; N, \tag{8.3.56}$$

relative to the unknowns A_n^s. Although for the approximate values of the unknowns B_n^s we also obtain the relationship

$$B_n^s = \Phi_n^s - \sum_{m=0}^{N} \gamma_{nm}^{(-s,s)} A_m^{-s}, \quad s = \pm 1,$$

however in this case on the strength of (8.3.16)

$$b_n^s \equiv 0 \tag{8.3.57}$$

and it makes no sense to compute B_n^s. For seeking the approximate values of the unknowns A_n^s it follows to solve system (8.3.56) when $\xi_s^0 = 0$. For bands of equal width, when $q_{-1} = q_{+1}$, the solution to system (8.3.56) when $\gamma = \beta = 0$ may be reduced to solving two systems of N-th order,

$$\begin{gathered} z_n + \sum_{m=0}^{N-1} (-1)^m \alpha_{nm}^{(-s,s)} z_m = f_n, \\ u_n - \sum_{m=0}^{\infty} (-1)^m \alpha_{nm}^{(-s,s)} u_m = \tilde{f}_n, \quad n = 0, \; 1, \; \ldots, \; N-1, \end{gathered} \tag{8.3.58}$$

relative to the unknowns z_n, u_n, connected with A_n^{-1} and A_n^{+1} by the formulas in (8.3.51).

4. As already noted, the convergence of series (4.5.13) - (4.5.16) is improved with increase in the values of the parameter kl (for given q_s and ξ_s^0). Therefore, taking into account the finiteness of the values of the indices n and m with numerical solution to the problem and that the dominant terms in the sums (4.5.13) - (4.5.17) will be terms with $r = n$, $p = m$, we find that when $kl >> 1$ ($kl >> n + m$) when the asymptotic formulas (2.2.24) are satisfied, the coefficients (4.5.13) - (4.5.17) will be approximately equal to

$$Q_{mn}^{(1)}(-s, s; \beta) = L\,\mathrm{ce}_m(\alpha_{-s,s}, q_{-s})\,\mathrm{ce}_n(\alpha_{-s,s}-\beta, q_s),$$

$$R_{mn}^{(1)}(-s, s; \beta) = L\,\mathrm{se}_n(\alpha_{-s,s}+\beta, q_s)\,\mathrm{ce}_m(\alpha_{-s,s}, q_{-s}),$$

$$\tilde{Q}_{mn}^{(1)}(-s, s; \beta) = L\,\mathrm{se}_m(\alpha_{-s,s}, q_{-s})\,\mathrm{ce}_n(\alpha_{-s,s}-\beta, q_s), \qquad (8.3.59)$$

$$\tilde{R}_{mn}^{(1)}(-s, s; \beta) = L\,\mathrm{se}_m(\alpha_{-s,s}, q_{-s})\,\mathrm{se}_n(\alpha_{-s,s}-\beta, q_s),$$

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where $L = \sqrt{\frac{2}{\pi kl}}\, e^{i\left(kl-\frac{\pi}{4}\right)}$, and then for the matrix elements instead of (8.3.18) we find the expressions

$$\alpha_{nm}^{(-s,s)} = 2L\,\frac{c_m(q_{-s})\,\mathrm{Ce}_m(\xi_{-s}^0, q_{-s})}{c_n(q_s)\,\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)} \times$$
$$\times \mathrm{ce}_n(\alpha_{-s,s}-\beta, q_s)\,\mathrm{ce}_m(\alpha_{-s,s}, q_{-s}),$$
$$\beta_{nm}^{(-s,s)} = 2L\,\frac{d_m(q_{-s})\,\mathrm{Se}_m(\xi_{-s}^0, q_{-s})}{c_n\,\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)} \times$$
$$\times \mathrm{ce}_n(\alpha_{-s,s}-\beta, q_s)\,\mathrm{se}_m(\alpha_{-s,s}, q_{-s}), \qquad (8.3.60)$$
$$\gamma_{nm}^{(-s,s)} = 2L\,\frac{c_m(q_{-s})\,\mathrm{Ce}_m(\xi_{-s}^0, q_{-s})}{d_n(q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s^0, q_s)} \times$$
$$\times \mathrm{se}_n(\alpha_{-s,s}+\beta, q_s)\,\mathrm{ce}_m(\alpha_{-s,s}, q_{-s}),$$
$$\delta_{nm}^{(-s,s)} = 2L\,\frac{d_m(q_{-s})\,\mathrm{Se}_m(\xi_{-s}^0, q_{-s})}{d_n(q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s^0, q_s)} \times$$
$$\times \mathrm{se}_n(\alpha_{-s,s}-\beta, q_s)\,\mathrm{se}_m(\alpha_{-s,s}, q_{-s}).$$

Just as in the case of circular cylinders, one of the criteria, in the satisfaction of which formulas (8.3.60) can be used, is the condition $l >> 4a$.

5. The formulas are simplified in the case when $q_{-1} \to 0$ or $q_{+1} \to 0$, or when simultaneously $q_s \to 0$, $s = \pm 1$. Then one of the elliptical cylinders or both are degenerated into circular cylinders of the respective radii. As a result from formulas (8.3.13) - (8.3.19) we find the solution to the problem of diffraction of a

plane electromagnetic wave on a diffracting system, composed of one elliptical and one circular cylinder with parallel longitudinal axes, or the solution to the problem of diffraction of a wave on two circular parallel cylinders.

6. Let us find the solution to system (8.3.17) by formal application to it of the method of successive approximations, after taking, as the inital values of the unknowns, the values of $A^s_{n,0} = 0$ and $B^s_{n,0} = 0$. Then

$$A^s_{n,1} = F^s_n, \quad B^s_{n,1} = \Phi^s_n,$$

$$A^s_{n,2} = A^s_{n,1} - \sum_{k=0}^{\infty} \{ \alpha^{-s,s}_{nk} A^{-s}_{k,1} + \beta^{(-s,s)}_{nk} B^{-s}_{k,1} \},$$

$$B^s_{n,2} = B^s_{n,1} - \sum_{k=0}^{\infty} \{ \gamma^{(-s,s)}_{nk} A^{-s}_{k,1} + \delta^{(-s,s)}_{nk} B^{-s}_{k,1} \}, \qquad (8.3.61)$$

. . . . . .

$$A^s_{n,t} = A^s_{n,1} - \sum_{k=0}^{\infty} \{ \alpha^{(-s,s)}_{nk} A^{-s}_{k,t-1} + \beta^{(-s,s)}_{nk} B^{-s}_{k,t-1} \},$$

$$B^s_{n,t} = B^s_{n,1} - \sum_{k=0}^{\infty} \{ \gamma^{(-s,s)}_{nk} A^{-s}_{k,t-1} + \delta^{(-s,s)}_{nk} B^{-s}_{k,t-1} \}.$$

. . . . . .

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Comparison of (8.3.61) with recurrent formulas (8.3.10 - (8.3.13) shows that

$$A^s_n = \lim_{t \to \infty} A^s_{n,t} = \sum_{m=1}^{\infty} {}^sA^m_n,$$

$$B^s_n = \lim_{t \to \infty} B^s_{n,t} = \sum_{m=1}^{\infty} {}^sB^m_n, \qquad (8.3.62)$$

since

$$A^s_{n,t} = {}^sA^1_n + {}^sA^2_n + \ldots + {}^sA^t_n, \qquad (8.3.63)$$

$$B^s_{n,t} = {}^sB^1_n + {}^sB^2_n + \ldots + {}^sB^t_n. \qquad (8.3.64)$$

Therefore, just as in all problems studied above on the interaction of two bodies, the effect of the interaction of two elliptical cylinders at any point of space outside the cylinders is completely determined by the diffraction coefficients A^s_n and B^s_n, the solution to system (8.3.17).

As noted earlier, the approximate values of the coefficients A_n^s and B_n^s are found in the general case by solving the truncated system obtained from the infinite system (8.3.17). In the case when $kl >> 1$ and the cross section dimensions of the cylinders are not high in comparison with the wave length of excitation, the approximate values of the coefficients A_n^s and B_n^s may be found also by summation in (8.3.62) of the approximate values of the scattering coefficients ${}^sA_n^m$ and ${}^sB_n^m$ of all successive secondary diffractions obtained from (8.3.61) using formula (2.2.24) which is asymptotic relative to the argument. For example, let the major axes of the cross section ellipses lie on one straight line (on the Ox axis). Then A_n^s and B_n^s must be the solution to systems (8.3.48) and (8.3.49), respectively, representable through the scattering coefficients of the secondary waves by formulas (8.3.63), (8.3.64), in which now /435

$$\begin{gathered} {}^sA_n^m = -\sum_{\kappa=0}^{\infty} \alpha_{n\kappa}^{(-s,s)} {}^sA_\kappa^{m-1}, \\ {}^sB_n^m = -\sum_{\kappa=0}^{\infty} \delta_{n\kappa}^{(-s,s)} {}^sB_\kappa^{m-1}, \quad m \geqslant 2, \\ {}^sA_n^1 = F_n^s, \ {}^sB_n^1 = \Phi_n^s. \end{gathered} \tag{8.3.65}$$

on the basis of (8.3.59) and (8.3.60) when $kl >> 1$ ($kl >> n + m$) we have approximately $\delta_{nk}^{(-s,s)} = 0$, since $\alpha_{-1,1} = 0$ and $\alpha_{+1,1} = \pi$ and $\beta = 0$,

$$\alpha_{n\kappa}^{(-s,s)} = L\,\alpha_n(s)\,\beta_\kappa(s), \tag{8.3.66}$$

where we have denoted

$$\begin{gathered} a_n(s) = \frac{2\mathrm{ce}_n(\alpha_{-s,s}, q_s)}{c_n(q_s)\,\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)}, \\ \beta_n(s) = c_n(q_{-s})\,\mathrm{Ce}_n(\xi_{-s}^0, q_{-s},)\ \mathrm{ce}_n(\alpha_{-s,s}, q_{-s}). \end{gathered} \tag{8.3.67}$$

Then ${}^sB_n^m = 0$ for all values of n and when $m \geq 2$, and the approximate values of the coefficients ${}^sA_n^m$ are written in the form,

$$\begin{gathered} {}^sA_n^{2\kappa+1} = -L^{2\kappa+1} F(-s)\,\alpha_n(s)\,[Q(s)\,Q(-s)]^\kappa, \\ {}^sA_n^{2\kappa} = L^{2\kappa} F(s)\,\alpha_n(s)\,[Q(s)\,Q(-s)]^{\kappa-1}, \\ \kappa = 1, 2, \ldots, \end{gathered} \tag{8.3.68}$$

where

$$Q(s) = \sum_{n=0}^{\infty} \alpha_n(s)\beta_n(-s) =$$

$$= 2\sum_{n=0}^{\infty} \frac{\mathrm{Ce}_n(\xi_s^0, q_s)}{\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)} \mathrm{ce}_n(\alpha_{-s,s}, q_s)\,\mathrm{ce}_n(\alpha_{s,-s}, q_s), \tag{8.3.69}$$

$$F(s) = \sum_{n=0}^{\infty} \beta_n(-s) F_n^s = -e^{iskl_0\cos\alpha} \times$$

$$\times \sum_{n=0}^{\infty} \frac{\mathrm{Ce}_n(\xi_s^0, q_s)}{\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)} \mathrm{ce}_n(\alpha_s, q_s)\,\mathrm{ce}_n(\alpha_{s,-s}, q_s).$$

It is obvious that the value of the parameter kl, which contains the /436
expressions of L in the denominator may always be taken such that we can satisfy the inequality,

$$|L^2 Q(s) Q(-s)| < 1 \tag{8.3.70}$$

(for given cross sectional dimensions of the cylinder) by virtue of the convergence of the first of the series (8.3.69). For condition (8.3.70) seeking the approximate values of A_n^s, $s = \pm 1$ is reduced to summing the infinitely decreasing geometric progression

$$A_n^s = F_n^s - LF(-s)\alpha_n(s)\sum_{m=1}^{\infty}[L^2 Q(s) Q(-s)]^{m-1} +$$

$$+ L^2 F(s) Q(-s)\alpha_n(s)\sum_{m=1}^{\infty}[L^2 Q(s) Q(-s)]^{m-1}, \tag{8.3.71}$$

as a result of which we find

$$A_n^s = F_n^s + \alpha_n(s) L \frac{LQ(-s) F(s) - F(-s)}{1 - [L^2 Q(s) Q(-s)]}. \tag{8.3.72}$$

We have approximately,

$$B_n^s = \Phi_n^s. \tag{8.3.73}$$

Substituting (8.3.72), (8.3.73) into (8.3.14), we find

$$u^1 = u_1 + u_{\text{II}}, \tag{8.3.74}$$

where

$$u_1 = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} [c_n^2 \mathrm{Ce}_n(\xi_s^0, q_s) F_n^s \mathrm{Me}_n^{(1)}(\xi_s, q_s) \mathrm{ce}_n(\eta_s, q_s) + d_n^2 \mathrm{Se}_n(\xi_s^0, q_s) \Phi_n^s \mathrm{Ne}_n^{(1)}(\xi_s^0, q_s) \mathrm{se}_n(\eta_s, q_s)], \quad (8.3.75)$$

but

$$u_2 = \frac{2EL}{k^2} \left\{ \sum_{s=\pm 1} \frac{LQ(-s)F(s) - F(-s)}{1 - [L^2 Q(s) Q(-s)]} \sum_{n=0}^{\infty} c_n^2 \times \mathrm{Ce}_n(\xi_s^0, q_s) \alpha_n(s) \mathrm{Me}_n^{(1)}(\xi_s, q_s) \mathrm{ce}_n(\eta_s, q_s) \right\}. \quad (8.3.76)$$

It is easy to see that the function u_1 determines the precise value of the Hertz potential of the field of the first order of scattering, without taking into account the influence of the cylinders on one another. The function u_2 approximately determines at the same point of observation the Hertz potential of the secondary field, which sums all diffraction secondary waves of subsequent orders of scattering.

The approximate values of the components of the electromagnetic field are found by the usual method from (1.6.19) by substituting (8.3.74) there.

As is obvious from (8.3.76) the absolute value of the function u_2 drops with increase in kl and, at the limit when $kl \to \infty$, it will be equal to zero. Therefore with sufficiently large values of kl at any point of observation the diffraction field of the given system of cylinders is composed only of waves of the first order of scattering, whereas the influence of the secondary waves of the subsequent orders of scattering on the values of the field becomes negligibly small. As shown in [82] the effect of the interaction of cylinders (for a given kl) at the point of observation is increased in proportion to the degeneration of the elliptical cylinders into circular ones with the retention of the cross section dimensions along the axes $O_s x_s$, $s = \pm 1$, and becomes least with their degeneration into bands. However, with decrease in the value of the ratio of the interfocal distance of the cross section ellipse of the cylinders to the wave length of excitation the form of the cross section of the cylinders is expressed to an ever smaller degree in the value of this effect. /43

In a similar manner, when $kl >> 1$ the computation of A_n^s and B_n^s in (8.3.61) may be reduced to a summation of the decreasing geometric progression with terms which give approximate values of the scattering coefficients A_{nm}^s and B_{nm}^s and with arbitrary orientation

of the respective semiaxes of the cross section ellipses of the cylinders.

TE Type Field. If we seek the Hertz potential of the stray field in the form,

$$v^1 = \frac{2E}{k^2} \sum_{s=\pm 1} v_s, \tag{8.3.77}$$

where

$$v_s = \sum_{n=0}^{\infty} \left[a_n^s c_n \mathrm{Me}_n^{(1)}(\xi_s, q_s) \mathrm{ce}_n(\eta_s, q_s) + b_n^s d_n \mathrm{Ne}_n^{(1)}(\xi_s, q_s) \mathrm{se}_n(\eta_s, q_s) \right], \tag{8.3.78}$$

then after substituting the coefficients of the series a_n^s and b_n^s by the new coefficients according to the formulas,

$$a_n^s = c_n \mathrm{Ce}_n'(\xi_s^0, q_s) A_n^s, \quad b_n^s = d_n \mathrm{Se}_n'(\xi_s^0, q_s) B_n^s \tag{8.3.79}$$

from boundary conditions (1.7.8), using formulas (4.5.8), we obtain for the determination of the unknowns A_n^s and B_n^s an infinite system of equations of the same form as (8.3.17) with the difference that now the matrix elements and the free terms have the expressions

$$\alpha_{nm}^{(-s,s)} = 2 \frac{c_m \mathrm{Ce}_m'(\xi_{-s}^0, q_{-s})}{c_n \mathrm{Me}_n^{(1)'}(\xi_s^0, q_s)} Q_{mn}^{(1)}(-s, s; \beta),$$

$$\beta_{nm}^{(-s,s)} = 2 \frac{d_m \mathrm{Se}_m'(\xi_{-s}^0, q_{-s})}{c_n \mathrm{Me}_n^{(1)'}(\xi_s^0, q_s)} \bar{Q}_{mn}^{(1)}(-s, s; \beta), \tag{8.3.80}$$

$$\gamma_{nm}^{(-s,s)} = 2 \frac{c_m \mathrm{Ce}_m'(\xi_{-s}^0, q_{-s})}{d_n \mathrm{Ne}_n^{(1)'}(\xi_s^0, q_s)} R_{mn}^{(1)}(-s, s; \beta),$$

$$\delta_{nm}^{(-s,s)} = 2 \frac{d_m \mathrm{Se}_m'(\xi_{-s}^0, q_{-s})}{d_n \mathrm{Ne}_n^{(1)'}(\xi_s^0, q_s)} \bar{R}_{mn}^{(1)}(-s, s; \beta),$$

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$$F_n^s = -\frac{e^{iskl_0 \cos\alpha} \mathrm{ce}_n(\alpha_s, q_s)}{c_n \mathrm{Me}_n^{(1)'}(\xi_s^0, q_s)}, \tag{8.3.81}$$

$$\Phi_n^s = -\frac{e^{iskl_0 \cos\alpha} \mathrm{se}_n(\alpha_s, q_s)}{d_n \mathrm{Ne}_n^{(1)'}(\xi_s^0, q_s)}$$

with the previous values of the parameters $\alpha_{-s,s}$, α_s, q_s and ($s = \pm 1$). However this does not change the properties of the system associated with its solvability, the coefficients of the series A_n^s

and B_n^s are determined uniquely from (8.3.17) with (8.3.80), (8.3.81), and $\sum_{n=0}^{\infty} |A_n^s|^2 < \infty$ and $\sum_{n=0}^{\infty} |B_n^s|^2 < \infty$. Their approximate values may be, with any degree of accuracy, found from the finite system of equations, obtained by truncation of the infinite system. Just as in the previous case we can show the different variants of the choice of particular values of the parameters of the problem, which permit simplification of the expressions appearing in the system when, with given cross sectional dimenions of the cross section ellipses of the cylinders, a decrease occurs in the order of the truncated systems. These cases are studied as in the problem for a TM-type field. With degeneration of the cylinders into bands ($\xi_s^0 = 0$, $s = \pm 1$) the matrix elements $\beta_{nm}^{(-s,s)}$, $\delta_{nm}^{(-s,s)}$ will vanish instead of the matrix elements $\alpha_{nm}^{(-s,s)}$ and $\gamma_{nm}^{(-s,s)}$, since $\mathrm{Se}_n'(0, q) \neq 0$, but $\mathrm{Ce}_n'(0, q) = 0$.

Seeking the quantities A_n^s and B_n^s will fully solve the problem on determining the potential v. Through v the components of the stray field are found from formulas (1.6.20).

On the basis of (8.3.79) we have

$$v^1 = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \{ c_n^2 A_n^s \mathrm{Ce}_n'(\xi_s^0, q_s) \mathrm{Me}_n^{(1)}(\xi_s, q_s) \mathrm{ce}_n(\eta_s, q_s) + d_n^2 B_n^s \mathrm{Se}_n'(\xi_s^0, q_s) \mathrm{Ne}_n^{(1)}(\xi_s, q_s) \mathrm{se}_n(\eta_s, q_s) \}. \quad (8.3.82)$$

Therefore in the vicinity of the wave band in polar coordinates ρ, ϕ with the origin at point O

$$E_\eta^1 = \sqrt{\frac{\mu}{\varepsilon}} H_z^1 = \frac{e^{ik\rho}}{\sqrt{\rho}} A^*(\varphi), \quad (8.3.83)$$

where the amplitude function /43

$$A^*(\varphi) = \sqrt{\frac{\mu}{\varepsilon}} 2E \sqrt{\frac{2}{\pi k}} e^{-\frac{i\pi}{4}} \sum_{s=\pm 1} e^{-iskl_0 \cos\varphi} \times \sum_{n=0}^{\infty} \{ c_n A_n^s \mathrm{Ce}_n'(\xi_s^0, q_s) \mathrm{ce}_n(\eta_s, q_s) + d_n B_n^s \mathrm{Se}_n'(\xi_s^0, q_s) \mathrm{se}_n(\eta_s, q_s) \}. \quad (8.3.84)$$

Here $\eta_{-1} = \phi + \gamma$, $\eta_{+1} = \phi + \gamma + \beta$ and $\rho = fe^{\xi/2}$. The linear cross section of back scattering (radar cross section) is computed from the formula ($\eta_{-1} = \alpha + \gamma$, $\eta_{+1} = \alpha + \gamma + \beta$)

$$\sigma_B = \frac{16}{k}\left|\sum_{s=\pm 1} e^{iskl_0\cos\alpha}\sum_{n=0}^{\infty}(-1)^n\{A_n^s c_n \mathrm{Ce}_n'(\xi_s^0, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + B_n^s d_n \mathrm{Se}_n'(\xi_s^0, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}\right|^2, \quad (8.3.85)$$

and in the case of two bands

$$\sigma_B = \frac{16}{k}\left|\sum_{s=\pm 1} e^{iskl_0\cos\alpha}\sum_{n=0}^{\infty}(-1)^n B_n^s d_n \mathrm{Se}_n'(0, q_s)\,\mathrm{se}_n(\eta_s, q_s)\right|^2.$$

The complete linear scattering cross section may be computed from formulas (8.3.30) in which instead of $A(\phi)$ if follows to take $A^*(\phi)$ from (8.3.84). In the particular case of two identical elliptical cylinders, when $\beta = 0$ and $q_{-1} = q_{+1}$, σ_S will be determined by formula (8.3.31), in which a_n^s and b_n^s are expressed through A_n^s and B_n^s by the relationships (8.3.79) and A_n^s, B_n^s are found from system (8.3.17) with (8.3.80), (8.3.81).

In the neighboring band the potential of the complete field in local coordinates of the s-th cylinder is represented in the form,

$$v = \frac{E}{k^2}\sum_{n=0}^{\infty}\{c_n^2 A_n^s[\mathrm{Ce}_n'(\xi_s^0, q_s)\,\mathrm{Me}_n^{(1)}(\xi_s, q_s) - \mathrm{Ce}_n(\xi_s^0, q_s)\,\mathrm{Me}_n'(\xi_s^0, q_s)]\,\mathrm{ce}_n(\eta_s, q_s) + d_n^2 B_n^s[\mathrm{Se}_n'(\xi_s^0, q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s, q_s) - \mathrm{Se}_n(\xi_s^0, q_s)\mathrm{Ne}_n^{(1)\prime}(\xi_s, q_s)]\,\mathrm{se}_n(\eta_s, q_s)\}. \quad (8.3.86)$$

Analysis of the field near the cylinders may be carried out from the formulas

$$H_z = k^2 v,$$

$$E_\xi = \frac{ik_0\mu}{f\sqrt{\mathrm{ch}^2\xi_s^0 - \cos^2\eta_s}}\cdot\frac{\partial v}{\partial \eta_s} \quad (8.3.87)$$

after substitution into their right-hand sides the expressions in (8.3.86). On the basis of (1.1.7), (8.3.87) and (8.3.86) for the components of the density of surface currents induced on the surface of the s-th cylinder, we have

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$$j_\xi = j_z = 0.$$

$$j_\eta = \frac{-icE}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\{A_n^s\,\mathrm{ce}_n(\eta_s, q_s) + B_n^s\,\mathrm{se}_n(\eta_s, q_s)\}. \quad (8.3.88)$$

§4. Field of a Longitudinal Emitter in the Presence of Two Parallel Elliptical Cylinders.

Now let us look at the problem of the field of a dipole emitter of electromagnetic waves (electric and magnetic) in the presence of two infinitely long parallel elliptical cylinders, occupying the regions $\xi_s \leq \xi_s^0$, $s = \pm 1$ in a uniform isotropic unbounded space with electromagnetic constants ε, μ, $\sigma(\sigma = 0)$ in local elliptical coordinates ξ_s, η_s, z_s ($s = \pm 1$) on the assumption that the cylinders are ideally conducting, the interfocal distances of their cross section ellipses in the general case are different ($2f_{-1}$ for one and $2f_{+1}$ for the other cylinder) and that the local coordinates of the $O_s x_s$ axis are directed along the major semiaxes of the cross section ellipses, intersecting at an angle β (Fig. 53,a). The dipole is found at point P, whose position in local elliptical coordinate systems ξ_s, η_s, z_s is given by the values $\xi_s = \xi_{s0}$, $\eta_s = \eta_{s0}$, $z_s = z_0$ ($z_{-1} = z_{+1}$, $s = \pm 1$). Its moment is directed parallel to the generatrices of the cylinders and is oriented to the side of increase in the coordinate z. As usual, the problem consists of seeking the secondary electromagnetic field according to the given dipole field in free space. In the case of an electric dipole this is represented by the potential function $u^0 = pe^{ikR}/R$ (p is the longitudinal component of the dipole moment $\mathrm{p} = \{0, 0, p\}$), which on the basis of (8.2.1) may be written in local coordinates of the s-th cylinder ($s = \pm 1$) in the form of a series of Fourier integrals,

$$u^0 = ip \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \left\{ \mathrm{ce}_n(\eta_s, q_s)\, \mathrm{ce}_n(\eta_{s0}, q_s) \times \right.$$
$$\times \begin{bmatrix} \mathrm{Me}_n^{(1)}(\xi_s, q_s)\, \mathrm{Ce}_n(\xi_{s0}, q_s) \\ \mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)\, \mathrm{Ce}_n(\xi_s, q_s) \end{bmatrix} c_n^2 + d_n^2\, \mathrm{se}_n(\eta_s, q_s)\, \mathrm{se}_n(\eta_{s0}, q_s) \times \qquad (8.4.1)$$
$$\left. \times \begin{bmatrix} \mathrm{Ne}_n^{(1)}(\xi_s, q_s)\, \mathrm{Se}_n(\xi_{s0}, q_s) \\ \mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)\, \mathrm{Se}_n(\xi_s, q_s) \end{bmatrix} \right\} e^{ih(z-z_0)}\, dh, \quad \begin{matrix} \xi_s > \xi_{s0}, \\ \xi_s < \xi_{s0}, \end{matrix}$$

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in which $q_s = f_s^2(k^2 - h^2)/4$, $p = I(z_0)dz_0/ - i4\pi\omega\varepsilon$, and $I(z_0)$ is the given electric current in the dipole. In the case of a magnetic dipole with the moment $\mathrm{m} = \{0, 0, m\}$, $m = K(z_0)dz_0/ - i4\pi\omega\mu$, where $K(z_0)$ is the given magnetic current in the dipole, the potential function v^0 of the primary field in local coordinates of the s-th cylinder may be written on the basis of (8.2.2) in the form of the series

$$v^0 = im \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \left\{ \mathrm{ce}_n(\eta_s, q_s)\, \mathrm{ce}_n(\eta_{s0}, q_s)\, c_n^2 \times \right.$$
$$\times \begin{bmatrix} \mathrm{Me}_n^{(1)}(\xi_s, q_s)\, \mathrm{Ce}_n(\xi_{s0}, q_s) \\ \mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)\, \mathrm{Ce}_n(\xi_s, q_s) \end{bmatrix} + \mathrm{se}_n(\eta_s, q_s)\, \mathrm{se}_n(\eta_{s0}, q_s)\, d_n^2 \times \qquad (8.4.2)$$
$$\left. \times \begin{bmatrix} \mathrm{Ne}_n^{(1)}(\xi_s, q_s)\, \mathrm{Se}_n(\xi_{s0}, q_s) \\ \mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)\, \mathrm{Se}_n(\xi_s, q_s) \end{bmatrix} \right\} e^{ih(z-z_0)}\, dh,$$

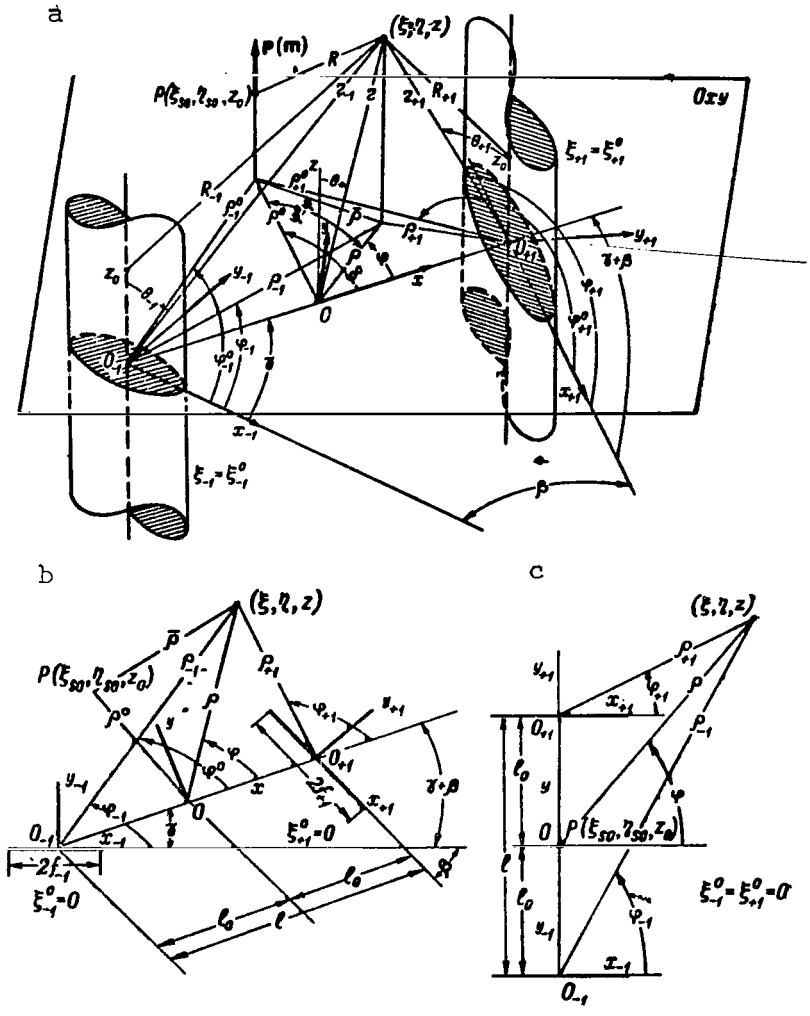

Fig. 53. Excitation of Two Elliptical Cylinders by the Longitudinal Dipole (a), the Case of Two Bands (Plane z = const) with Arbitrary Values of the Parameters (b) and the Case when $f_{-1} = f_{+1} = f$, $\gamma = \pi/2$, $\beta = 0$ (c).

where $q_s = f_s^2(k^2 - h^2)/4$, $s = \pm 1$. Just as in the case of one elliptical cylinder, the secondary electromagnetic field is found through /442 the respective potential u^1 or v^1 (depending on the type of field) from relationships (1.6.19) or (1.6.20) [106, 107]).

Electric Dipole. If we seek the potential u^1 of the secondary field in the form of a series of Fourier integrals,

$$u^1 = ip \sum_{s=\pm 1} u_s, \qquad (8.4.3)$$

where

$$u_s = \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} \left[a_n^s(h)\, c_n \mathrm{Me}_n^{(1)}(\xi_s, q_s)\, \mathrm{ce}_n(\eta_s, q_s) + \right.$$
$$\left. + b_n^s(h)\, d_n \mathrm{Ne}_n^{(1)}(\xi_s, q_s)\, \mathrm{se}_n(\eta_s, q_s) \right] e^{ih(z-z_s)}\, dh, \qquad (8.4.4)$$

$q_s = f_s^2(k^2 - h^2)$ $(s = \pm 1)$, but $a_n^s(h)$ and $b_n^s(h)$ are unknowns subject to determination that depend on h, then from the boundary condition (1.7.7) on the surface of each cylinder with the aid of the addition theorem (4.5.8) we find that (1.7.7) will be satisfied if the quantities $A_n^s(h)$ and $B_n^s(h)$ associated with $a_n^s(h)$ and $b_n^s(h)$ by the formulas,

$$a_n^s(h) = c_n \mathrm{Ce}_n(\xi_s^0, q_s)\, A_n^s, \quad b_n^s(h) = d_n \mathrm{Se}_n(\xi_s^0, q_s)\, B_n^s \qquad (8.4.5)$$

$(s = \pm 1)$, will be the solution to the infinite system of linear equations,

$$A_n^s + \sum_{m=0}^{\infty} \left\{ \alpha_{nm}^{(-s,s)} A_m^{-s} + \beta_{nm}^{(-s,s)} B_m^{-s} \right\} = F_n^s,$$
$$B_n^s + \sum_{m=0}^{\infty} \left\{ \gamma_{nm}^{(-s,s)} A_m^{-s} + \delta_{nm}^{(-s,s)} B_m^{-s} \right\} = \Phi_n^s, \qquad (8.4.6)$$

in which the matrix elements are determined by formulas (8.3.18) and the free terms are given by the expressions,

$$F_n^s = -(-1)^n \frac{\mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)}{\mathrm{Me}_n^{(1)}(\xi_n^0, q_s)}\, \mathrm{ce}_n(\eta_{s0}, q_s),$$
$$\Phi_n^s = -(-1)^n \frac{\mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)}{\mathrm{Ne}_n^{(1)}(\xi_s^0, q_s)}\, \mathrm{se}_n(\eta_{s0}, q_s). \qquad (8.4.7)$$

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Here unlike (8.3.18) the matrix elements $\alpha_{nm}^{(-s,s)}$, $\beta_{nm}^{(-s,s)}$, $\gamma_{nm}^{(-s,s)}$, $\delta_{nm}^{(-s,s)}$ are functions of the parameter h, which varies in the range from $-\infty$ to $+\infty$, since $q_s = f_s^2(k^2 - h^2)/4$. However, just as before we can show that for any fixed value of h the coefficients of the system will satisfy the condition (1.9.19) and its right-hand sides will satisfy condition (1.9.20) if the cylinders do not touch one another, and the dipole does not lie on the surface of one of the cylinders (in the opposite case $|F_n^s|^2$, $|\Phi_n^s|^2 \to$ const, if $\xi_{s0} = \xi_s^0$, where $s = -1$ or $s = +1$. Satisfaction of these conditions allows establishing the unique solvability of the system (8.4.6) by the

method of truncation, and for each h we will have $\sum_{n=0}^{\infty} |A_n^s|^2 < \infty$ and $\sum_{n=0}^{\infty} |B_n^s|^2 < \infty$.

By determining the values of A_n^s and B_n^s the formal solution to the problem is finished.

Formulas (8.4.3), (8.4.6) which give the solution to the problem are precise. However in practice in the general case they are suitable only for an approximate computation of the potential u^1 by numerical methods. If rather than the general case, we look only at the field in the wave band for which we assume $\xi_s \to \infty$, $kf \operatorname{ch} \xi_s \to \infty$, then we can simplify the terms in the series in (8.4.3) and be freed of the functional dependence in the matrix elements and the free terms of system (8.4.6) on the parameter h. Thus, in the wave band the Fourier integrals

$$J_1 = \int_{-\infty}^{\infty} \left[a_n^s(h) c_n \operatorname{Me}_n^{(1)}(\xi_s, q_s) \operatorname{ce}_n(\eta_s, q_s) \right] e^{ih(z-z_0)} dh,$$
$$J_2 = \int_{-\infty}^{\infty} \left[b_n^s(h) d_n \operatorname{Ne}_n^{(1)}(\xi_s, q_s) \operatorname{se}_n(\eta_s, q_s) \right] e^{ih(z-z_0)} dh, \qquad (8.4.8)$$

which contain in (8.4.4) after substitution in them of the radial Mathieu functions $\operatorname{Me}_n^{(1)}(\xi_s, q_s)$ and $\operatorname{Ne}_n^{(1)}(\xi_s, q_s)$ by the expressions which are asymptotic relative to the argument and th- subsequent application of the method of steepest descents to the integrals thus obtained (just as was done in the case of one cylinder), will be approximately equal to

$$J_1 = \frac{2(-1)^n}{i} \cdot \frac{e^{ikR_s}}{R_s} a_n^s(h) \operatorname{ce}_n(\eta_s, q_s),$$
$$J_2 = \frac{2(-1)^n}{i} \cdot \frac{e^{ikR_s}}{R_s} b_n^s(h) \operatorname{se}_n(\eta_s, q_s), \qquad (8.4.9)$$

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where $R_s = r_s - z_0 \cos\theta$, $r_s = r - s l_0 \sin\theta \cos\phi$, $s = \pm 1$, and where we have omitted the terms which contain higher powers of the quantity R_s^{-1}. In the right-hand sides of (8.4.9) now $h = k \cos\theta$, and $q_s = f_s^2 k^2 \sin^2\theta/4$, $s = \pm 1$. Therefore, system (8.4.6) with a fixed value of the parameter θ must be solved only when $h = k\cos\theta$.

After substituting (8.4.9) into (8.4.4) and subsequent conversion to spherical coordinates r, θ, ϕ with the origin at point O for the function u^1, we obtain the approximate expression,

$$u^1 = 2p\frac{e^{ikr}}{r}\, e^{-ikz_0\cos\theta} \sum_{s=\pm 1} e^{-iskl_0\sin\theta\cos\varphi} \sum_{n=0}^{\infty} (-1)^n \times$$
$$\times \{A_n^s c_n \mathrm{Ce}_n(\xi_s^0, q_s)\,\mathrm{ce}_n(\varphi_s, q_s) +$$
$$+ B_n^s d_n \mathrm{Se}_n(\xi_s^0, q_s)\,\mathrm{se}_n(\varphi_s, q_s)\}, \quad \eta_s \approx \varphi_s; \tag{8.4.10}$$

In these same coordinates the dipole field in free space is determined by the formula (in the wave band)

$$E_\theta^0 = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^0 =$$
$$= -pk^2\frac{e^{ikr}}{r}\, e^{-ikz_0\cos\theta - ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}\sin\theta, \tag{8.4.11}$$

where ρ^0, ϕ^0, z_0 are the polar coordinates of point P with the dipole in the system with the origin at point O. Therefore, from (1.6.19) in this approximation for the components of the complete field,

$$E_\theta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi = -k^2 p\sin\theta\,\frac{e^{ikr}}{r}\times$$
$$\times e^{-ikz^0\cos\theta - ik\rho^0\sin\theta\cos(\varphi-\varphi^0)} W(\theta, \varphi), \tag{8.4.12}$$

where through $W(\theta, \phi)$ we denote the correction factor (damping factor) to the characteristics of the directionality of the electric dipole taking into account the presence in the dipole field of two parallel ideally conducting cylinders of elliptical cross section. The function $W(\theta, \phi)$ is determined by the expression,

$$W(\theta, \varphi) = 1 + e^{ik\rho^0\sin\theta\cos(\varphi-\varphi^0)} \sum_{s=\pm 1} e^{-iskl_0\sin\theta\cos\varphi} \sum_{n=0}^{\infty} (-1)^n \times$$
$$\times \{A_n^s c_n \mathrm{Ce}_n(\xi_s^0, q_s)\,\mathrm{ce}_n(\varphi_s, q_s) +$$
$$+ B_n^s d_n \mathrm{Se}_n(\xi_s^0, q_s)\,\mathrm{se}_n(\varphi_s, q_s)\} \tag{8.4.13}$$

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and is, for a fixed value of $q_s = f_s^2 k^2 \sin^2\theta/4$, $s = \pm 1$, a function of the angle ϕ, which characterizes the azimuth directionality of emission from the system, consisting of an electric dipole and two elliptical cylinders which play the role of passive reflectors. The resistance of the emission from this system as well as the coefficient of directionality of the emission may be computed from the computations of the radiative power transferred through the infinite sphere with the center at the origin O according to formulas (1.4.14) and (1.4.17), respectively, if we apply to the integrals encountered in these formulas the numerical methods of integration.

With a fixed value of the angle θ the matrix elements of system (8.4.6) possess all the properties of (8.3.17). Therefore, by

the choice of the particular values of the parameters, which determine the solution to the problem, we can to a significant degree simplify the formulas for A_n^s and B_n^s and the form of the function $W(\theta, \phi)$ (see, for example, §3, Chapter 8). The function $W(\theta, \phi)$ assumes a comparatively simple form in the case when the elliptical cylinders are degenerated into bands ($\xi_s^0 = 0$, $s = \pm 1$, Fig. 53,b). Then

$$W(\theta, \varphi) = 1 + e^{ikp^0 \sin\theta \cos(\varphi - \varphi^0)} \sum_{s=\pm 1} e^{-iskl_0 \sin\theta\cos\varphi} \times$$
$$\times \sum_{n=0}^{\infty} (-1)^n A_n^s c_n \operatorname{ce}_n(0, q_s) \operatorname{ce}_n(\varphi_s, q_s) \qquad (8.4.14)$$

regardless of the value of the angle β, at which the planes $O_{-1}x_{-1}z_{-1}$ and $O_{+1}x_{+1}z_{+1}$ intersect with the bands located in them having a width of $2f_{-1}$ and $2f_{+1}$, respectively. The coefficients of the series in (8.4.14) are found from the system,

$$A_n^s + \sum_{m=0}^{\infty} \alpha_{nm}^{(-s,s)} A_m^{-s} = F_n^s, \qquad (8.4.15)$$

in which

$$\alpha_{nm}^{(-s,s)} = 2 \frac{c_m \operatorname{Ce}_m(0, q_{-s})}{c_n \operatorname{Me}_n^{(1)}(0, q_s)} Q_{mn}^{(1)}(-s, s; \beta), \qquad (8.4.16)$$

$$F_n^s = -(-1)^n \frac{\operatorname{Me}_n^{(1)}(\xi_{s0}, q_s)}{\operatorname{Me}_n^{(1)}(0, q_s)} \operatorname{ce}_n(\eta_{s0}, q_s). \qquad (8.4.17)$$

When $\beta = 0$ (the plane $O_{-1}x_{-1}z_{-1}$ and $O_{+1}x_{+1}z_{+1}$ are parallel) and $q_{-1} = q_{+1}$,

$$\alpha_{nm}^{(-s,s)} = (-1)^{n+m} \alpha_{nm}^{(s,-s)}, \; s = \pm 1. \qquad (8.4.18)$$

If, furthermore, the bands are arranged as shown in Fig. 53,c when /446
$\alpha_{-1,+1} = \pi/2$, $\alpha_{+1,-1} = 3\pi/2$, $\eta_{-1,0} = \pi/2$, $\eta_{+1,0} = 3\pi/2$, then summation in the expression for $Q_{mn}^{(1)}(-s, s; 0)$ in fact will be carried out only over r and p of the same parity [if $r \pm p$ is odd, then $Q_{mn}^{(1)}(-s, s; 0) = 0$], on the strength of which $p \pm r$, and consequently $n + m$ also will be even numbers. In the case instead of (8.4.18),

$$\alpha_{nm}^{(-s,s)} = \alpha_{nm}^{(s,-s)}. \qquad (8.4.19)$$

For the same reason ($ce_n(\eta_{s0}, q_s) = 0$, if $\eta_{-1,0} = \pi/2$, $\eta_{+1,0} = 3\pi/2$ with odd n) we also find that

$$F_n^s = F_n^{-s}, \tag{8.4.20}$$

if the dipole is found midway between the bands on the line of the centers of their cross section, when $\xi_{-10} = \xi_{+10}$. As a result from (8.4.15) where for given values of the parameters the summation is carried out over m of the same parity as n, we find the equation,

$$A_n^{-1} = A_n^{+1} = A_n, \tag{8.4.21}$$

the form,

$$W(\theta, \varphi) = 1 + e^{ik\rho^0 \sin\theta\cos(\varphi-\varphi^0)}\, 2\cos[kl_0 \sin\theta\cos\varphi] \sum_{n=0}^{\infty} \times \\ \times (-1)^n A_n c_n \mathrm{Ce}_n(0, q)\, \mathrm{ce}_n(\varphi, q), \tag{8.4.22}$$

where An is found from system (8.4.16) with $s = +1$ .r $s = -1$. Hence it is clear that at the points of the wave band for which

$$kl_0 \sin\theta\cos\varphi = \frac{\pi}{2}(2m+1), \quad m = 0, \pm 1, \ldots, \tag{8.4.23}$$

the presence of bands in the field of the electric dipole located as shown above with respect to them, is not expressed in the characteristics of the dipole emission.

If $\xi_{-1}^0 = \xi_{+1}^0 = 0$ and $\alpha_{-1,+1} = 0$, $\alpha_{+1,-1} = \pi$, and the position of the dipole is determined by the coordinates $\xi_{-1,0} = \xi_{+1,0}$, $\eta_{-1,0} = \alpha$, $\eta_{+1,0} = \pi - \alpha$, then when $q_{-1} = q_{+1} = q$ (8.4.18) is valid and

$$F_n^s = (-1)^n F_n^{(s)}, \quad s = \pm 1. \tag{8.4.24}$$

As a result from (8.4.15) we can establish that then

$$A_n^{+1} = (-1)^n A_n^{-1} = A_n \tag{8.4.25}$$

and the damping factor (8.4.14) is written in the form,

$$W(\theta, \varphi) = 1 + e^{ik\rho^0 \sin\theta\cos(\varphi-\varphi^0)}\, 2\{\cos(kl_0 \sin\theta\cos\varphi \times \\ \times \sum_{n=0,2,\ldots} (-1)^n A_n c_n \mathrm{Ce}_n(0, q)\, \mathrm{ce}_n(\varphi, q) - i\sin(kl_0 \sin\theta\cos\varphi) \times \\ \times \sum_{n=1,3,\ldots} (-1)^n A_n c_n \mathrm{Ce}_n(0, q)\, \mathrm{ce}_n(\varphi, q)\}. \tag{8.4.26}$$

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Returning to the general case, let us note that the law of the density distribution of surface currents induced on the surface of the s-th cylinder by the field of the electric dipole, taking into account the interaction of the cylinders is determined on the basis of (1.1.7), (1.6.19), (8.4.1), (8.4.3), (4.5.8), (8.4.6), (8.4.5) by the formula,

$$j_z^s = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{ck}{2\pi^2}\cdot\frac{1}{f_s\sqrt{\operatorname{ch}^2\xi_s^0-\cos^2\eta_s}}\times \\ \times\sum_{n=0}^{\infty}(-1)^n\int_{-\infty}^{\infty}\{A_n^s(h)\operatorname{ce}_n(\eta_s,\ q_s)+B_n^s(h)\operatorname{se}_n(\eta_s,\ q_s)\}\,e^{ih(z-z_0)}\,dh \qquad (8.4.27)$$

($j_\xi^s = j_\eta^s = 0$), where $q_s = f_s^2(k^2 - h^2)/4$, $s = \pm 1$. With degeneration of the cylinders into bands ($\xi_s^0 = 0$, $s = \pm 1$).here we will have $B_n^s = 0$, and $\sqrt{\operatorname{ch}^2\xi_s^0 - \cos^2\eta_s} = \sin\eta_s$. The approximate values of j_z^s may be found by the methods of numerical integration.

From the solution to the problem for the dipole source we can obtain generalizations to the case of a linear antenna and to the case of an infinitely long filament with an electric current in it. Thus, by integrating (8.4.12) over z_0 from z_1 to z_2, we find for the field of the linear antenna,

$$E_\theta = -\frac{ik^2}{4\pi\omega\varepsilon}\cdot\frac{e^{ikr}}{r}\sin\theta\left\{\int_{z_1}^{z_2} I(z_0)\,e^{-ikz_0\cos\theta}\,dz_0\right\}\times \\ \times W(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}. \qquad (8.4.28)$$

If we set here $z_1 = -\bar{l}$, and $z_2 = \bar{l}$ and assume that the distribution of the current in the antenna is given in the form,

$$I(z_0) = \begin{cases} \dfrac{I_0}{\sin k\bar{l}}\sin k(\bar{l}-z_0), & 0 \leqslant z_0 \leqslant \bar{l}, \\ \dfrac{I_0}{\sin k\bar{l}}\sin k(\bar{l}+z_0), & -\bar{l} \leqslant z_0 \leqslant 0, \end{cases} \qquad (8.4.29)$$

then for such a distribution the electric component

$$E_\theta = -\frac{I_0 k}{2\pi\omega\varepsilon}\cdot\frac{e^{ikr}}{r}\cdot\frac{[\cos(k\bar{l}\cos\theta)-\cos k\bar{l}]}{\sin k\bar{l}\sin\theta}\times \\ \times W(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}.$$

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In particular, for the half-wave antenna ($2\overline{l} = \lambda_0/2$) we will have

$$E_\theta = -\frac{I_0 k}{2\pi\omega\varepsilon}\cdot\frac{e^{ikr}}{r}\cdot\frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta}\times$$
$$\times W(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}.$$

Of definite interest is the case when the source of the primary field is an infinitely long filament with an electric current, parallel to the generatrices of the cylinders. For such a case the solution to the problem is obtained by integration in (8.4.1) and (8.4.3) over z_0 from $-\infty$ to $+\infty$. As a result we find the expressions

$$u^0 = -\frac{I\sqrt{\frac{\mu}{\varepsilon}}}{2kc}\sum_{n=0}^{\infty}(-1)^n\Big\{c_n^2\,\mathrm{ce}_n(\eta_s,\ q_s)\,\mathrm{ce}_n(\eta_{s0},\ q_s)\times$$
$$\times\begin{bmatrix}\mathrm{Me}_n^{(1)}(\xi_s,\ q_s)\,\mathrm{Ce}_n(\xi_{s0},\ q_s)\\ \mathrm{Me}_n^{(1)}(\xi_{s0},\ q_s)\,\mathrm{Ce}_n(\xi_s,\ q_s)\end{bmatrix} + d_n^2\,\mathrm{se}_n(\eta_s,\ q_s)\,\mathrm{se}_n(\eta_{s0},\ q_s)\times$$
$$\times\begin{bmatrix}\mathrm{Ne}_n^{(1)}(\xi_s,\ q_s)\,\mathrm{Se}_n(\xi_{s0},\ q_s)\\ \mathrm{Ne}_n^{(1)}(\xi_{s0},\ q_s)\,\mathrm{Se}_n(\xi_s,\ q_s)\end{bmatrix}\Big\},\quad \begin{matrix}\xi_{s0}<\xi_s,\\ \xi_{s0}>\xi_s,\end{matrix} \qquad (8.4.30)$$

$$u^1 = -\frac{I\sqrt{\frac{\mu}{\varepsilon}}}{2kc}\sum_{s=\pm1}\sum_{n=0}^{\infty}\{c_n^2 A_n^s\,\mathrm{Ce}_n(\xi_s^0,\ q_s)\times$$
$$\times\,\mathrm{Me}_n^{(1)}(\xi_s,\ q_s)\,\mathrm{ce}_n(\eta_s,\ q_s) +$$
$$+ d_n^2 B_n^s\,\mathrm{Se}_n(\xi_s^0,\ q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s,\ q_s)\,\mathrm{se}_n(\eta_s,\ q_s)\},$$

in which $q_s = (f_s k/2)^2$ ($h = 0$), and the quantities A_n^s, B_n^s are solution to the system (8.4.6) when $h = 0$. Expansion for the potential u^0 represents, as we can easily see, the expansion of the cylindrical wave (4.4.9) according to the Mathieu wave functions. Therefore the expression for the potential u^1 will be similar to the expression which is obtained for u^1 with a strict solution to the problem on diffraction of a cylindrical wave on two ideally conducting parallel cylinders of elliptical cross section.

In the vicinity of the wave band the only non-zero component of the electric field is equal to

$$E_z^1 = -\sqrt{\frac{\mu}{\varepsilon}}\cdot\frac{Ie^{ik\rho}k}{2c\sqrt{\rho}}\sqrt{\frac{2}{\pi k}}\times \qquad (8.4.31)$$
$$\times\sum_{s=\pm1}\sum_{n=0}^{\infty}\{c_n A_n^s\,\mathrm{Ce}_n(\xi_s^0,\ q_s)\,\mathrm{ce}_n(\varphi,\ q_s) +$$
$$+ d_n B_n^s\,\mathrm{Se}_n(\xi_s^0,\ q_s)\,\mathrm{se}_n(\varphi,\ q_s)\}\,e^{-iskl_s\cos\varphi - i\pi/4}$$

Using the addition theorem (4.5.8) of system (8.4.6) and the formulas in (8.4.30) for the potential of the complete field $u = u^0 + u^1$ we find an expression (in coordinates of the s-th cylinder)

$$u = -\frac{I\sqrt{\frac{\mu}{\varepsilon}}}{2kc}\sum_{n=0}^{\infty}\{c_n^2 A_n^s[\mathrm{Ce}_n(\xi_s^0, q_s)\mathrm{Me}_n^{(1)}(\xi_s, q_s) - \mathrm{Ce}_n(\xi_s, q_s)\mathrm{Me}_n^{(1)}(\xi_s^0, q_s)]\,\mathrm{ce}_n(\eta_s, q_s) + d_n^2 B_n^s[\mathrm{Se}_n(\xi_s^0, q_s)\mathrm{Ne}_n^{(1)}(\xi_s, q_s) - \mathrm{Se}_n(\xi_s, q_s)\mathrm{Ne}_n^{(1)}(\xi_s^0, q_s)] \times \mathrm{se}_n(\eta_s, q_s)\},$$

which permits along with (1.6.19) studying the electromagnetic field in the neighboring band. On the basis of (1.1.7) the law of density distribution of the surface currents, induced on the surface of the s-th cylinder by the field of the filament with the current, will be given by the formulas,

$$j_\xi^s = j_\eta^s = 0,$$
$$j_z^s = \frac{I}{4\pi^2}\cdot\frac{1}{f_s\sqrt{\mathrm{ch}^2\xi_s^0 - \cos^2\eta_s}}\times \times\sum_{n=0}^{\infty}(-1)^n\{A_n^s\,\mathrm{ce}_n(\eta_s, q_s) + B_n^s\,\mathrm{se}_n(\eta_s, q_s)\}. \qquad (8.4.32)$$

Magnetic Dipole. At point P let there be found a magnetic dipole with the moment $\mathfrak{m}$, directed toward the positive values of z parallel to the generatrices of the cylinders, whose configuration remains as before. Then if we seek the potential v^1 of the secondary field in the form of a series of Fourier integrals,

$$v^1 = im\sum_{s=\pm 1} v_s,$$
$$v_s = \sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\{a_n^s(h)\,c_n\mathrm{Me}_n^{(1)}(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + \qquad (8.4.33)$$
$$+ b_n^s(h)\,d_n\mathrm{Ne}_n^{(1)}(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}\,e^{ih(z-z_s)}\,dh, \qquad (8.4.34)$$

where $q_s = f_s^2(k^2 - h^2)/4$, $s = \pm 1$, then for determination of the unknowns A_n^s, B_n^s coupled with a_n^s and b_n^s by the formulas, /450

$$a_n^s = c_n\mathrm{Ce}_n'(\xi_s^0, q_s)A_n^s, \quad b_n^s = d_n\mathrm{Se}_n'(\xi_s^0, q_s)B_n^s, \qquad (8.4.35)$$

we find from the boundary conditions (1.7.8) on the basis of (8.4.2) an infinite system of linear equations (8.4.6) with matrix elements (8.3.80) and with the free terms,

$$F_n^s = -(-1)^n \frac{\mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)}{\mathrm{Me}_n^{(1)'}(\xi_s^0, q_s)} \mathrm{ce}_n(\eta_{s0}, q_s),$$

$$\Phi_n^s = -(-1)^n \frac{\mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)}{\mathrm{Ne}_n^{(1)'}(\xi_s, q_s)} \mathrm{se}_n(\eta_{s0}, q_s), \qquad (8.4.36)$$

$$s = \pm 1.$$

In (8.3.80) just as in (8.4.36), $q_s = f_s^2(k^2 - h^2)/4$, $s = \pm 1$, on the strength of which the matrix elements of the system and its free terms are functions of the parameter h, which varies in the range from $-\infty$ to ∞. The properties of the system referring to its solvability, remain as before; for each h the system is uniquely solvable by the method of truncation, and $\sum_{n=0}^{\infty} |A_n^s|^2 < \infty$ and $\sum_{n=0}^{\infty} |B_n^s|^2 < \infty$ for each h.

On the basis of (8.4.9) in the vicinity of the wave band in spherical coordinates r, θ, ϕ with the origin at point O the potential v^0 of the primary dipole field

$$v^0 = ikm \frac{e^{ikr - ikz_0 \cos\theta}}{r} \sin\theta\, e^{-ik\rho^0 \sin\theta \cos(\varphi^0 - \varphi)}, \qquad (8.4.37)$$

and the potential v^1 of the secondary field is approximately equal to

$$v^1 = 2m \frac{e^{ikr - ikz_0 \cos\theta}}{r} \sum_{s=\pm 1} e^{-iskl_0 \cos\varphi \sin\theta} \times$$

$$\times \sum_{n=0}^{\infty} (-1)^n \left[A_n^s c_n \mathrm{Ce}_n'(\xi_s^0, q_s) \mathrm{ce}_n(\eta_s, q_s) + \right. \qquad (8.4.38)$$

$$\left. + B_n^s d_n \mathrm{Se}_n(\xi_s^0, q_s) \mathrm{se}_n(\eta_s, q_s)\right]$$

(ρ^0, ϕ^0, z_0 are polar coordinates of point P with the origin at O). From (1.6.20) we find that in this approximation in the wave band for the complete field,

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\theta = \sqrt{\frac{\mu}{\varepsilon}} k^2 m \frac{e^{ikr - ikz_0 \cos\theta}}{r} \times$$

$$\times \sin\theta\, W^*(\theta, \varphi) e^{-ik\rho^0 \sin\theta \cos(\varphi - \varphi^0)}, \qquad (8.4.39)$$

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where the damping factor $W(\theta, \phi)$, for the characteristics of directionality of the magnetic dipole in free space, is equal to

$$W^*(\theta, \varphi) = 1 + e^{ik\rho^0 \sin\theta\cos(\varphi-\varphi^0)} \sum_{s=\pm 1} e^{-is\hat{k}l_0 \cos\varphi \sin\theta} \times$$
$$\times \sum_{n=0}^{\infty} (-1)^n \left[A_n^s c_n \mathrm{Ce}_n'(\xi_s^0, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + \right. \tag{8.4.40}$$
$$\left. + B_n^s d_n \mathrm{Se}_n'(\xi_s^0, q_s)\,\mathrm{se}_n(\eta_s, q_s)\right].$$

Just as before, the function $W^*(\theta, \phi)$ with a fixed value of θ characterizes the azimuth directionality of the emission from the system, consisting of a magnetic dipole and two elliptical cylinders, which play the role of passive reflectors. The resistance of the emission and the coefficient of the directed action of this system may be computed on the basis of the methods of numerical integration according to formulas (1.4.14) and (1.4.17), respectively.

For a numerical solution to the problem the approximate values of the quantities A_n^s and B_n^s are found from the finite system,

$$A_n^s + \sum_{m=0}^{N} \left\{\alpha_{nm}^{(-s,s)} A_m^{-s} + \beta_{nm}^{(-s,s)} B_m^{-s}\right\} = F_n^s,$$
$$B_n^s + \sum_{m=0}^{N} \left\{\gamma_{nm}^{(-s,s)} A_m^{-s} + \delta_{nm}^{(-s,s)} B_m^{-s}\right\} = \Phi_n^s, \tag{8.4.41}$$

in which $q_s = (f_s k \sin\theta)^2/4$, and $N \approx [2ka]$, where $ka = \max\limits_{s=\pm 1} (kf_s \mathrm{ch}\xi_s^0)$ (in the wave band $h = k \cos\theta$).

Just as in the previous problems we can significantly simplify the expressions which determine A_n^s, B_n^s and $W^*(\theta, \phi)$, due to the choice of special values of the parameters of the problem. In particular, in the case of two parallel bands, arranged as shown on Figure 53, c, expression (8.4.40) is substantially simplified. Thus, if $\alpha_{-1} = \pi/2$, $\alpha_{+1} = 3\pi/2$ and $\xi_{-1,0} = \xi_{+1,0}$, $f_{-1} = f_{+1}$, then

$$B_n = B_n^{+1} = -B_n^{-1}, \tag{8.4.42}$$

and then

$$W^*(\theta, \varphi) = 1 - 2i\, e^{ik\rho^0 \sin\theta\cos(\varphi-\varphi^0)} \sin(kl_0 \sin\theta\cos\varphi) \times$$
$$\times \sum_{n=1}^{\infty} (-1)^n B_n d_n \mathrm{Se}_n(0, q)\,\mathrm{se}_n(\varphi, q). \tag{8.4.43}$$

Hence it follows that the characteristics of emission from the magnetic dipole, located midway between the two parallel ideally conducting bands and possessing a moment parallel to them, do not depend on the presence of bands in the directions for which /452

$$kl_0 \sin s\theta \cos\varphi = 2\pi m, \quad m = 0, \pm 1, \ldots \tag{8.4.44}$$

If these same bands are arranged as shown in Figure 53,c, then

$$\delta_{nm}^{(-s,s)} = (-1)^{n+m}\delta_{nm}^{(s,-s)}, \tag{8.4.45}$$

$$F_n^{+1} = -(-1)^n F_n^{-1}, \qquad B_n = B_n^{+1} = -(-1)^n B_n^{-1}. \tag{8.4.46}$$

Therefore for such a case

$$W^*(\theta, \varphi) = 1 - 2ie^{ikp^0 \sin\theta\cos(\varphi-\varphi^0)} \times$$
$$\times \Big\{ \sin(kl_0 \sin\theta\cos\varphi) \sum_{n=0,2,\ldots} B_n d_n \mathrm{Se}_n'(0, q)\,\mathrm{se}_n(\varphi, q) -$$
$$- i\cos(kl_0 \sin\theta\cos\varphi) \sum_{n=1,3,\ldots} B_n d_n \mathrm{Se}_n'(0, q)\,\mathrm{se}_n(\varphi, q)\Big\}. \tag{8.4.47}$$

The law of density distribution of the surface currents induced on the surface of the s-the cylinder by the field of the magnetic dipole is determined as usual from (1.1.7), (1.6.20) on the basis of (8.4.2), (8.4.33) of the infinite system (8.4.6) with (8.3.80) ($h = k\cos\theta$), (8.4.36) and has in the general case the form,

$$j_\eta^s = \frac{mc}{2\pi^2} \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \{A_n^s(h)\,\mathrm{ce}_n(\eta_s, q_s) +$$
$$+ B_n^s(h)\,\mathrm{se}_n(\eta_s, q_s)\}(k^2 - h^2)e^{ih(z-z_0)}\,dh, \tag{8.4.48}$$

$$j_z^s = \frac{-imc}{2\pi^2 f_s \sqrt{\mathrm{ch}^2\xi_s^0 - \cos^2\eta_s}} \times$$
$$\times \sum_{n=0}^{\infty} (-1)^n \int_{-\infty}^{\infty} \{A_n^s(h)\,\mathrm{ce}_n'(\eta_s, q_s) + B_n^s(h)\,\mathrm{se}_n'(\eta_s, q_s)\}\,he^{ih(z-z_0)}\,dh$$

($j_\xi^s = 0$), where $q_s = f_s^2(k^2 - h^2)/4$, $s = \pm 1$. With degeneration of the cylinders into bands ($\xi_s^0 = 0$, $s = \pm 1$) $A_n^s(h) \neq 0$ and then

$$j_\eta^s \triangleq \frac{mc}{2\pi^2}\sum_{n=0}^{\infty}(-1)^n\int_{-\infty}^{\infty}B_n^s(h)\,\mathrm{se}_n(\eta_s,\ q_s)(k^2-h^2)\,e^{ih(z-z_0)}\,dh,$$

$$j_z^s = \frac{-imc}{2\pi^2 f_s \sin\eta_s}\sum_{n=0}^{\infty}(-1)^n\int_{-\infty}^{\infty}B_n^s(h)\,\mathrm{se}_n'(\eta_s,\ q_s)\,he^{ih(z-z_0)}\,dh. \qquad (8.4.49)$$

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Expressions (8.4.49) indicate an automatic satisfaction of the Meixner condition at the edge with transition from cylinders to bands.

The approximate values j_η^s, f_z^s may be found by the methods of numerical integration in (8.4.48), (8.4.49).

By integration in (8.4.3) over z_0 from $z_0 = z_1$ to $z_0 = z_2$, we find the generalization of the solution of the problem to the case of a linear antenna of finite length, extending from $z_0 = z_1$ to $z_0 = z_2$ parallel to the generatrices of the cylinders with the given current $K = K(z_0)$. The component of the electric field for such a case is equal to

$$E_\varphi = \frac{ick}{4\pi}\cdot\frac{e^{ikr}}{r}\sin\theta\left\{\int_{z_1}^{z_2}K(z_0)\,e^{-ikz_0\cos\theta}dz_0\right\}\times$$
$$\times W^*(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}. \qquad (8.4.50)$$

If $z_1 = -\bar{l}$, $z_2 = \bar{l}$ and the antenna is excited in the middle such that the current in it is distributed symmetrically according to the rule,

$$K(z_0) = \begin{cases}\dfrac{K_0}{\sin k\bar{l}}\sin k(\bar{l}-z_0), & 0\leqslant z_0\leqslant\bar{l},\\ \dfrac{K_0}{\sin k\bar{l}}\sin k(\bar{l}+z_0), & -\bar{l}\leqslant z_0\leqslant 0,\end{cases}$$

where K_0 is the current in the loop (the maximum value of $K(z_0)$), then for such a case,

$$E_\varphi = \frac{iK_0c}{2\pi}\cdot\frac{e^{ikr}}{r}\cdot\frac{\left[\cos(k\bar{l}\cos\theta)-\cos k\bar{l}\right]}{\sin\theta\sin k\bar{l}}\times$$
$$\times W^*(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)} \qquad (8.4.51)$$

or if the length of the antenna is equal to half the wave ($\bar{l} = \lambda_0/4$).

$$E_\varphi = \frac{iK_0c}{2\pi}\cdot\frac{e^{ikr}}{r}\cdot\frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta}\times \qquad (8.4.52)$$
$$\times W^*(\theta,\ \varphi)\,e^{-ik\rho^0\sin\theta\cos(\varphi-\varphi^0)}$$

With integration in (8.4.2), (8.4.33) over z_0 from $-\infty$ to $+\infty$ we find the generalization of the solution of the problem to the case when the exciter of the cylinders is an infinite linear source with a given magnetic current K, parallel to the generatrices of the cylinders. In this case for the potentials of the primary and secondary fields we find the expressions /454

$$v^0 = -\frac{\sqrt{\frac{\varepsilon}{\mu}}K}{2kc}\sum_{n=0}^{\infty}(-1)^n\{c_n^2\,\mathrm{ce}_n(\eta_s, q_s)\,\mathrm{ce}_n(\eta_{s0}, q_s)\times$$
$$\times\begin{bmatrix}\mathrm{Me}_n^{(1)}(\xi_s, q_s)\,\mathrm{Ce}_n(\xi_{s0}, q_s)\\ \mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)\,\mathrm{Ce}_n(\xi_s, q_s)\end{bmatrix} + d_n^2\,\mathrm{se}_n(\eta_s, q_s)\,\mathrm{se}_n(\eta_{s0}, q_s)\times$$
$$\times\begin{bmatrix}\mathrm{Ne}_n^{(1)}(\xi_s, q_s)\,\mathrm{Se}_n(\xi_{s0}, q_s)\\ \mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)\,\mathrm{Se}_n(\xi_s, q_s)\end{bmatrix}\Bigg\}, \quad \begin{matrix}\xi_s > \xi_{s0},\\ \xi_s < \xi_{s0},\end{matrix} \qquad (8.4.53)$$

$$v^1 = -\frac{\sqrt{\frac{\varepsilon}{\mu}}K}{2kc}\sum_{n=0}^{\infty}\sum_{s=\pm 1}\{A_n^s c_n^2\,\mathrm{Ce}_n'(\xi_s^0, q_s)\times$$
$$\times\,\mathrm{Me}_n^{(1)}(\xi_s, q_s)\,\mathrm{ce}_n(\eta_s, q_s) + \qquad (8.4.54)$$
$$+ B_n^s d_n^2\,\mathrm{Se}_n'(\xi_s^0, q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s, q_s)\,\mathrm{se}_n(\eta_s, q_s)\},$$

where $q_s = (f_s k)^2/4$, and A_n^s and B_n^s are found from system (8.4.6) with the matrix elements (8.3.30) and with right-hand sides (8.4.36) with the values of $h = 0$. Solution to (8.4.54) is similar to the solution to the problem on diffraction of a cylindrical wave (4.4.9) on two parallel ideally conducting cylinders of elliptical cross section. In the vicinity of the wave band the only non-zero longitudinal component of the secondary magnetic field in polar coordinates ρ, ϕ with the origin at point O is equal to

$$H_z^1 = -\frac{K\sqrt{\frac{\varepsilon}{\mu}}}{c}\sqrt{\frac{k}{2\pi}}\frac{e^{ik\rho}}{\sqrt{\rho}}e^{-\frac{i\pi}{4}}\sum_{s=\pm 1}e^{-iskl_s\cos\varphi}\times \qquad (8.4.55)$$
$$\times\sum_{n=0}^{\infty}\{A_n^s c_n\,\mathrm{Ce}_n'(\xi_s^0, q_s)\,\mathrm{ce}_n(\eta_s, q_s) +$$
$$+ B_n^s d_n\,\mathrm{Se}_n(\xi_s^0, q_s)\,\mathrm{se}_n(\eta_s, q_s)\}.$$

The complete field near the s-th cylinder may be studied using the formulas in (1.6.20) in which the potential $v = v^0 + v^1$ on the basis of the addition theorems (4.5.7), (4.5.8) and (8.4.6), (8.4.53), (8.4.54) is represented in the form,

$$v = \frac{\sqrt{\frac{\varepsilon}{\mu}}\,K}{2kc}\sum_{n=0}^{\infty}\{c_n^2 A_n^s\,[\mathrm{Me}_n^{(1)\prime}(\xi_s^0,\ q_s)\,\mathrm{Ce}_n(\xi_s,\ q_s) -$$
$$-\mathrm{Ce}_n'(\xi_s^0,\ q_s)\,\mathrm{Me}_n^{(1)}(\xi_s,\ q_s)]\,\mathrm{ce}_n(\eta_s,\ q_s) +$$
$$+ d_n^2 B_n^s\,[\mathrm{Ne}_n^{(1)\prime}(\xi_s^0,\ q_s)\,\mathrm{Se}_n(\xi_s,\ q_s) -$$
$$-\mathrm{Se}_n'(\xi_s^0,\ q_s)\,\mathrm{Ne}_n^{(1)}(\xi_s,\ q_s)]\,\mathrm{se}_n(\eta_s,\ q_s)\} \qquad (8.4.56)$$

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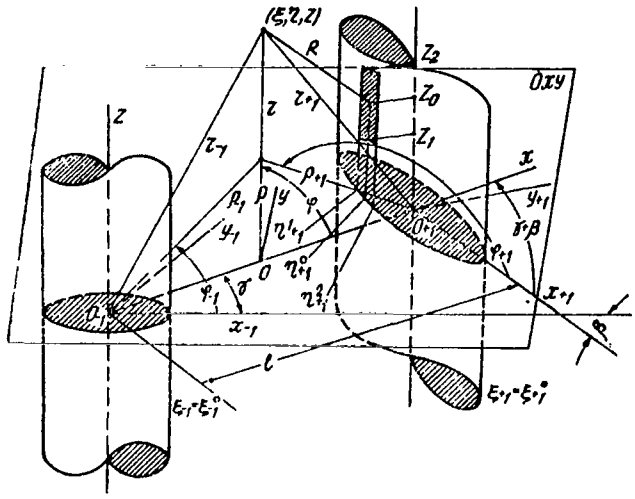

Fig. 54. Generalized Problems for the Case of a Longitudinal Slot. Located on One of the Elliptical Cylinders.

On the basis of (8.4.56) and (1.1.7), (1.6.20) for the density distribution of the surface currents induced on the surface of the s-th cylinder, we find the formula,

$$j_\eta^s = -\frac{k^2 mc}{\pi}\sum_{n=0}^{\infty}(-1)^n\{A_n^s\,\mathrm{ce}_n(\eta_s,\ q_s) + B_n^s\,\mathrm{se}_n(\eta_s,\ q_s)\}, \qquad (8.4.57)$$
$$q_s = (f_s k)^2/4,$$

written in the coordinates of the s-th cylinder.

This problem allows significant generalization to the case of a longitudinal slot (Fig. 54), located on the surface of one of the elliptical cylinders in the presence in the field of emission from the slot of the other elliptical cylinder, playing (with respect to the first cylinder) the role of the passive reflector. Such a generalization is obtained by the limiting conversion in (8.4.50) over /456 ξ_{s0} when ξ_{s0} tends to ξ_s^0 ($s = -1$ or $s = +1$). After completing the limiting conversion we obtain a problem concerning the source, equivalent in the electrodynamic sense to the narrow longitudinal emitting slot with a width $h_{\eta_0} d\eta_0$, located on the surface of the

s-th cylinder and extending in length from $z_0 = z_1$ to $z_0 = z_2$. If we compare the coordinate system $Oxyz$, in which expression (8.4.50) is written with the local coordinate system $O_s x_s y_s z_s$ of the s-th cylinder, then in spherical coordinates r, θ, ϕ with the origin at point O when $\xi_{s0} = \xi_s^0$ ($s = -1$ or $s = +1$) we will have

$$E_\varphi = \frac{ick}{4\pi} \cdot \frac{e^{ikr}}{r} \sin\theta \left\{ \int_{z_1}^{z_2} K(z_0) e^{-ikz_0 \cos\theta} dz^0 \right\} \times \\ \times W^*(\theta, \varphi) e^{-ik\rho^0 \sin\theta \cos[(\varphi - \varphi^0)}, \tag{8.4.58}$$

where

$$W^*(\theta, \varphi) = 1 + e^{ik\rho^0 \sin\theta \cos(\varphi - \varphi^0)} \times \\ \times \left\{ \sum_{n=0}^{\infty} (-1)^n \left[A_n^s c_n \mathrm{Ce}_n'(\xi_s^0, q_s) \mathrm{ce}_n(\varphi_s, q_s) + \right.\right. \\ \left. + B_n^s d_n \mathrm{Se}_n'(\xi_s^0, q_s) \mathrm{se}_n(\varphi_s, q_s) \right] + \\ + e^{sikl \sin\theta \cos\varphi} \sum_{n=0}^{\infty} (-1)^n \left[A_n^{-s} c_n \mathrm{Ce}_n(\xi_{-s}^0, q_{-s}) \times \right. \\ \left.\left. \times \mathrm{ce}_n(\varphi_{-s}, q_{-s}) + B_n^{-s} d_n \mathrm{Se}_n(\xi_{-s}^0, q_{-s}) \mathrm{se}_n(\varphi_{-s}, q_{-s}) \right] \right\} \tag{8.4.59}$$

If we substitute the distribution of the current $K(z_0)$ in (8.4.58) by the distribution of voltage on the slot $E_\eta(\xi_s^0, \eta_{s0}, z_0) h_{\eta_{s0}} d\eta_{s0}$, then after integrating (8.4.58) over $\eta_s^0 = \eta_{s0}$ in the range from η_s^1 to η_s^2 we find the generalization of the solutions to the case of a rectangular slot of finite width with the given distribution of the field $E_\eta(\xi_s^0, \eta_s^0, z_0)$ in the opening of the slot. If we then assume that the voltage on the slot is

$$V(z_0) = E_\eta(\xi_s^0, \eta_s^0, z_0) h_{\eta_s^0} (h_s^2 - h_s^1),$$

we obtain

$$E_\varphi = \frac{e^{ikr}}{r} S(\theta) M(\theta, \varphi), \tag{8.4.60}$$

where

$$S(\theta) = \frac{ick}{4\pi} \sin\theta \int_{z_1}^{z_2} V(z_0) e^{-ikz^0 \cos\theta} dz_0, \tag{8.4.61}$$

$$M(\theta, \varphi) = \int_{\eta_s^1}^{\eta_s^2} \frac{e^{-ik\rho^0 \sin\theta \cos(\varphi - \varphi^0)}}{(\eta_s^2 - \eta_s^1)} W^*(\theta, \varphi) d\eta_s^0. \tag{8.4.62}$$

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If, in the antenna, the distribution of the magnetic current along the slot is sinusoidal, then this case will be represented by the distribution of voltage along the slot,

$$V(z_0) = V_0 \sin k(\bar{l} - |z_0|), \quad z_1 = -\bar{l} \leqslant z_0 \leqslant \bar{l} = z_2,$$

which can be written in the form,

$$V(z_0) = V_0 \frac{\sin k(\bar{l} - |z_0|)}{\sin k\bar{l}}, \quad -\bar{l} \leqslant z_0 \leqslant \bar{l},$$

if instead of the voltage in the loop V_0 we take the voltage at the supply points V_0, applied in the middle of the slot. Then

$$S(\theta) = \frac{icV_0}{\pi} \cdot \frac{\cos(k\bar{l}\cos\theta) - \cos k\bar{l}}{\sin\theta \sin k\bar{l}}$$

or when $\bar{l} = \lambda_0/4$

$$S(\theta) = \frac{icV_0}{\pi} \cdot \frac{\cos\left[\left(\frac{\pi}{2}\right)\cos\theta\right]}{\sin\theta}.$$

In expression (8.4.62) both A_n^s, B_n^s and A_n^{-s}, B_n^{-s}, through which the function $W^*(\theta, \phi)$ is determined, depend on the integration variable η_s^0. Furthermore, $A_n^{\pm s}$ and $B_n^{\pm s}$ are also functions of the variable η_{-s}^0 which depends on η_s^0. Therefore, in order to carry out the integration in (8.4.62) over η_s^0 we must first explicity express the functional dependence of the quantities $A_n^{\pm s}$ and $B_n^{\pm s}$ on η_s^0. This can be done in the following manner.

Let us write the right-hand sides of the truncated system (8.4.41) from which we find the approximate values of the quantities $A_n^{\pm s}$, $B_n^{\pm s}$, in the form,

$$\begin{aligned} F_n^s &= f_n^s c_n \mathrm{Me}_n^{(1)}(\xi_{s0}, q_s)\,\mathrm{ce}_n(\eta_s^0, q_s), \\ \Phi_n &= \varphi_n^s d_n \mathrm{Ne}_n^{(1)}(\xi_{s0}, q_s)\,\mathrm{se}_n(\eta_s^0, q_s), \end{aligned} \qquad (8.4.63)$$

where, as is clear from (8.4.36),

$$f_n^s = -\frac{(-1)^n}{c_n \mathrm{Me}_n^{(1)'}(\xi_s^0, q_s)}, \quad \varphi_n^s = \frac{(-1)^n}{d_n \mathrm{Ne}_n^{(1)'}(\xi_s^0, q_s)}.$$

System (8.4.41) can be written in normal form, /458

$$z_i + \sum_{j=1}^{4N+2} c_{ij} z_j = f_i, \quad i = 1, 2, \ldots, 4N+2, \tag{8.4.64}$$

if we introduce the designations (8.3.20), (8.3.22), (8.3.23). Let Δ be the determinant of the system (8.4.64). Then

$$z_i = \frac{\Delta_i}{\Delta}, \quad i = 1, 2, \ldots, 4N+2, \tag{8.4.65}$$

where Δ_i is the determinant obtained from Δ by substitution in it of the elements of the i-th column with the column of free terms. Expanding Δ_i over elements of the j-th column and writing the result in the form,

$$\Delta_i = \sum_j f_{4j+1} \Delta_{ij}^{(1)} + \sum_j f_{4j+2} \Delta_{ij}^{(2)} + \\ + \sum_j f_{4j+3} \Delta_{ij}^{(3)} + \sum_j f_{4j+4} \Delta_{ij}^{(4)},$$

where $\Delta_{ij}^{(1)-(4)}$ are algebraic complements, corresponding to the elements f_i we will have on the basis of the definitions used above for F_n^s and Φ_n^s,

$$z_i = \sum_j (f_{4j+1}^{+1} \Delta_{ij}^{(1)}) c_{4j+1} \mathrm{Me}_{4j+1}^{(1)}(\xi_{+1,0}, q_{+1}) \mathrm{ce}_{4j+1}(\eta_{+1}^0, q_{+1}) + \\ + \sum_j (f_{4j+2}^{-1} \Delta_{ij}^{(2)}) c_{4j+2} \mathrm{Me}_{4j+2}^{(1)}(\xi_{-1,0}, q_{-1}) \mathrm{ce}_{4j+2}(\eta_{-1}^0, q_{-1}) + \\ + \sum_j (f_{4j+3}^{+1} \Delta_{ij}^{(3)}) d_{4j+3} \mathrm{Ne}_{4j+3}^{(1)}(\xi_{+1,0}, q_{+1}) \mathrm{se}_{4j+3}(\eta_{+1}^0, q_{+1}) + \\ + \sum_j (f_{4j+4}^{-1} \Delta_{ij}^{(4)}) d_{4j+4} \mathrm{Ne}_{4j+4}^{(1)}(\xi_{-1,0}, q_{-1}) \mathrm{se}_{4j+4}(\eta_{-1}^0, q_{-1}). \tag{8.4.66}$$

Here the functions of the angles η_{+1}^0, η_{-1}^0 are only the clearly written angular Mathieu functions ($s = -1$ or $s = +1$). Since the slot is located on the surface of the s-th cylinder ($s = -1$ or $s = +1$) and integration in (8.4.62) is carried over η_{-1}^0 or only over η_{+1}^0, $\xi_{-s,0}$, η_{-s}^0 in (8.4.66) it is sufficient to apply the addition theorems (4.5.7), (4.5.8) to the clearly written elliptical wave functions.

The formulas obtained may be simplified by choosing special

particular values of the parameters determining the solution to the problem.

CHAPTER 9

DIFFRACTION OF ELECTROMAGNETIC WAVES ON SPHEROIDS AND DISKS

§1. Solution to the Problem on Spheroids

Unlike the problems studied in the previous chapters, the use /459
of the method of separation of variables for solving the diffraction problems with the boundary conditions given on the surface of a spheroid (prolate or oblate) involves a number of specific difficulties. These difficulties consist of the fact that in the general case in spheroidal coordinates we can not introduce auxiliary potential functions which, by satisfying the Helmholtz equation, would also solve simple (individual) boundary conditions. For this reason there are still insufficient data on the diffraction of electromagnetic waves on spheroids. Among the solutions to the problems is the problem on the diffraction of a plane electromagnetic wave on an ideally conducting spheroid under the condition that the wave is propagated along the polar axis of the spheroid [165, 166].[1] The problem is solved on the axisymmetric excitation of a prolate (oblate) ideally conducting spheroid by an electric dipole, parallel to the polar axis of the spheroid and located on it [40, 50]. The problem on the diffraction of electromagnetic waves is studied in greater detail on an ideally conducting circular disk, i.e., the degenerated case of an oblate spheroid. This case is exceptional in the sense that it allows complete separation of the Maxwell equations in the coordinates of an oblate spheroid (see, for example, [48, 50, 167]). The diffraction problems for a disk may be solved in the general case using two scalar potential functions, which satisfy the Helmholtz equation and the individual boundary conditions on the surface of the disk. The problem on the diffraction of the dipole field of the source, for example a horizontal magnetic dipole, located on the axis of rotation of a prolate or oblate spheroid, also permits the introduction of two auxiliary scalar /460
potentials, through which the sought field vectors are expressed. They are similar to those which were introduced in solving the problem on the diffraction of a field of a horizontal magnetic dipole on a disk (the dipole is located on the axis of the disk [50]). However for these potentials it is impossible to obtain individual boundary conditions on the surface of the spheroid as a result of which for determining the expansion coefficients of the potentials into infinite series of spheroidal wave functions we must solve an infinite system of linear equations.

The problems on the diffraction of electromagnetic waves on two spheroids and disks with a common axis of symmetry are solved

[1] In [189] the problem was studied of the diffraction of a plane electromagnetic wave incident at an arbitrary angle on the *Oz* axis.

ısing the auxiliary potentials introduced on the basis of formulas 1.2.13), (1.2.17), (1.3.9), (1.3.10), which couple the electric ›r magnetic Hertz vectors with the field vectors. For this reason .he method of solving the problem, based on the use of the Hertz ·ectors, is sometimes called in the literature the Hertz vector ıethod. Although the problem is exciting a spheroid by a vertical .ipole located on the axis of the spheroid and directed along it, ›ermits introduction of the Abraham potentials, here it is solved ›y the Hertz vector method.

§2. Field of a Vertical Electric Dipole in the Presence of a Prolate (Oblate) Spheroid.

In an unbounded uniform and isotropic space with electric and agnetic permeabilities ε, μ and a conductivity σ ◘ 0, let there be prolate (oblate) ideally conducting spheroid which, in the coor- inates of a prolate (oblate) spheroid ξ, η, ϕ, occupies the region f space $\xi \leq \xi^0$ (origin of the coordinate system *Oxyz* matches the enter of the spheroid and the *Oz* axis is directed along the axis f rotation), on whose axis at point *P* we find the unit electric scillation dipole with the moment p, directed along the axis of otation toward the positive values of the coordinate *z* (Fig. 55). n this case the primary dipole field is determined through the lectric Hertz vector,

$$\vec{\Pi}^0 = \mathbf{p}\frac{e^{ikR}}{R}, \tag{9.2.1}$$

hich has only a non-zero longitudinal component,

$$\Pi_z^0 = \Pi^0 = p\frac{e^{ikR}}{R}, \qquad p = p_z = |\mathbf{p}|, \tag{9.2.2}$$

here *R* is the distance from point *P* to the point of observation /461 hat is independent in spheroidal coordinates of the azimuth angle ($\frac{\partial}{\partial\phi} \equiv 0$). Let us look at the case of a prolate spheroid. Let $\vec{\Pi}^{(1)} =$ 0, 0, Π_z^1} be the electric Hertz vector of the secondary electro- agnetic field scattered on the spheroid that is also independent f the angle ϕ (the problem is axisymmetric). Then using (1.2.13) e can establish that in the coordinates of a prolate spheroid 1.5.40) the components of the resulting electromagnetic field are xpressed through the only non-zero azimuth component H_ϕ of the ector of magnetic intensity by the formulas

$$E_\xi = \frac{i}{k_0\varepsilon}\cdot\frac{1}{f(\xi^2-\eta^2)^{1/2}}\cdot\frac{\partial}{\partial\eta}[(1-\eta^2)^{1/2}H_\varphi],$$
$$E_\eta = \frac{-i}{k_0\varepsilon}\cdot\frac{1}{f(\xi^2-\eta^2)^{1/2}}\cdot\frac{\partial}{\partial\xi}[(\xi^2-1)^{1/2}H_\varphi], \tag{9.2.3}$$

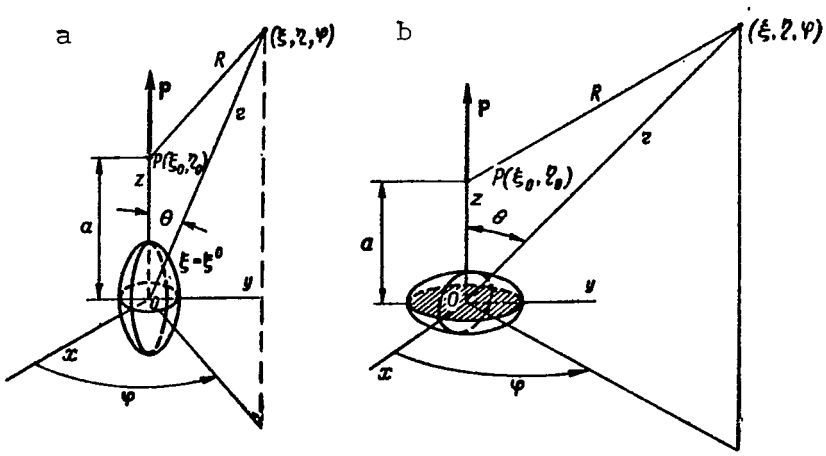

Fig. 55. Excitation of a Prolate (a) and Oblate (b) Spheroid by a Vertical Dipole.

in which H_ϕ is the sum of the components H_ϕ^0 and H_ϕ^1, respectively, of the primary dipole field and the secondary field scattered on the spheroid. As shown in [40, 50], the component H_ϕ^0 may be expanded into a series of spheroidal wave functions,

$$H_\varphi^0 = \frac{4k^2p}{f(\xi_0^2-1)^{1/2}} \sum_{n=1}^{\infty} \frac{\sigma_{1n}(c)}{N_{1n}(c)} S_{1n}(c,\eta) \times$$

$$\times \begin{cases} R_{1n}^{(1)}(c,\xi) R_{1n}^{(3)}(c,\xi_0), & \xi_0 > \xi, \\ R_{1n}^{(1)}(c,\xi_0) R_{1n}^{(3)}(c,\xi), & \xi_0 < \xi, \end{cases} \quad (c = kf) \qquad (9.2.4)$$

in which ξ_0 and $\eta_0 = 1$ are coordinates which determine the position of point P with the dipole on the axis Oz, and $\sigma_{1n}(c) = \frac{1}{2} \sum_{r=0,1}' \frac{(r+2)!}{r!} d_r^{1n}(c)$

Seeking the resulting electromagnetic field in this problem is reduced to determining the component H_ϕ^1 of the secondary field. The components E_ξ^1, E_η^1 of the vector of the electric intensity $\mathbf{E}^1$ of the secondary field will be expressed through H_ϕ^1 by the same relationships as (9.2.3). Therefore if we rewrite the Maxwell equation $\operatorname{rot} \mathbf{E}^1 = ik_0\mu \mathbf{H}^{(1)}$ in spheroidal coordinates and then drop from the expression obtained for H_ϕ^1 the components E_ξ^1, E_η^1 we finally obtain the differential equation,

$$\sqrt{\xi^2-1}\,\frac{\partial^2}{\partial\xi^2}\{\sqrt{\xi^2-1}\,H_\varphi^1\} + \sqrt{1-\eta^2}\times$$

$$\times \frac{\partial^2}{\partial\eta^2}\{\sqrt{1-\eta^2}\,H_\varphi^1\} + c^2(\xi^2-\eta^2)H_\varphi^1 = 0 \qquad (9.2.5)$$

for the function H_{ϕ}^{1} which as we can easily see is the Helmholtz equation with the azimuth number $m = 1$. Consequently, in mathematical formulation the problem consists of seeking a unique solution to the Helmholtz equation (9.2.5) which satisfies on the surface of the spheroid when $\xi = \xi^0$ the boundary condition,

$$\frac{\partial}{\partial \xi}\left[(\xi^2-1)^{1/2}\left(H_{\varphi}^{0}+H_{\varphi}^{1}\right)\right]=0, \tag{9.2.6}$$

which is found from the condition of vanishing of the tangential component E_{η} on the spheroid, and the supplementary condition of radiation at infinity. Solution to the problem may be sought in the form,

$$H_{\varphi}^{1}=\frac{4k^2p}{f(\xi_0^2-1)^{1/2}}\sum_{n=1}^{\infty}a_n R_{1n}^{(3)}(c,\ \xi)S_{1n}(c,\ \eta). \tag{9.2.7}$$

Then condition (9.2.6) will be satisfied if, as the expansion coefficients, we take the expressions,

$$a_n=-\frac{\sigma_{1n}(c)}{N_{1n}(c)}R_{1n}^{(3)}(c,\ \xi_0)\frac{\left[\sqrt{\xi^{02}-1}\,R_{1n}^{(1)}(c,\ \xi^0)\right]'}{\left[\sqrt{\xi^{02}-1}\,R_{1n}^{(3)}(c,\ \xi^0)\right]'} \tag{9.2.8}$$

Thus, for the resulting field when $\xi > \xi_0$, /463

$$H_{\varphi}=\frac{4k^2p}{f(\xi_0^2-1)^{1/2}}\sum_{n=1}^{\infty}A_n(c;\ \xi_0,\ \xi^0)R_{1n}^{(3)}(c,\ \xi)S_{1n}(c,\ \eta), \tag{9.2.9}$$

where

$$A_n(c;\ \xi_0,\ \xi^0)=\frac{\sigma_{1n}(c)}{N_{1n}(c)}\times$$
$$\times\left\{R_{1n}^{(1)}(c,\ \xi_0)-\frac{\left[\sqrt{\xi^{02}-1}\,R_{1n}^{(1)}(c,\ \xi^0)\right]'}{\left[\sqrt{\xi^{02}-1}\,R_{1n}^{(3)}(c,\ \xi^0)\right]'}R_{1n}^{(3)}(c,\ \xi_0)\right\}. \tag{9.2.10}$$

If we set here $\xi_0 = \xi^0$, then

$$A_n(c;\ \xi^0,\ \xi^0)=\frac{i\sigma_{1n}(c)}{c\sqrt{\xi^{02}-1}\,N_{1n}(c)\left[\sqrt{\xi^{02}-1}\,R_{1n}^{(3)}(c,\ \xi^0)\right]'}. \tag{9.2.11}$$

In the vicinity of the wave band,

$$E_\eta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi = k^2 p \frac{e^{ikR}}{R} W(\theta), \qquad (9.2.12)$$

where

$$W(\theta) = -i\sqrt{\frac{\mu}{\varepsilon}}\,\frac{4e^{ika\cos\theta}}{c(\xi^{02}-1)^{1/2}} \times \times \sum_{n=1}^{\infty}(-i)^n A_n(c,\ \xi^0,\ \xi^0) S_{1n}(c\cos\theta) \qquad (9.2.13)$$

($a = z_0 = f\xi^0$). In the electrodynamic sense the case $\xi_0 = \xi^0$ is equivalent to the case of an annular-shaped symmetrical emitting slot ("magnetic ring") cut off at the pole of the spheroid, if the radius of the ring is small in comparison with the wave length and the dimensions of the spheroid. The function $W(\theta)$ is a complex characteristic of the emission from such a slot (damping factor of the electric vertical dipole in the presence of a prolate spheroid).

After using the value of the Wronskian determinant for the radial spheroidal functions, we find that when $\xi_0 > \xi^0$ the law of the density distribution of surface currents on the surface of the spheroid induced by the field of the raised vertical electric dipole, will be given by the formula,

$$j_\eta = -\frac{\omega\sqrt{\varepsilon\mu}pi}{\pi f^2(\xi_0^2-1)^{1/2}(\xi^{02}-1)^{1/2}} \sum_{n=1}^{\infty}\frac{\sigma_{1n}(c)}{N_{1n}(c)} \times \times \frac{R_{1n}^{(3)}(c,\ \xi_0)}{\sqrt{\xi^{02}-1}\,R_{1n}^{(3)}(c,\ \xi^0)]'} S_{1n}(c,\ \eta). \qquad (9.2.14)$$

The resistance of the radiation from the dipole field in the presence of a spheroid may be computed according to formula (1.4.16) where in the case when the electric dipole is found on the surface of the spheroid, the total power emitted by the spheroidal antenna is equal to (c_0 is the speed of light), /464

$$\bar{P} = 4k^2p^2\frac{c_0\sqrt{\frac{\mu}{\varepsilon}}}{f^2(\xi^{02}-1)}\sum_{n=1}^{\infty}|A_n(c,\xi^0,\xi^0)|^2 N_{1n}(c). \qquad (9.2.15)$$

Now let us look at the case of an oblate spheroid. The substitution

$$\left.\begin{aligned}\xi &\to i\xi,\\ \eta &\to \eta,\\ f &\to -if\end{aligned}\right\} \qquad (9.2.16)$$

will formally transform the prolate spheroidal system of coordinates (1.5.40) into an oblate spheroidal system of coordinates (1.5.48), Here the Helmholtz wave equation in system (1.5.40) is transformed into the Helmholtz wave equation in system (1.5.48) (the parameter c of the wave equation is transformed into the parameter $-ic$). Therefore any solution to the wave equation in the prolate spheroidal system of coordinates and any formula that is valid in this system transforms into the respective solution to the wave equation or to the formula in the coordinates of an oblate spheroid with the substitution (9.2.16) (the substitution, inverse to (9.2.16), leads to the opposite). Therefore after substituting (9.2.16) in all the formulas obtained above, we find the solution to the problem on diffraction of the field of a vertical electric dipole on an oblate ideally conducting spheroid. In particular, the characteristics of the emission from the oblate spheroid $\xi = \xi^0$ excited by the vertical electric dipole located in it, will be equal to

$$W(\theta) = -i\sqrt{\frac{\mu}{\varepsilon}}\,\frac{4e^{ika\cos\theta}}{c(\xi^{02}+1)^{1/2}}\times \\ \times\sum_{n=1}^{\infty}(-i)^n A_n(-ic,\ i\xi^0,\ i\xi^0)S_{1n}(-ic,\ \cos\theta). \qquad (9.2.17)$$

Converting to the limit when $\xi^0 \to 0$ (the oblate spheroid transforms to a disk with radius f), we find the characteristics of the emission from an infinitely thin ideally conducting circular disk, excited by the vertical electric dipole located in its center

$$W(\theta) = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{4}{c^2}\sum_{n=1}^{\infty}(-i)^n\,\frac{\sigma_{1n}(-ic)}{N_{1n}(-ic)R_{1n}^{(3)\prime}(-ic,\ i0)}\times \\ \times S_{1n}(-ic,\ \cos\theta). \qquad (9.2.18)$$

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The results of computing the function $W(\theta)$ for the series of values of the parameter $c = kf$ are shown in [50].

The law of the density distribution of surface currents induced on the surface of an ideally conducting oblate spheroid by the field of the vertical electric dipole is determined by the formula,

$$j_\eta = -\frac{\omega\sqrt{\varepsilon\mu}\,pi}{\pi f^2(\xi_0^2+1)^{1/2}(\xi^{02}+1)^{1/2}}\times \\ \times\sum_{n=1}^{\infty}\frac{\sigma_{1n}(-ic)}{N_{1n}(-ic)}\,\frac{R_{1n}^{(3)}(-ic,\ i\xi_0)S_{1n}(-ic,\ \eta)}{\left[\sqrt{\xi^{02}+1}\,R_{1n}^{(3)}(-ic,\ i\xi^0)\right]'}, \qquad (9.2.19)$$

which, in the case of degeneration of the spheroid into a circular infinitely thin disk with radius f, transforms into

$$j_\eta = -\frac{\omega\sqrt{\varepsilon\mu}\,pi}{\pi f^2(\xi_0^2+1)^{1/2}}\sum_{n=1}^{\infty}\frac{\sigma_{1n}(-ic)}{N_{1n}(-ic)}\times$$
$$\times\frac{R_{1n}^{(3)}(-ic,\ i\xi_0)}{R_{1n}^{(3)'}(-ic,\ i0)}S_{1n}(-ic,\ \eta). \qquad (9.2.20)$$

The resistance of the emission from the dipole in the presence of an oblate spheroid, at whose pole we find the dipole, is found according to formula (1.4.14), where

$$\bar{P} = 4k^2p^2\frac{c_0\sqrt{\frac{\mu}{\varepsilon}}}{f^2(\xi^{02}+1)}\times$$
$$\times\sum_{n=1}^{\infty}|A_n(-ic,\ i\xi^0,\ i\xi^0)|^2N_{1n}(-ic), \qquad (9.2.21)$$

and in the case of a disk

$$\bar{P} = 4p^2\frac{c_0\sqrt{\frac{\mu}{\varepsilon}}}{f^4}\times$$
$$\times\sum_{n=1}^{\infty}\left|\frac{\sigma_{1n}(-ic)}{R_{1n}^{(3)'}(-ic,\ i0)}\right|^2\frac{1}{N_{1n}(-ic)}. \qquad (9.2.22)$$

§3. Field of a Horizontal Magnetic Dipole in the Presence of a Prolate (Oblate) Spheroid

Now let us look at the problem of diffraction of the field of a horizontal magnetic dipole located at point P on the axis of /46

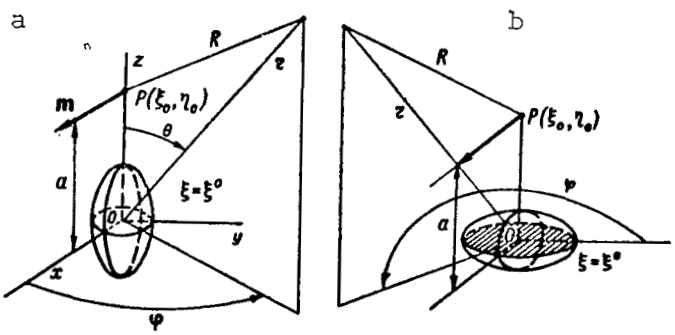

Fig. 56. Excitation of a Prolate (a) and an Oblate (b) Spheroid by a Horizontal Dipole.

rotation of the spheroid and possessing the moment m, directed perpendicular to it (Fig. 56). It is assumed that the spheroid is an ideally conducting one and that the moment m lies in the plane Oxz. Therefore for the magnetic Hertz vector of the dipole field,

$$\vec{\Pi}^{*0} = \mathbf{m}\frac{e^{ikR}}{R} \tag{9.3.1}$$

we will have $\Pi_y^{*0} = \Pi_z^{*0} = 0$, and

$$\Pi_x^{*0} = m\frac{e^{ikR}}{R}, \qquad m = m_x. \tag{9.3.2}$$

The coordinates of point P remain as before: $\xi = \xi^0$, $\eta = \eta_0 = 1$ (or $x = y = 0$, $z = z_1$). The component Π_x^{*0} obviously does not depend on ϕ, since $R = f(\xi^2 \mp 1 \pm \eta^2)^{1/2}$. The primary field of the magnetic dipole is found through $\vec{\Pi}^{*0}$ according to formulas (1.2.17),

$$\begin{aligned} \mathbf{E}^0 &= ik_0\mu \operatorname{rot}\vec{\Pi}^{*0}, \\ \mathbf{H}^0 &= \operatorname{rot}\operatorname{rot}\vec{\Pi}^{*0} = \operatorname{grad}\operatorname{div}\vec{\Pi}^{*0} + k^2\vec{\Pi}^{*0}. \end{aligned} \tag{9.3.3}$$

If we introduce into study the magnetic Hertz vector $\vec{\Pi}^*$ of the resulting field E, H and set $\vec{\Pi}^* = \vec{\Pi}^{*0} + \vec{\Pi}^{*1}$, where $\vec{\Pi}^{*1}$ is the Hertz vector of the secondary field, then the vectors E, H will be related with $\vec{\Pi}^*$ by the formulas /467

$$\begin{aligned} \mathbf{E} &= ik_0\mu \operatorname{rot}\vec{\Pi}^*, \\ \mathbf{H} &= \operatorname{rot}\operatorname{rot}\vec{\Pi}^* = \operatorname{grad}\operatorname{div}\vec{\Pi}^* + k^2\vec{\Pi}^*. \end{aligned} \tag{9.3.4}$$

The desired value here is the Hertz vector $\vec{\Pi}^{*1}$ of the secondary field. The problem is fully solved by determining it. Its solution permits a series of particular cases which are of interest from the view point of their applications. Thus, for example, when the dipole source is moved to infinity, we find the solution to the problem on the diffraction of a plane electromagnetic wave on a spheroid, propagating along the axis of rotation of the spheroid. In the case of an oblate spheroid, when $\xi_0 = \xi^0 = 0$, from it we obtain the solution to the problem on the emitting unit slot located on the disk in its center, etc.

Let us look first at the case of a prolate spheroid. In spheroidal coordinates $\Pi_x^{*0} = m\frac{e^{ikR}}{R} = \Phi^0(\xi, \eta)$.. The Hertz vector $\vec{\Pi}^{*1}$ of the primary field is sought in the form $\vec{\Pi}^{*1} = \{\Pi_x^{*1}, 0, \Pi_z^{*1}\}$, , where by virtue

of the symmetry we must have $\Pi_x^{*1} = \Phi_1(\xi, \eta)$ (Π_x^{*1} does not depend on ϕ). The component Π_z^{*1} is represented in the form $\Pi_z^{*1} = \Psi(\xi, \eta)\cos\varphi$. Then the components of the vector $\vec{\Pi}^*$ of the complete field are found to equal

$$\Pi_x^* = \Phi^0(\xi, \eta) + \Phi_1(\xi, \eta) = \Phi(\xi, \eta),$$
$$\Pi_y^* = 0, \qquad (9.3.5)$$
$$\Pi_z^* = \Pi_z^{*1} = \Psi(\xi, \eta)\cos\varphi.$$

Since the Cartesian rectangular components Π_x^*, Π_z^* of the vector Π^* satisfy the homogeneous Helmholtz equation, then after conversion in it to spheroidal coordinates it will be satisfied also by the function $\Phi(\xi, \eta)$ (with the azimuth number $m = 0$), $\Psi(\xi, \eta)$(with the azimuth number $m = 1$). From (9.3.4) we find that in the coordinates of a prolate spheroid,

$$E_\eta = \frac{ik_0\mu}{h_\xi h_\varphi}\left[\frac{\partial}{\partial\varphi}(h_\xi \Pi_\xi^*) - \frac{\partial}{\partial\xi}(h_\varphi \Pi_\varphi^*)\right],$$
$$E_\varphi = \frac{ik_0\mu}{h_\xi h_\eta}\left[\frac{\partial}{\partial\xi}(h_\eta \Pi_\eta^*) - \frac{\partial}{\partial\eta}(h_\xi \Pi_\xi^*)\right], \qquad (9.3.6)$$

where h_ξ, h_η, h_ϕ are the Lame metric coefficients (1.5.41). Hence it follows that on the surface of the spheroid (when $\xi = \xi^0$) the following relationship must be satisfied for any η and ϕ, /468

$$\frac{\partial}{\partial\varphi}(h_\xi \Pi_\xi^*) = \frac{\partial}{\partial\xi}(h_\varphi \Pi_\varphi^*),$$
$$\frac{\partial}{\partial\xi}(h_\eta \Pi_\eta^*) = \frac{\partial}{\partial\eta}(h_\xi \Pi_\xi^*). \qquad (9.3.7)$$

On the basis of (1.5.43)

$$\Pi_\xi^* = \xi\left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2}\cos\varphi\Phi + \eta\left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2}\cos\varphi\Psi,$$
$$\Pi_\eta^* = -\eta\left(\frac{\xi^2-1}{\xi^2-\eta^2}\right)^{1/2}\cos\varphi\Phi + \xi\left(\frac{1-\eta^2}{\xi^2-\eta^2}\right)^{1/2}\cos\varphi\Psi, \qquad (9.3.8)$$
$$\Pi_\varphi^* = -\sin\varphi\Phi.$$

Therefore after substituting (9.3.8) into (9.3.7) as the boundary conditions for the potential functions Φ, Ψ we obtain the equations (when $\xi = \xi^0$),

$$\frac{\eta}{\sqrt{1-\eta^2}}\Psi = \sqrt{\xi^2-1}\,\frac{\partial\Phi}{\partial\xi},$$
$$\eta\frac{\partial\Psi}{\partial\eta} - \xi\frac{\partial\Psi}{\partial\xi} = -\xi\left(\frac{1-\eta^2}{\xi^2-1}\right)^{1/2}\times \tag{9.3.9}$$
$$\times\frac{\partial\Phi}{\partial\eta} - \eta\left(\frac{\xi^2-1}{1-\eta^2}\right)^{1/2}\frac{\partial\Phi}{\partial\xi}.$$

Thus, in mathematical formulation the problem for the magnetic horizontal dipole is reduced to seeking the Hertz potentials $\Phi(\xi,\eta)$, $\Psi(\xi,\eta)$, which satisfy the Helmholtz equation with the azimuth number $m = 0$ and $m = 1$, respectively, the boundary condition (9.3.9), and the condition of radiation at infinity.

We shall seek the function Φ_1, Ψ in the form of a series,

$$\Phi_1(\xi,\eta) = 2ikm\sum_{n=0}^{\infty}\alpha_n R_{0n}^{(3)}(c,\xi)S_{0n}(c,\eta), \tag{9.3.10}$$

$$\Psi(\xi,\eta) = 2ikm\sum_{n=1}^{\infty}\beta_n R_{1n}^{(3)}(c,\xi)S_{1n}(c,\eta) \tag{9.3.11}$$

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for the spheroidal wave functions with the azimuth numbers $m = 0$ and $m = 1$, respectively. On the basis of (5.4.1) the potential of the primary field Φ^0 may be represented by the series

$$\Phi^0(\xi,\eta) = 2ikm\sum_{n=0}^{\infty}\frac{S_{0n}(c,\eta_0)}{N_{0n}(c)}\times$$
$$\times R_{0n}^{(3)}(c,\xi_0)R_{0n}^{(1)}(c,\xi)S_{0n}(c,\eta). \tag{9.3.12}$$

Although in the boundary conditions of (9.3.9) the functions Φ and Ψ are not separated, these conditions may be used for finding the coefficients α_n, β_n. If in (9.3.9) we substitute the expansions (9.3.10) - (9.3.12) and then multiply the obtained equations by $S_{0m}(c,\eta)$ (where m is any integral non-negative number), we find after integration over η from -1 to +1 that (terms with β_0 should be omitted everywhere),

$$\sum_{n=0}^{\infty}(\alpha_n C_{mn} + \beta_n D_{mn}) = \sum_{n=0}^{\infty}B_{mn},$$
$$\sum_{n=0}^{\infty}(\alpha_n V_{mn} + \beta_n W_{mn}) = \sum_{n=0}^{\infty}U_{mn}, \tag{9.3.13}$$

where

$$
\begin{aligned}
B_{mn} &= -A_n(\xi^{02}-1)^{1/2} R_{0n}^{(1)'}(c,\ \xi^0) I_1^{mn},\\
C_{mn} &= (\xi^{02}-1)^{1/2} R_{0n}^{(3)'}(c,\ \xi^0) I_1^{mn},\\
D_{mn} &= -R_{1n}^{(3)}(c,\ \xi^0) I_2^{mn},\\
U_{mn} &= -A_n\{(\xi^{02}-1) R_{0n}^{(1)'}(c,\ \xi^0) I_3^{mn} + \xi^0 R_{0n}^{(1)}(c,\ \xi^0) I_4^{mn}\},\\
V_{mn} &= (\xi^{02}-1) R_{0n}^{(3)'}(c,\ \xi^0) I_3^{mn} + \xi^0 R_{0n}^{(3)}(c,\ \xi^0) I_4^{mn},\\
W_{mn} &= -\xi^0(\xi^{02}-1)^{1/2} R_{1n}^{(3)'}(c,\ \xi^0) I_5^{mn} +\\
&\quad + (\xi^{02}-1)^{1/2} R_{1n}^{(3)}(c,\ \xi^0) I_6^{mn},\\
I_1^{mn} &= \int_{-1}^{1} S_{0n}(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta = \delta_{nm} N_{0n}(c)
\end{aligned}
\qquad (9.3.14)
$$

(δ_{nm} is the Kronecker symbol),

$$I_2^{mn} = \int_{-1}^{1} \frac{\eta}{\sqrt{1-\eta^2}} S_{1n}(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta,$$

$$I_3^{mn} = \int_{-1}^{1} \eta\, S_{0n}(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta,$$

$$I_4^{mn} = \int_{-1}^{1} (1-\eta^2) S_{0n}'(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta,$$

$$I_5^{mn} = \int_{-1}^{1} \sqrt{1-\eta^2}\, S_{1n}(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta, \qquad /470$$

$$I_6^{mn} = \int_{-1}^{1} \eta \sqrt{1-\eta^2}\, S_{1n}'(c,\ \eta) S_{0m}(c,\ \eta)\, d\eta.$$

The integrals I_1^{mn} - I_6^{mn} and the coefficients C_{mn}, D_{mn}, V_{mn}, W_{mn} of the system in (9.3.13) of an infinite number of linear equations (relative to the unknowns α_n, β_n) agree in form with those given in [165]. The system in (9.3.13) itself differs from the infinite system of linear equations obtained in [165] only in the expressions for A_n, i.e., the expansion coefficient of the potential of the primary field in the series of spheroidal wave functions. Here

$$A_n = \frac{S_{0n}(c,\ 1)}{N_{0n}(c)} R_{0n}^{(3)}(c,\ \xi_0). \qquad (9.3.15)$$

hen the dipole is moved along the axis Oz to infinity ($\xi_0 \to \infty$)

$$A_n = (-i)^{n+1} \frac{S_{0n}(c, 1)}{c\,\xi_0 N_{0n}(c)} e^{ic\xi_0}, \tag{9.3.16}$$

nd in this case system (9.3.13) does not differ essentially from ystem (71) cited in [165].

System (9.3.13) is a system of $2m$ equations with $2n$ unknowns m, n = 0, 1, ...). When the region of variation n, m is bounded y a certain number N, we obtain from it a finite system of linear quations for the approximate values of α_n, β_n, which approximate he precise values of α_n, β_n. The degree of accuracy of the ap- roximation, will obviously increase with increase in the number N in [166] in computing the scattering cross section of a prolate pheroid, the author confined himself to values of n, m = 0, 1, 2,).

The first of the equations in (9.3.13) may be written in the orm,

$$\alpha_m + \frac{1}{c_{mm}} \sum_{n=1}^{\infty} D_{mn} \beta_n = \frac{B_{mm}}{C_{mm}}. \tag{9.3.17}$$

'herefore, after omitting from the second equation in (9.3.13) the nknown α_n, using (9.3.17) we find a system of equations containing nly the unknowns β_n. /471

The formal solution to the problem is finished by finding α_n nd β_n.

The components of the electromagnetic field are found from 9.3.4) and (9.3.8) and have the form

$$\begin{aligned}
E_\eta &= \frac{ik_0 \mu \sin\varphi}{f\sqrt{\xi^2-\eta^2}} \left\{ \sqrt{\xi^2-1}\, \frac{\partial \Phi}{\partial \xi} - \frac{\eta}{\sqrt{1-\eta^2}} \Psi \right\}, \\
E_\varphi &= -\frac{ik_0 \mu \cos\varphi}{f(\xi^2-\eta^2)} \left\{ \eta(\xi^2-1) \frac{\partial \Phi}{\partial \xi} + \right. \\
&\left. + \xi(1-\eta^2) \frac{\partial \Phi}{\partial \eta} - \sqrt{(1-\eta^2)(\xi^2-1)} \left[\xi \frac{\partial \Psi}{\partial \xi} - \eta \frac{\partial \Psi}{\partial \eta} \right] \right\}, \\
H_\eta &= \frac{\cos\varphi}{f} \sqrt{\frac{1-\eta^2}{\xi^2-\eta^2}} \times \\
&\times \left\{ \frac{\partial}{\partial \eta} \left[\frac{\sqrt{(1-\eta^2)(\xi^2-1)}}{\xi^2-\eta^2} \left(\xi \frac{\partial \Phi}{\partial \xi} - \eta \frac{\partial \Phi}{\partial \eta} \right) + \right. \right.
\end{aligned} \tag{9.3.18}$$

$$+\frac{1}{\xi^2-\eta^2}\left(\eta(\xi^2-1)\frac{\partial\Psi}{\partial\xi}+\xi(1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right)\Bigg]+$$
$$+k^2\left(\xi\Psi-\frac{\eta\sqrt{\xi^2-1}}{\sqrt{1-\eta^2}}\Phi\right)\Bigg\},$$
$$H_\varphi=-\frac{\sin\varphi}{f^2(\xi^2-\eta^2)}\left\{\xi\frac{\partial\Phi}{\partial\xi}-\eta\frac{\partial\Phi}{\partial\eta}+\frac{1}{\sqrt{(1-\eta^2)(\xi^2-1)}}\times\right.$$
$$\left.\times\left(\eta(\xi^2-1)\frac{\partial\Psi}{\partial\xi}+\xi(1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right)+k^2\Phi\right\}. \qquad (9.3.19)$$

In the vicinity of the wave band in spherical coordinates r, θ, ϕ, $\left(f\xi\approx r,\ \eta\approx\cos\theta,\ \frac{\partial}{\partial\xi}\approx f\frac{\partial}{\partial r},\ \frac{\partial}{\partial\eta}\approx-\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\right)$ we have

$$E_\varphi=ik_0\mu\cos\varphi\left[\sin\theta\frac{\partial\Psi}{\partial r}-\cos\theta\frac{\partial\Phi}{\partial r}\right], \qquad (9.3.20)$$

Furthermore, $$H_\varphi=-k^2\sin\varphi\Phi.$$

$$E_\varphi=-\sqrt{\frac{\mu}{\varepsilon}}H_\eta=-k^2\sqrt{\frac{\mu}{\varepsilon}}m\frac{e^{ikR}}{R}\times$$
$$\times\cos\theta\cos\varphi\,W_1(\theta), \qquad (9.3.21)$$
$$E_\eta=\sqrt{\frac{\mu}{\varepsilon}}H_\varphi=-k^2\sqrt{\frac{\mu}{\varepsilon}}m\frac{e^{ikR}}{R}\sin\varphi\,W_2(\theta),$$

where the function $W_1(\theta)$, $W_2(\theta)$ are the damping coefficients, which take into account the presence in the dipole field of a spheroid. They are equal to /47

$$W_1(\theta)=1+2e^{ika\cos\theta}\left\{\sum_{n=0}^{\infty}(-i)^n\alpha_n S_{0n}(c,\cos\theta)-\right.$$
$$\left.-\operatorname{tg}\theta\sum_{n=1}^{\infty}(-i)^n\beta_n S_{1n}(c,\cos\theta)\right\}, \qquad (9.3.22)$$
$$W_2(\theta)=1+2e^{ika\cos\theta}\sum_{n=0}^{\infty}(-i)^n\alpha_n S_{0n}(c,\cos\theta)$$

($R=r=a\cos\theta$, where a is the distance from the center of the spheroid to the dipole). The density distribution of the surface currents induced on the prolate ideally conducting spheroid by the field of the horizontal magnetic dipole located at the point $z=a$ on the Oz axis, may be found from the formulas

$$j_\eta = -\frac{c_0}{4\pi} H_\varphi, \quad j_\varphi = \frac{c_0}{4\pi} H_\eta, \qquad (9.3.23)$$

$$j_\xi = 0,$$

where H_ϕ and H_η are determined by the formulas in (9.3.19) with the value of $\xi = \xi 0$.

If the dipole approaches the surface of the spheroid, causing ξ_0 to tend to ξ^0, then at the limit we find from formulas (9.3.10) - (9.3.22) the solution to the problem on the field of the emitting unit slot, located at the pole $\eta_0 = 1$, $\xi = \xi^0$ of the prolate ideally conducting spheroid $\xi = \xi^0$.

If the horizontal dipole is moved from the spheroid along the Oz axis without changing its orientation toward the positive values of z, then when $\xi_0 \to \infty$, $\xi_0 \to \infty$, from (9.3.10) - (9.3.22) we find the solution to the problem on the diffraction of a plane electromagnetic wave,

$$E_y^0 = \sqrt{\frac{\mu}{\varepsilon}} H_x^0 = k^2 \sqrt{\frac{\mu}{\varepsilon}} \Pi_x^{*0} = Ee^{-ikz} \qquad (9.3.24)$$

with the amplitude,

$$E = k^2 \sqrt{\frac{\mu}{\varepsilon}} m \frac{e^{ikz_1}}{z_1}, \qquad z_1 = f\xi_0, \qquad (9.3.25)$$

propagating in the direction of the negative values of z along the Oz axis on the ideally conducting prolate spheroid $\xi = \xi^0$. In this case in (9.3.14) as A_η it follows to take expression (9.3.16). The electric vector of the wave (9.3.24) is polarized in the direction of positive values of the Oz axis. Therefore in computing the radar cross section for a prolate spheroid we find

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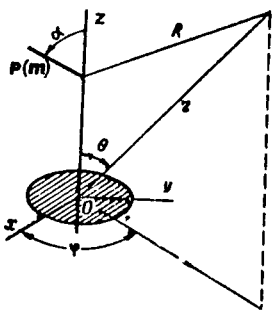

Fig. 57. Excitation of a Circular Disk by a Dipole.

$$\sigma_B = \frac{4\pi}{k^2} |W(0)|^2 \qquad (9.3.26)$$

$\left(\phi = \frac{\pi}{2}, \ \theta = 0\right)$, where

$$W(0) = 2kz_1 \sum_{n=0}^{\infty} (-i)^n a_{n1} S_{0n}(c, 1).$$

With degeneration of the prolate spheroid coordinates ξ, η, ϕ into spherical coordinates r, θ, ϕ, when $f \to 0$, $\xi \to \infty$ such that $f\xi \to r$, $\eta \to \cos\theta$, at the limit from (9.3.10) - (9.3.22) we find the

solution to the problem of the field of a horizontal magnetic dipole, located over an ideally conducting sphere with radius $r = r_0$ and a distance a away from the center of the sphere.

In formulas (9.3.10) - (9.3.22) if we make the substitution of coordinates of the prolate spheroid by the coordinates of an oblate spheroid according to formulas (9.2.16), as a result we then find the solution to the problem on the diffraction of the field of a horizontal magnetic dipole on an ideally conducting oblate spheroid $\xi = \xi^0$. Since the degenerate case of an oblate spheroid is that of an infinitely thin circular disk, then at the limit when $\xi^0 \to 0$, from (9.3.10) - (9.3.22) after conversion in these formulas to the coordinates of an oblate spheroid we find the solution to the problem of diffraction of an electromagnetic field on a circular ideally conducting disk of radius f (Fig. 57).

In the coordinates of an oblate spheroid, /47

$$\begin{aligned}
\Phi_1(\xi, \eta) &= 2ikm \sum_{n=0}^{\infty} \alpha_n(\xi_0) R_{0n}^{(3)}(-ic, i\xi) S_{0n}(-ic, \eta), \\
\Psi(\xi, \eta) &= 2ikm \sum_{n=1}^{\infty} \beta_n(\xi_0) R_{1n}^{(3)}(-ic, i\xi) S_{1n}(-ic, \eta), \\
\Phi_0(\xi, \eta) &= 2ikm \sum_{n=0}^{\infty} \frac{S_{0n}(-ic, 1)}{N_{0n}(-ic)} \times \\
&\times R_{0n}^{(3)}(-ic, i\xi_0) R_{0n}^{(1)}(-ic, i\xi) S_{0n}(-ic, \eta)
\end{aligned} \qquad (9.3.27)$$

(for convenience, in the future the coefficients of the series α_n and β_n, which depend on ξ_0, are denoted as the functions of the parameter ξ_0).

Let us look in more detail at the case $\xi^0 = 0$. When $\xi^0 = 0$ the boundary conditions (9.3.9) are written in the form,

$$\left.\begin{aligned}
\frac{\eta}{\sqrt{1-\eta^2}} \Psi &= \frac{\partial \Phi}{\partial \xi}, \\
\frac{\partial \Psi}{\partial \eta} &= -\frac{1}{\sqrt{1-\eta^2}} \cdot \frac{\partial \Phi}{\partial \xi},
\end{aligned}\right\} \xi^0 = 0, \quad -1 \leqslant \eta \leqslant 1, \qquad (9.3.28)$$

whence after dropping the derivative $\frac{\partial \Phi}{\partial \xi}$ from the second equation we find

$$\frac{\partial \Psi}{\partial \eta} = -\frac{\eta}{1-\eta^2} \Psi, \quad \xi^0 = 0.$$

f we integrate this equation over η, we find that

$$\Psi = (1-\eta^2)^{1/2} C, \tag{9.3.29}$$

here C is an arbitrary integration constant. On the basis of 9.3.29) from the first equation in (9.3.28) for the function Φ we ind

$$\frac{\partial \Phi}{\partial \xi} = \eta C. \tag{9.3.30}$$

Thus, in the coordinates of an oblate spheroid when $\xi^0 = 0$ the oundary conditions for the potential functions Φ and Ψ are separated and assume the form,

$$\left.\begin{aligned} \frac{\partial \Phi}{\partial \xi} &= \eta C, \\ \Psi &= (1-\eta^2)^{1/2} C, \end{aligned}\right\} \xi^0 = 0. \tag{9.3.31}$$

The integration constant C is determined from the supplementary conditions, as which, when $\xi_0 \neq 0$, we can take the Meixner conditions at the edge of the disk ($\xi = 0$, $\eta = 0$). For this it is sufficient to require that the radial component of the density vector of the total surface current, induced on the disk, vanish at the edge of the disk. In mathematical formulation the Meixner conditions may be written in the form, /475

$$j_\rho = \frac{c_0}{4\pi}\left\{H_\varphi\Big|_{\substack{\xi=0\\ \eta=-0}} - H_\varphi\Big|_{\substack{\xi=0\\ \eta=+0}}\right\} = 0. \tag{9.3.32}$$

Conditions (9.3.31), (9.3.32) make it possible to find the precise expressions for the coefficients $\alpha_n(\xi_0)$, $\beta_n(\xi_0)$ which have a closed form. Thus, after substitution of (9.3.27) into (9.3.31) using the properties of orthogonality of the angular spheroidal functions we find

$$\begin{aligned} \alpha_n(\xi_0) &= -\frac{S_{0n}(-ic,\ 1)\, R_{0n}^{(1)'}(-ic,\ i0)}{N_{0n}(-ic)\, R_{0n}^{(3)'}(-ic,\ i0)} R_{0n}^{(3)}(-ic,\ i\xi_0) + \\ &\quad + \frac{d_1^{0n}(-ic)\, C}{3ikmN_{0n}(-ic)\, R_{0n}^{(3)'}(-ic,\ i0)}, \\ \beta_n(\xi_0) &= \frac{2d_0^{1n}(-ic)\, C}{3ikmN_{1n}(-ic)\, R_{1n}^{(3)}(-ic,\ i0)}. \end{aligned} \tag{9.3.33}$$

From the properties of the functions S_{mn}, R_{mn} and their derivatives, and also from the properties of the coefficients d_r^{mn} it follows that

$$\alpha_n = 0, \ \beta_n = 0 \text{ when } n = 0, 2, \ldots \tag{9.3.34}$$

Consequently, in (9.3.27) the series for the functions Φ_1 and Ψ are summed only over the odd numbers of n.

After repeating the arguments similar to those given in [50] (pp. 157-158), from (9.3.32) we obtain the relationship,

$$\begin{aligned} &\sum_{n=1,\,3\ldots} \alpha_n(\xi_0) R_{0n}^{(3)}(-ic,\ i0) S'_{0n}(-ic,\ 0) = \\ &= \sum_{n=1,\,3\ldots} \beta_n(\xi_0) R_{1n}^{(3)'}(-ic,\ i0) S_{1n}(-ic,\ 0), \end{aligned} \tag{9.3.35}$$

from which the constant C is also determined when $\xi_0 \neq 0$. After substituting expression (9.3.33) into (9.3.35), we find that

$$C = \frac{3km}{c} C_1(\xi_0), \tag{9.3.36}$$

where

$$C_1(\xi_0) = S_1(\xi_0)/[S_2 + S_3],$$

and

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$$\begin{aligned} S_1(\xi_0) &= \sum_{n=1,\,3,\ldots} \frac{S_{0n}(-ic,\ 1) R_{0n}^{(3)}(-ic,\ i\xi_0)}{N_{0n}(-ic) R_{0n}^{(3)'}(-ic,\ i0)} \dot{S}'_{0n}(-ic,\ 0), \\ S_2 &= \sum_{n=1,\,3,\ldots} \frac{d_1^{0n}(-ic) R_{0n}^{(3)}(-ic,\ i0),}{N_{0n}(-ic) R_{0n}^{(3)'}(-ic,\ i0)} S'_{0n}(-ic,\ 0), \\ S_3 &= -2 \sum_{n=1,\,3,\ldots} \frac{d_0^{1n}(-ic) R_{1n}^{(3)'}(-ic,\ i0)}{N_{1n}(-ic) R_{1n}^{(3)}(-ic,\ i0)} S_{1n}(-ic,\ 0). \end{aligned} \tag{9.3.37}$$

In the same manner the problem is solved on the field of the horizontal magnetic dipole located at the point $\xi = \xi_0$ of the Oz axis over the disk ($\xi_0 \neq 0$). In the wave band in spherical coordinates r, θ, ϕ, we have approximately

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\eta =$$

$$= -k^2 m \sqrt{\frac{\mu}{\varepsilon}} \frac{e^{ikR}}{R} \cos\theta \cos\varphi W_1(\theta),$$

$$E_\eta = \sqrt{\frac{\mu}{\varepsilon}} H_\varphi = \qquad (9.3.38)$$

$$= -k^2 m \sqrt{\frac{\mu}{\varepsilon}} \frac{e^{ikR}}{R} \sin\varphi W_2(\theta),$$

here $R = r - a\cos\theta$ (a is the distance from the center of the disk o the dipole),

$$W_1(\theta) = 1 - 2e^{ika\cos\theta}\left\{\sum_{n=1,3,\ldots} a_n(\xi_0) S_{0n}(-ic, \cos\theta) - \right.$$
$$- \frac{iC_1(\xi_0)}{c} \sum_{n=1,3,\ldots} b_n S_{0n}(-ic, \cos\theta) -$$
$$\left. - \frac{iC_1(\xi_0)\operatorname{tg}\theta}{c} \sum_{n=1,3,\ldots} c_n S_{1n}(-ic, \cos\theta)\right\},$$
$$W_2(\theta) = 1 + 2e^{ika\cos\theta}\left\{- \sum_{n=1,3,\ldots} a_n(\xi_0) S_{0n}(-ic, \cos\theta) + \right. \qquad (9.3.39)$$
$$\left. + \frac{iC_1(\xi_0)}{c} \sum_{n=1,3,\ldots} b_n S_{0n}(-ic, \cos\theta)\right\}.$$

ere

$$a_n(\xi_0) = -i^n \frac{S_{0n}(-ic, 1)}{N_{0n}(-ic)} \cdot \frac{R_{0n}^{(1)'}(-ic, i0) R_{0n}^{(3)}(-ic, i\xi_0)}{R_{0n}^{(3)'}(-ic, i0)},$$

$$b_n = i^n \frac{d_1^{0n}(-ic)}{N_{0n}(-ic) R_{0n}^{(3)'}(-ic, i0)}, \qquad (9.3.40)$$

$$c_n = -i^n \frac{2d_0^{1n}(-ic)}{N_{1n}(-ic) R_{1n}^{(3)'}(-ic, i0)},$$

and $C_1(\xi_0)$ is found from (9.3.36). /477

The density distribution of surface currents induced on the disk may be found from formulas (9.3.23) in which, when $\xi^0 = 0$,

$$H_\eta = \frac{\cos\varphi}{\eta f}\left\{\frac{1}{\eta^2} \cdot \frac{\partial\Phi}{\partial\eta} - \frac{1-\eta^2}{\eta} \cdot \frac{\partial^2\Phi}{\partial\eta^2} - \right.$$
$$- \frac{\sqrt{1-\eta^2}}{\eta^2} \cdot \frac{\partial\Psi}{\partial\xi} + \frac{\sqrt{1-\eta^2}}{\eta} \times$$
$$\left. \times \frac{\partial^2\Psi}{\partial\xi\,\partial\eta} - k^2\eta\Phi\right\}\Big|_{\xi=\xi^0=0}, \qquad (9.3.41)$$

$$H_\varphi = -\sin\varphi\left\{\frac{-1}{f^2\eta}\cdot\frac{\partial\Phi}{\partial\eta}+\right.$$
$$\left.+\frac{1}{f^2\eta\sqrt{1-\eta^2}}\cdot\frac{\partial\Psi}{\partial\xi}+k^2\Phi\right\}\Big|_{\xi=\xi'=0} \qquad (9.3.42)$$

When the dipole is moved from the surface of the disk along the axis Oz in its positive direction ($\xi_0 \to \infty$, $c\xi_0 \to \infty$) from (9.3.27) - (9.3.42) we find the solution to the problem of diffraction of a plane electromagnetic wave (9.3.24), (9.3.25), propagating in the direction of the Oz axis, on a infinitely thin ideally conducting circular disk of radius $\rho = f$. For such a case, in the vicinity of the wave band, the components of the secondary field scattered on the disk will be determined by the expressions,

$$E^1_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}H^1_\eta = E\frac{e^{ikr}}{r}\cos\theta\cos\varphi\, W_1(\theta),$$
$$E^1_\eta = \sqrt{\frac{\mu}{\varepsilon}}H^1_\varphi = E\frac{e^{ikr}}{r}\sin\varphi\, W_2(\theta), \qquad (9.3.43)$$

in which

$$W_1(\theta) = -\frac{2}{k}\left\{\sum_{n=1,3,\ldots}\bar{a}_n S_{0n}(-ic,\cos\theta)+\right.$$
$$+\frac{C_1}{c}\sum_{n=1,3,\ldots}b_n S_{0n}(-ic,\cos\theta)+$$
$$\left.+\frac{C_1\operatorname{tg}\theta}{c}\sum_{n=1,3,\ldots}c_n S_{1n}(-ic,\cos\theta)\right\}, \qquad (9.3.44)$$
$$W_2(\theta) = \frac{2}{k}\left\{\sum_{n=1,3,\ldots}\bar{a}_n S_{0n}(-ic,\cos\theta)+\right.$$
$$\left.+\frac{C_1}{c}\sum_{n=1,3,\ldots}b_n S_{0n}(-ic,\cos\theta)\right\},$$

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where

$$\bar{a}_n = i\,\frac{S_{0n}(-ic,\,1)\,R^{(1)\prime}_{0n}(-ic,\,i0)}{N_{0n}(-ic)\,R^{(3)\prime}_{0n}(-ic,\,i0)}, \qquad (9.3.45)$$

and b_n and c_n are the same as in (9.3.40). Here $C_1 = C_1(\infty)$ and is found from (9.3.36), (9.3.37) in which

$$S_1 = \sum_{n=1,3,\ldots} i^n\,\frac{S_{0n}(-ic,\,1)\,S'_{0n}(-ic,\,0)}{N_{0n}(-ic)\,R^{(3)\prime}_{0n}(-ic,\,i0)}.$$

By virtue of the character of polarization of the vector of the electric intensity given above for the plane wave the radar cross section for an infinitely thin ideally conducting circular disk of radius f computed from formula (1.4.6) will be equal to ($\theta = 0$, $\phi = \frac{\pi}{2}$)

$$\sigma_B = \frac{4\pi}{k^2} |W(0)|^2, \tag{9.3.46}$$

where

$$W(0) = 2 \sum_{n=1,\ 3,\ldots} \left\{ \frac{C_1}{c} b_n + \bar{a}_n \right\} S_{0n}(-ic,\ 1).$$

The two-position scattering cross section for the disk may be computed from the formula,

$$\sigma = \frac{4\pi}{k^2} \{\cos^2\theta \cos^2\varphi |\widetilde{W}_1(\theta)|^2 + \sin^2\varphi |\widetilde{W}_2(\theta)|^2\}, \tag{9.3.47}$$

where θ and ϕ are taken in the direction of the source of the stray field $\tilde{W}_{1,2} = kW_{1,2}$.

In the plane $\theta = \frac{\pi}{2}$ we will have $\sigma = 0$ since in the wave band of the plane $\theta = \frac{\pi}{2}$ in which the disk is located the secondary field is equal to zero (in our approximation).

The solution to the problem of diffraction of a plane electromagnetic wave obtained before may be used for determining the constant C in the case when the horizontal dipole is located on the surface of the disk in its center ($\xi_0 = \xi^0 = 0$, $\eta_0 = 1$). /479

Let us set $\xi_0 = \xi^0 = 0$ into (9.3.27) - (9.3.31), (9.3.33) and (9.3.38) - (9.3.42). For such a case in (9.3.27)

$$\alpha_n(0) = \frac{i[S_{0n}(-ic,\ 1) - d_1^{0n}(-ic) C_1(0)]}{cN_{0n}(-ic) R_{0n}^{(3)'}(-ic,\ i0)},$$
$$n = 1,\ 3,\ \ldots$$
$$\beta_n(0) = \frac{-2id_0^{1n}(-ic) C_1(0)}{cN_{1n}(-ic) R_{1n}^{(3)}(-ic,\ i0)}, \tag{9.3.48}$$

However the integration constant $C_1(0)$ which appears in (9.3.48), can not be determined from (9.3.35), since substitution of (9.3.48) into (9.3.35) will cause the series $S_1(\xi_0)$ to be divergent when

$\xi_0 = 0$. To seek the constant C in this case it is feasible to use the reciprocity theorem (1.3.14) for the magnetic dipoles. Let us assume that, at points O_1 with the coordinates $\xi = 0$, $\eta = 1$ and at O_2 with the coordinates $\xi = +\infty$, $\eta = 1$, the magnetic dipoles are found with equal moments $m_1 = m_2 = m$, directed toward the positive values of x parallel to the Ox axis (Fig. 58). Then from theorem (1.3.14) it follows that

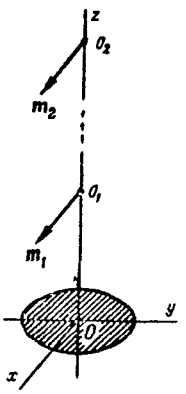

Fig. 58. Reciprocity Theorem in the Problem of Diffraction of a Dipole Field on a Disk.

$$H^1_{2,\varphi}(O_1)\big|_{\theta=0,\ \varphi=\frac{\pi}{2},\ \xi=0} = H^{(1)}_{1,\varphi}(O_2)\big|_{\theta=0,\ \varphi=\frac{\pi}{2},\ \xi=+\infty}, \qquad (9.3.49)$$

where H^1_1, ϕ is the secondary magnetic field of the dipole, located in the center of the disk, in the wave band in the direction of the Oz axis, but H^1_2, ϕ is the secondary field of the plane wave approaching the center of the disk (at the point $\theta = 0$, $\xi = 0$) from infinity in the direction of negative values of z. /480

The component H^1_ϕ of the secondary dipole field located in the center of the disk and oriented along the Ox axis, in the wave band in spherical coordinates is determined by the formula,

$$H^1_\varphi = -k^2 m \frac{e^{ikr}}{r} W(\theta)\sin\varphi, \qquad (9.3.50)$$

where when $\theta = 0$,

$$W(0) = -2\sum_{n=1,3,\ldots} i^n \alpha_n(0) S_{0n}(-ic,\ 1).$$

The component H^1_ϕ of the secondary field of the plane wave at any point is given by the expression,

$$H^1_\varphi = -\sin\varphi\left\{\frac{1}{f^2(\xi^2+\eta^2)}\left[\xi\frac{\partial\Phi^1}{\partial\xi} - \right.\right.$$
$$\left.-\eta\frac{\partial\Phi^1}{\partial\eta}\right] + \frac{1}{f^2(\xi^2+\eta^2)\sqrt{(1-\eta^2)(\xi^2+1)}}\times$$
$$\left.\times\left[\eta(1+\xi^2)\frac{\partial\Psi}{\partial\xi} + \xi(1-\eta^2)\frac{\partial\Psi}{\partial\eta}\right] + k^2\Phi^1\right\}.$$

Therefore in the center of the disk with values of $\eta = 1$, $\phi = \frac{\pi}{2}$, $\xi = \xi^0 = 0$, we will have

$$H_\varphi^1 = -2i\,\frac{E\sqrt{\frac{\varepsilon}{\mu}}}{c}\,D\sin\varphi, \tag{9.3.51}$$

where

$$D = \frac{1}{c}\left\{\sum_{n=1,3,\ldots} i^n\left[\bar{a}_n + \frac{C_1}{c}\,b_n\right]R_{0n}^{(3)}(-ic,\ i0)\,S_{0n}'(-ic,\ 1) + \right.$$
$$+ \frac{C_1}{c}\sum_{n=1,3,\ldots} i^n c_n R_{1n}^{(3)'}(-ic,\ i0)\,\sigma_{1n}(-ic) - \tag{9.3.52}$$
$$\left. - c^2\sum_{n=1,3,\ldots} i^n\left[\bar{a}_n + \frac{C_1}{c}\,b_n\right]R_{0n}^{(3)}(-ic,\ i0)\,S_{0n}(-ic,\ 1)\right\}.$$

Here $\bar{a}_n$, b_n, c_n are determined by the formulas in (9.3.40), (9.3.45), $C_1 = C_1(\infty)$, and /481

$$\sigma_{1n}(-ic) = \lim_{\eta\to 1}\frac{S_{1n}(-ic,\ \eta)}{\sqrt{1-\eta^2}} =$$
$$= -\lim_{\eta\to 1}(1-\eta^2)^{1/2}\frac{d}{d\eta}S_{1n}(-ic,\ \eta) = \tag{9.3.53}$$
$$= \frac{1}{2}\sum_{r=0,\ 1}^{\infty}(r+1)(r+2)\,d_r^{1n}(-ic).$$

If we write the function $W(0)$ from (9.3.50) in the form,

$$W(0) = -\frac{2i}{c}(A - C_1(0)\,B), \tag{9.3.54}$$

where

$$A = \sum_{n=1,3,\ldots} i^n\,\frac{S_{0n}^2(-ic,\ 1)}{N_{0n}(-ic)\,R_{0n}^{(3)'}(-ic,\ i0)},$$
$$B = \sum_{n=1,3,\ldots} i^n\,\frac{d_1^{0n}(-ic)\,S_{0n}(-ic,1)}{N_{0n}(-ic)\,R_{0n}^{(3)'}(-ic,\ i0)}, \tag{9.3.55}$$

then by substitution of (9.3.50) and (9.3.51) into (9.3.49) we find for the computation of $C_1(0)$ the expression

$$C_1(0) = (D + A)/B. \tag{9.3.56}$$

In the same way the coefficients (9.3.48) of series (9.3.27) are found; they represent, with values of $\xi^0 = \xi_0 = 0$, the diffraction

field of the magnetic dipole lying on the disk at its center and directed along the Ox axis. In the electrodynamic sense this case, as noted above, corresponds to the problem on the field of emission of a self-contained unit slot, located on the surface of an ideally conducting circular disk of radius f at its center. The complex characteristics of the emission from such a slot, i.e., the functions $W_1(\theta)$, $W_2(\theta)$, may be computed from formulas (9.3.38) - (9.3.40), if we set there $\xi_0 = 0$ and take $C_1(0)$ from (9.3.56). Furthermore in the formulas in (9.3.39) it follows to set $a = 0$.

We can familiarize ourselves with certain results of the numerical computation of the complex characteristics of emission from a self-contained slot on the disk, for example, in [50].

§4. Oblique Incidence of a Plane Wave on a Circular Disk

As was just shown, the solution to the problem of scattering of a plane electromagnetic wave on an infinitely thin ideally conducting disk with normal incidence on the disk (the wave is propagated along the axis of the disk) may be obtained as part of the solution to the problem on the diffraction of a field of the horizontal magnetic dipole on such a disk. In the coordinates of an oblate spheroid ξ, η, ϕ with focal distance $f = a$ (a is the radius of the disk) we can also solve the problem of scattering of a plane electromagnetic wave with oblique incidence on the disk by the method of separation of variables [48, 167]. /482

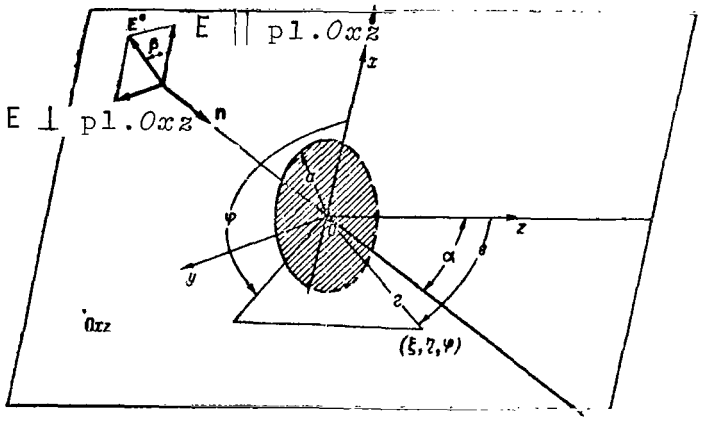

Fig. 59. Diffraction of a Plane Wave With Oblique Incidence on the Disk.

Let the incident wave

$$\mathbf{E}^0 = \mathbf{E}' e^{ik(\mathbf{nr})}, \quad k = \frac{\omega}{c_0}\sqrt{\varepsilon\mu}, \tag{9.4.1}$$

with the amplitude $E = |\mathbf{E}'|$, E = const be propagated in the direction of the unit vector $\mathbf{n}$ which forms the angle α with the positive direction of the Oz axis (the Oz axis is the axis of the disk whose

center coincides with the origin (Fig. 59). Furthermore, let the coordinate plane Oxz be parallel to the vector $\mathbf{n}$ and the wave (9.4.1) be polarized such that the vector $\mathbf{E}^0$:

(1) is parallel to the plane Oxz;

(2) is perpendicular to the plane Oxz (it is obvious that the solution to the problem with an arbitrary orientation of the vector $\mathbf{E}^0$ relative to the plane Oxz may be obtained from the solutions to the problems for cases 1, 2). In both cases,

$$(\mathbf{nr}) = -x\sin\alpha + z\cos\alpha.$$

If $\mathbf{E}^0 \perp$ pl.Oxz, then

$$\mathbf{E}^0 = E\{0,\ \Psi,\ 0\},$$
$$\mathbf{H}^0 = \sqrt{\frac{\varepsilon}{\mu}}\,E\{-\Psi\cos\alpha,\ 0,\ -\Psi\sin\alpha\}, \qquad (9.4.2)$$

and if $\mathbf{E}^0 \parallel$ pl.Oxz, then /483

$$\mathbf{E}^0 = E\{\Psi\cos\alpha,\ 0,\ \Psi\sin\alpha\},$$
$$\mathbf{H}^0 = \sqrt{\frac{\varepsilon}{\mu}}\,E\{0,\ \Psi,\ 0\}, \qquad (9.4.3)$$

where

$$\Psi = e^{ik(\mathbf{nr})}. \qquad (9.4.4)$$

This case of polarization of the wave (9.4.1) is represented by the electric Hertz vectors

$$\vec{\Pi}^0 = \frac{E}{k^2}\{0,\ \Psi,\ 0\} \qquad (9.4.5)$$

and

$$\vec{\Pi}^0 = \frac{E}{k^2\cos\alpha}\{\Psi,\ 0,\ 0\}. \qquad (9.4.6)$$

We formulate the problem of seeking the secondary electromagnetic field $\mathbf{E}^1$, $\mathbf{H}^1$ scattered on the disk. For its solution we introduce into study the Hertz vector $\vec{\Pi}^1 = \{\Pi^1_x,\ \Pi^1_y,\ 0\}$, parallel to the plane Oxy. Then the total electromagnetic field is found through

the Hertz vector $\vec{\Pi} = \vec{\Pi}^0 + \vec{\Pi}^1$ of the total field from the relationships

$$\mathbf{E} = \operatorname{grad}\operatorname{div}\vec{\Pi} + k^2\vec{\Pi},$$

$$\mathbf{H} = -ik_0\varepsilon\operatorname{rot}\vec{\Pi}, \qquad (9.4.7)$$

where $\Pi_x = 0$ in the first of the cases of polarization of the wave and $\Pi_y = 0$ in the second case.

In mathematical formulation this problem consists of finding the solution to the Helmholtz equation,

$$\Delta\Pi^1_{x,y} + k^2\Pi^1_{x,y} = 0,$$

which satisfies on the surface and the edge of the disk the given boundary conditions and the supplementary condition of radiation at infinity. The boundary conditions for the unknown functions Π^1_x and Π^1_y are obtained as usual from the requirement that the tangential components of the vector E of the total field on the surface of the disk vanish. In the general case in Cartesian rectangular coordinates

$$E_x = -\frac{\partial\Phi}{\partial x} + k^2\Pi_x,$$

$$E_y = -\frac{\partial\Phi}{\partial y} + k^2\Pi_y, \qquad (9.4.8)$$

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where

$$\Phi = -\frac{\partial\Pi_y}{\partial y} - \frac{\partial\Pi_x}{\partial x}. \qquad (9.4.9)$$

Therefore the boundary conditions are written in the form,

$$\left.\begin{aligned} \Pi_x = \frac{1}{k^2}\cdot\frac{\partial\Phi}{\partial x}, \quad \Pi_y = \frac{1}{k^2}\cdot\frac{\partial\Phi}{\partial y}, \\ \Delta\Phi + k^2\Phi = 0, \end{aligned}\right\}\ \xi = 0. \qquad (9.4.10)$$

In the coordinates of an oblate spheroid the Meixner conditions are written in the form,

$$\left.\begin{matrix} \dfrac{\partial}{\partial\xi} \\ \dfrac{\partial}{\partial\eta} \end{matrix}\right\}(\Pi^1_x\cos\varphi + \Pi^1_y\sin\varphi) = 0, \quad \xi = \eta = 0 \qquad (9.4.11)$$

(condition (9.4.11) is automatically satisfied by the primary field). To solve the problem let us represent the Hertz vector $\vec{\Pi}^1$ of the secondary field in the form of a sum of two vectors

$$\vec{\Pi}^1 = \vec{\Pi}_1 + \vec{\Pi}_2 \qquad (9.4.12)$$

(the stray field is sought in the form of a superposition of two particular fields, one of which is determined through the vector $\vec{\Pi}_1$, and the second through the vector $\vec{\Pi}_2$) and we require that on the surface of the disk the vector $\vec{\Pi}_2$ satisfy the condition

$$\vec{\Pi}^0 + \vec{\Pi}_1 = 0, \quad \xi = 0 \qquad (9.4.13)$$

(as a result condition (9.4.10) will be satisfied only by the vector $\vec{\Pi}_2$).

On the basis of (5.5.5) and (9.2.17) the functions Π_x^0 and Π_y^0 in the coordinates of an oblate spheroid may be written in the form of a series

$$\Pi_x^0 = 0,$$

$$\Pi_y^0 = \frac{2E}{k^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{i^n}{N_{mn}(-ic)} S_{mn}(-ic, \cos\alpha) \times \qquad (9.4.14)$$

$$\times S_{mn}(-ic, \eta) R^{(1)}_{|m|n}(-ic, i\xi) e^{im\varphi}$$

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(the vector $E^0 \perp$ pl. Oxz) and

$$\Pi_y^0 = 0,$$

$$\Pi_x^0 = \frac{2E}{k^2 \cos\alpha} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{i^n}{N_{mn}(-ic)} \times \qquad (9.4.15)$$

(the vector $E^0 \parallel$ pl. Oxz, respectively. Therefore, if we seek the vector $\vec{\Pi}_1$ in the form,

$$\Pi_{1,x} = 0,$$

$$\Pi_{1,y} = E \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a^1_{mn} R^{(3)}_{|m|n}(-ic, i\xi) S_{mn}(-ic, \eta) e^{im\varphi} \qquad (9.4.16)$$

or in the form,

$$\Pi_{1,y} = 0,$$

$$\Pi_{1,x} = E\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{mn}^{1} R_{|m|n}^{(3)}(-ic,\ i\xi) S_{mn}(-ic,\ \eta) e^{im\varphi} \qquad (9.4.17)$$

as a function of this case of polarization of the wave (9.4.1), then from (9.4.13), we find

$$a_{mn}^{1} = -\frac{2i^n}{k^2} \cdot \frac{S_{mn}(-ic,\ \cos\alpha) R_{|m|n}^{(1)}(-ic,\ i0)}{N_{mn}(-ic) R_{|m|n}^{(3)}(-ic,\ i0)} \qquad (9.4.18)$$

or

$$b_{mn}^{1} = -\frac{2i^n}{k^2\cos\alpha} \cdot \frac{S_{mn}(-ic,\ \cos\alpha) R_{|m|n}^{(1)}(-ic,\ i0)}{N_{mn}(-ic) R_{|m|n}^{(3)}(-ic,\ i0)} \qquad (9.4.19)$$

respectively. In the same way the vector $\vec{\Pi}_1$ is determined.

Let us seek the vector $\vec{\Pi}_2$ in the form of a series

$$\Pi_{2,x} = 0,$$

$$\Pi_{2,y} = E\sum_{n=0}^{\infty}\sum_{m=-n}^{n} a_{mn}^{2} R_{|m|n}^{(3)}(-ic,\ i\xi) S_{mn}(-ic,\ \eta) e^{im\varphi} \qquad (9.4.20)$$

or /486

$$\Pi_{2,y} = 0,$$

$$\Pi_{2,x} = E\sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{mn}^{2} R_{|m|n}^{(3)}(-ic,\ i\xi) S_{mn}(-ic,\ \eta) e^{im\varphi}. \qquad (9.4.21)$$

To determine the expansion coefficients let us expand the function Φ in the form of a series of Bessel functions

$$\Phi(\rho,\ \varphi) = E\sum_{m=-\infty}^{\infty} c_m J_m(k\rho) e^{im\varphi}, \qquad (9.4.22)$$

(on the surface of the disk the function Φ is the solution to the Helmholtz equation (9.4.10)), where ρ and ϕ are polar coordinates with the origin at the center of the disk. Since

$$\frac{\partial}{\partial x} = \cos\varphi\frac{\partial}{\partial\rho} - \frac{\sin\varphi}{\rho}\cdot\frac{\partial}{\partial\varphi},$$

$$\frac{\partial}{\partial y} = \sin\varphi\frac{\partial}{\partial\rho} + \frac{\cos\varphi}{\rho}\cdot\frac{\partial}{\partial\varphi},$$

then after substitution of (9.4.22) into the first two equations in (9.4.10) on the basis of the recurrent relationship

$$J'_m(x) \pm \frac{m}{x} J_m(x) = \pm J_{m\mp1}(x)$$

for the Bessel functions we will have

$$\Pi_{2,y} = \frac{ikE}{2} \sum_{m=-\infty}^{\infty} (c_{m+1} + c_{m-1}) J_m(k\rho) e^{im\varphi} \qquad (9.4.23)$$

or

$$\Pi_{2,x} = \frac{kE}{2} \sum_{m=-\infty}^{\infty} (c_{m+1} - c_{m-1}) J_m(k\rho) e^{im\varphi} \qquad (9.4.24)$$

corresponding to the case studied for polarization of the field (9.4.1). Since

$$J_m(k\rho) = 2 \sum_{n=|m|}^{\infty} i^{n-m} \frac{S_{mn}(-ic, 0)}{N_{mn}(-ic)} \times \times S_{mn}(-ic, \eta) R^{(1)}_{|m|n}(-ic, i\xi), \qquad (9.4.25)$$

then after substitution of (9.4.25) into (9.4.23) or (9.4.24) and comparison of the results thus obtained with (9.4.20) or (9.4.21) we find that /487

$$a^2_{mn} = i^{n-m+1} k (c_{m+1} + c_{m-1}) \times \times \frac{R^{(1)}_{|m|n}(-ic, i0)}{N_{mn}(-ic) R^{(3)}_{|m|n}(-ic, i0)} S_{mn}(-ic, 0) \qquad (9.4.26)$$

or

$$b^2_{mn} = i^{n-m} k (c_{m+1} - c_{m-1}) \times \times \frac{R^{(1)}_{|m|n}(-ic, i0)}{N_{mn}(-ic) R^{(3)}_{|m|n}(-ic, i0)} S_{mn}(-ic, 0). \qquad (9.4.27)$$

Thus, seeking the vector $\vec{\Pi}_2$ is reduced to determining the coefficients c_m of the expansion of the function Φ into the series (9.4.22). From (9.4.18), (9.4.19), (9.4.26), (9.4.27) it follows directly that

$$a^{1,2}_{mn} = b^{1,2}_{mn} = 0, \quad \text{if } (n-m) \text{ is odd.} \qquad (9.4.28)$$

From the Meixner condition applied to the Π_x^1 or Π_y^1 of vector (9.4.12) of the complete stray field, we find that

$$c_{m+1} + c_{m-1} = -\frac{A_m}{B_m}, \tag{9.4.29}$$

where

$$A_m = \sum_{n=|m|}^{\infty} a_{mn}^1 R_{|m|n}^{(3)'}(-ic, i0) S_{mn}(-ic, 0),$$
$$B_m = ik \sum_{n=|m|}^{\infty} i^{n-m} \frac{S_{mn}^2(-ic, 0)}{N_{mn}(-ic)} R_{|m|n}^{(3)'}(-ic, i0) \tag{9.4.30}$$

or

$$c_{m+1} - c_{m-1} = -\frac{iA_m}{\cos\alpha B_m}, \tag{9.4.31}$$

where A_m and B_m are determined just as in (9.4.29). As a result,

$$a_{mn}^2 = -i^{n-m+1} k \frac{A_m R_{|m|n}^{(1)}(-ic, i0) S_{mn}(-ic, 0)}{B_m N_{mn}(-ic) R_{|m|n}^{(3)}(-ic, i0)} \tag{9.4.32}$$

or

$$b_{mn}^2 = -i^{n-m+1} k \frac{A_m R_{|m|n}^{(1)}(-ic, i0) S_{mn}(-ic, 0)}{\cos\alpha B_m N_{mn}(-ic) R_{|m|n}^{(3)}(-ic, i0)} \tag{9.4.33}$$

corresponding to one or another case of polarization of the wave (9.4.1). In the same way the Hertz vector of the secondary field, scattered on the disk with oblique incidence of the wave (9.4.1) on it is determined. In the coordinates of an oblate spheroid the components of the stray electromagnetic field are expressed through Π_z^1 and Π_y^1 by the formulas, /488

$$E_\eta^1 = \frac{\sqrt{(1-\eta^2)(\xi^2+1)}}{a^2(\xi^2+\eta^2)^{1/2}} \cos\varphi \frac{\partial}{\partial\eta}\left\{\frac{\sqrt{1-\eta^2}}{\xi^2+\eta^2} \times \right.$$
$$\left. \times \left(\xi \frac{\partial \Pi_x^1}{\partial \xi} - \eta \frac{\partial \Pi_x^1}{\partial \eta}\right)\right\} - k^2 \eta \sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}} \cos\varphi \Pi_x^1,$$
$$E_\varphi^1 = \frac{-\sin\varphi}{a^2(\xi^2+\eta^2)}\left(\xi \frac{\partial \Pi_x^1}{\partial \xi} - \eta \frac{\partial \Pi_x^1}{\partial \eta}\right) - k^2 \sin\varphi \Pi_x^1,$$

$$H_\eta^1 = \frac{ik_0 \varepsilon \sin\varphi}{a(\xi^2+\eta^2)^{1/2}} \left\{ \frac{\xi}{\sqrt{\xi^2+1}} \Pi_x^1 - \right.$$
$$\left. - \frac{\partial}{\partial \xi} (\sqrt{1+\xi^2}\, \Pi_x^1) \right\}.$$

$$H_\varphi^1 = \frac{ik_0 \varepsilon \sqrt{(1-\eta^2)(\xi^2+1)}}{a(\xi^2+\eta^2)} \cos\varphi \left\{ \frac{\eta}{\sqrt{1-\eta^2}} \times \right. \qquad (9.4.34)$$
$$\left. \times \frac{\partial}{\partial \xi} (\sqrt{1+\xi^2}\, \Pi_x^1) - \frac{\xi}{\sqrt{1+\xi^2}} \cdot \frac{\partial}{\partial \eta} (\sqrt{1-\eta^2}\, \Pi_x^1) \right\}$$

in the case if $E^0 \parallel$ pl. Oxz, and

$$E_\eta^1 = \frac{\sqrt{(1-\eta^2)(\xi^2+1)}}{a^2(\xi^2+\eta^2)^{1/2}} \sin\varphi \frac{\partial}{\partial \eta} \left\{ \frac{\sqrt{1-\eta^2}}{\xi^2+\eta^2} \left(\xi \frac{\partial \Pi_y^1}{\partial \xi} - \right. \right.$$
$$\left. \left. - \eta \frac{\partial \Pi_y^1}{\partial \eta} \right) \right\} - k^2 \eta \sqrt{\frac{\xi^2+1}{\xi^2+\eta^2}} \sin\varphi\, \Pi_y^1,$$

$$E_\varphi^1 = - \frac{\cos\varphi}{a^2(\xi^2+\eta^2)} \left(\xi \frac{\partial \Pi_y^1}{\partial \xi} - \eta \frac{\partial \Pi_y^1}{\partial \eta} \right) + k^2 \cos\varphi\, \Pi_y^1,$$

$$H_\eta^1 = \frac{-ik_0 \varepsilon \cos\varphi}{a(\xi^2+\eta^2)^{1/2}} \times \qquad (9.4.35)$$
$$\times \left\{ \frac{\xi}{\sqrt{\xi^2+1}} \Pi_y^1 - \frac{\partial}{\partial \xi} (\sqrt{1+\xi^2}\, \Pi_y^1) \right\},$$

$$H_\varphi^1 = \frac{ik_0 \varepsilon \sqrt{(1-\eta^2)(\xi^2+1)}}{a(\xi^2+\eta^2)} \sin\varphi \left\{ \frac{\eta}{\sqrt{1-\eta^2}} \times \right.$$
$$\left. \times \frac{\partial}{\partial \xi} (\sqrt{1+\xi^2}\, \Pi_y^1) - \frac{\xi}{\sqrt{1+\xi^2}} \cdot \frac{\partial}{\partial \eta} (\sqrt{1-\eta^2}\, \Pi_y^1) \right\}$$

in the case if $E^0 \perp$ pl. Oxz. /489

For the total stray field,

$$\Pi_x^1 = E \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \beta_{mn} R_{|m|n}^{(3)}(-ic,\ i\xi) S_{mn}(-ic,\ \eta) e^{im\varphi} \qquad (9.4.36)$$

and

$$\Pi_y^1 = E \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \alpha_{mn} R_{|m|n}^{(3)}(-ic,\ i\xi) S_{mn}(-ic,\ \eta) e^{im\varphi} \qquad (9.4.37)$$

respectively, where

$$\beta_{mn} = b^1_{mn} + b^2_{mn} =$$

$$= -\left(\frac{2}{k^2} S_{mn}(-ic, \cos\alpha) + \frac{i^{1-m} kA_m}{B_m} S_{mn}(-ic, 0) \times \right.$$

$$\left. \times \frac{i^n R^{(1)}_{|m|n}(-ic, i0)}{N_{mn}(-ic) R^{(3)}_{|m|n}(-ic, i0)\cos\alpha}\right), \qquad (9.4.38)$$

$$\alpha_{mn} = \alpha^1_{mn} + a^2_{mn} =$$

$$= -\left(\frac{2}{k^2} S_{mn}(-ic, \cos\alpha) + \frac{i^{1-m} kA_m}{B_m} S_{mn}(-ic, 0)\right) \times$$

$$\times \frac{i^n R^{(1)}_{|m|n}(-ic, i0)}{N_{mn}(-ic) R^{(3)}_{|m|n}(-ic, i0)}. \qquad (9.4.39)$$

In particular, when $\alpha = 0$, when the wave is propagated along the axis of the disk in the direction of positive values of z,

$$a^{1,2}_{mn} = b^{1,2}_{mn} = 0,$$

if $m \neq 0$ and then $\alpha_{mn} = \beta_{mn} = 0$ if $m \neq 0$, but

$$\alpha_{0n} = -\left(\frac{2}{k^2} S_{0n}(-ic, 1) + \right.$$

$$\left. + \frac{ikA_0}{B_0} S_{0n}(-ic, 0)\right) \frac{i^n R^{(1)}_{0n}(-ic, i0)}{N_{0n}(-ic) R^{(3)}_{0n}(-ic, i0)}, \qquad (9.4.40)$$

$$\beta_{0n} = \alpha_{0n}.$$

In this case $\Pi^1_x \equiv \Pi^1_y$. /490

In the general case in the vicinity of the wave band,

$$E^1_\eta = \sqrt{\frac{\mu}{\varepsilon}} H^1_\varphi = -Ek\cos\theta\cos\varphi \frac{e^{ikr}}{ir} S_1(\theta, \varphi),$$

$$E^1_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H^1_\eta = -Ek\sin\varphi \frac{e^{ikr}}{ir} S_1(\theta, \varphi), \qquad (9.4.41)$$

if $E^0 \parallel$ pl. Oxz, where

$$S_1(\theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (-i)^n \beta_{mn} S_{mn}(-ic, \cos\theta) e^{im\varphi}, \qquad (9.4.42)$$

id

$$E_\eta^1 = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = -Ek\cos\theta\sin\varphi\,\frac{e^{ikr}}{ir}\,S_2(\theta,\ \varphi),$$

$$E_\varphi^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\eta^1 = Ek\cos\varphi\,\frac{e^{ikr}}{ir}\,S_2(\theta,\ \varphi), \qquad (9.4.43)$$

f $E^0 \perp$ pl. Oxz, where

$$S_2(\theta,\ \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(-i)^n\,\alpha_{mn}\,S_{mn}(-ic,\ \cos\theta)\,e^{im\varphi}. \qquad (9.4.44)$$

ıe value of the back scattering cross section for the disk with blique incidence of the plane wave is computed from the formula

$$\sigma_B = 4\pi k^2\,|S_1(\pi-\alpha,\ 0)|^2\cos^2\alpha \qquad (9.4.45)$$

r in accordance with the case of polarization of an incident wave rom the formula,

$$\sigma_B = 4\pi k^2\,|S_2(\pi-\alpha,\ 0)|^2. \qquad (9.4.46)$$

he density distribution of surface currents, induced on the disk ay be found from the expressions

$$j_\eta = -\frac{Ec_0\sqrt{\frac{\varepsilon}{\mu}}\cos\varphi}{2\pi c^2\eta\cos\alpha}\,\Psi, \qquad (9.4.47)$$

$$j_\varphi = -\frac{Ec_0\sqrt{\frac{\varepsilon}{\mu}}\sin\varphi}{2\pi c^2\eta\cos\alpha}\,\Psi$$

r

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$$j_\eta = -\frac{Ec_0\sqrt{\frac{\varepsilon}{\mu}}\sin\varphi}{2\pi c^2\eta}\,\Psi,$$

$$j_\varphi = \frac{Ec_0\sqrt{\frac{\varepsilon}{\mu}}\cos\varphi}{2\pi c^2\eta}\,\Psi, \qquad (9.4.48)$$

where

$$\Psi = \Phi_1 - \frac{ik^4 a}{2}\Phi_2,$$

but

$$\Phi_1 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{i^n S_{mn}(-ic, \cos\alpha) S_{mn}(-ic, \eta)}{N_{mn}(-ic) R^{(3)}_{|m|n}(-ic, i0)} e^{im\varphi},$$

$$\Phi_2 = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \frac{i^{n-m+1} A_m S_{mn}(-ic, 0) R^{(3)'}_{|m|n}(-ic, i0) S_{mn}(-ic, \eta)}{N_{mn}(-ic) B_m R^{(3)}_{|m|n}(-ic, i0)} e^{im\psi}. \quad (9.4.49)$$

Let us note that the method used for solving the problem on the oblique incidence of the plane electromagnetic wave on the disk may be extended also to the case of a dipole source of the primary field located at any point of space, if such source is an electric dipole with the moment p, parallel to the plane Oxz, in which the disk is located.

§5. Field of a Vertical Electric Dipole in the Presence of Two Coaxial Spheroids

Let there be two prolate ideally conducting spheroids with different focal distances f_{-1} and f_{+1} and with a common axis of rotation located in a uniform isotropic unbounded space with the constants ε, μ and $\sigma = 0$ such as shown on Figure 60. The centers of the spheroids are assumed for the origin of the local Cartesian coordinates x_s, y_s, z_s, $s = \pm 1$, and the local coordinates of the prolate spheroid ξ_s, η_s, $\phi_s = \phi$, $s = \pm 1$, associated with them by the relationships in (1.5.40). The distance between the spheroid centers is equal to l (it is assumed that the spheroids do not touch one another). In local spheroidal coordinates the surfaces of the spheroids are given by the equations $\xi_{-1} = \xi^0_{-1}$, $\xi_{+1} = \xi^0_{+1}$. Let there be found furthermore at a certain point P of the z axis a unit electric dipole with the moment p, directed along the Oz axis in the positive direction. Then for the primary field /49

$$\vec{\Pi}^0 = \mathbf{p}\frac{e^{ikR}}{R}, \text{ where } \mathbf{p} = \{0, 0, p\}, \quad (9.5.1)$$

and $\Pi^0_z = p\frac{e^{ikR}}{R}$ does not depend on the angle ϕ (the primary field E^0, H^0 of the dipole is symmetric relative to the Oz axis). It is obvious then that the secondary field E^1, H^1 produced by the presence in space of the spheroids will also be independent of ϕ. Therefore the complete field is determined by the vectors,

$$\mathbf{E} = \{E_\xi,\ E_\eta,\ 0\},$$
$$\mathbf{H} = \{0,\ 0,\ H_\varphi\}, \qquad (9.5.2)$$

where $E_\eta = E_\eta^0 + E_\eta^1$, $E_\xi = E_\xi^0 + E_\xi^1$, $H_\phi = H_\phi^0 + H_\phi^1$ (it is assumed that the vectors in (9.5.2) are written in one of the coordinate systems).

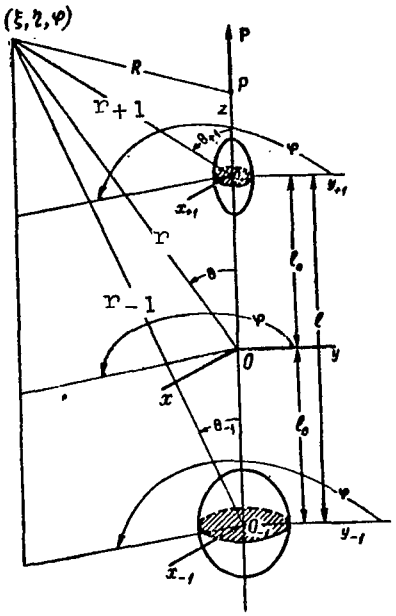

Fig. 60. Excitation of Two Coaxial Spheroids by a Vertical Dipole.

From the first equation of the Maxwell system, just as in the case of one spheroid, it follows that both nonzero components E_ζ, E_η of the vector E are determined through H_ϕ by the formulas in (9.2.3)

$$E_\xi = \frac{i\sqrt{\frac{\mu}{\varepsilon}}}{c(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial}{\partial \eta}[(1-\eta^2)^{1/2} H_\varphi], \qquad (9.5.3)$$

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$$E_\eta = \frac{-i\sqrt{\frac{\mu}{\varepsilon}}}{c(\xi^2-\eta^2)^{1/2}} \cdot \frac{\partial}{\partial \xi}[(\xi^2-1)^{1/2} H_\varphi]. \qquad (9.5.4)$$

Using the second equation of the Maxwell system, we can establish that the function H_ϕ^1 must be the solution to the Helmholtz equation (9.2.5) with the azimuth number $m = 1$. As a result the problem on seeking the diffraction field in mathematical formulation is reduced to finding the unique solution H_ϕ^1 of equation (9.2.5), which satisfies along with H_ϕ^0 on the surface of each spheroid, the boundary condition

$$\frac{\partial}{\partial \xi_s}[(\xi_s^2-1)^{1/2} H_\varphi]\Big|_{\xi_s=\xi_{0s}} = 0, \qquad s = \pm 1, \qquad (9.5.5)$$

and the supplementary condition of radiation at infinity. In the coordinates of the s-th spheroid the function H_ϕ^0 is represented [see (9.2.4)] by the expansion,

$$H_\varphi^0 = \frac{4k^2 p}{f_s(\xi_{0s}^2-1)^{1/2}} \sum_{n=1}^{\infty} \frac{\sigma_{1n}(c_s)}{N_{1n}(c_s)} \times$$
$$\times S_{1n}(c_s, \eta_s) \begin{cases} R_{1n}^{(1)}(c_s, \xi_s) R_{1n}^{(3)}(c_s, \xi_{0s}), & \xi_{0s} > \xi_s, \\ R_{1n}^{(1)}(c_s, \xi_{0s}) R_{1n}^{(3)}(c_s, \xi_s), & \xi_{0s} < \xi_s, \end{cases} \qquad (9.5.6)$$

where $\sigma_{1n}(c_s) = \lim_{\eta \to h_{0s}=\pm 1} \frac{S_{1n}(c_s, \eta_s)}{(1-\eta_s^2)^{1/2}}$, , and through ξ_{0_s}, η_{0_s} we denote the spheroidal coordinates of the point P with the dipole in local coordinates of the s-th spheroid.

The function H_ϕ^1 is sought in the form of the sum

$$H_\varphi^1 = \sum_{s=\pm 1} H_s, \tag{9.5.7}$$

where

$$H_s = k^2 p \sum_{n=1}^{\infty} a_n^s R_{1n}^{(3)}(c_s, \xi_s) S_{1n}(c_s, \eta_s) \tag{9.5.8}$$

is attributed to the secondary field, produced by the s-th spheroid. The expansion coefficients a_n^s are determined from the boundary conditions (9.5.5) using the addition theorem (5.5.18), which is written in the form,

$$R_{1n}^{(3)}(c_{-s}, \xi_{-s}) S_{1n}(c_{-s}, \eta_{-s}) = \\ = \sum_{m=1}^{\infty} Q_{1m1n}^{(3)}(c_s, c_{-s}; l, \theta_{-s,s}) R_{1m}^{(1)}(c_s, \xi_s) S_{1m}(c_s, \eta_s), \tag{9.5.9}$$

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where Q_{1m1n} is determined by formula (5.5.10). Here $\theta_{-1,1} = 0$, $\theta_{+1,-1} = \pi$. Carrying out the substitution of the unknowns a_n^s by the unknowns A_n^s according to the formula,

$$a_n^s = n[z_n^1(c_s, \xi_s^0)]' A_n^s,$$
$$[z_n^i]' = \frac{d}{d\xi}[(\xi^2-1)^{1/2} R_{1n}^i(c, \xi)]|_{\xi=\xi_s}, \qquad i = 1, 3, \tag{9.5.10}$$

and using (9.5.9) we find from (9.5.5) an infinite system of linear equations,

$$A_n^s + \sum_{m=1}^{\infty} \alpha_{mn}^{(-s,s)} A_m^{-s} = f_n^s,$$
$$n = 1, 2, \ldots; \quad s = \pm 1, \tag{9.5.11}$$

in which

$$\alpha_{mn}^{(-s,s)} = \frac{m}{n} \cdot \frac{[z_m^{(1)}(c_{-s}, \xi_{-s}^0)]'}{[z_n^{(3)}(c_s, \xi_s^0)]'} Q_{1n1m}(c_s, c_{-s}; l, \theta_{-s,s}), \tag{9.5.12}$$

$$f_n^s = -\frac{4\sigma_{1n}(c_s)}{f_s(\xi_{0s}^2-1)^{1/2} n N_{1n}(c_s)} \cdot \frac{R_{1n}^{(3)}(c_s, \xi_{0s})}{[z_n^{(3)}(c_s, \xi_s^0)]'}. \tag{9.5.13}$$

he dimensionless determinant of system (9.5.11) can be written in form similar to (6.3.23), i.e.,

$$\begin{vmatrix}
1 & \alpha_{11}^{--} & 0 & \alpha_{12}^{--} & 0 & \alpha_{13}^{--} & 0 & \alpha_{14}^{--} & 0 & \alpha_{15}^{--} & \cdots \\
\alpha_{11}^{++} & 1 & \alpha_{12}^{++} & 0 & \alpha_{13}^{++} & 0 & \alpha_{14}^{++} & 0 & \alpha_{15}^{++} & 0 & \cdots \\
0 & \alpha_{21}^{--} & 1 & \alpha_{22}^{--} & 0 & \alpha_{23}^{--} & 0 & \alpha_{24}^{--} & 0 & \alpha_{25}^{--} & \cdots \\
\alpha_{21}^{++} & 0 & \alpha_{22}^{++} & 1 & \alpha_{23}^{++} & 0 & \alpha_{24}^{++} & 0 & \alpha_{25}^{++} & 0 & \cdots \\
0 & \alpha_{31}^{--} & 0 & \alpha_{32}^{--} & 1 & \alpha_{33}^{--} & 0 & \alpha_{34}^{--} & 0 & \alpha_{35}^{--} & \cdots \\
\alpha_{31}^{++} & 0 & \alpha_{32}^{++} & 0 & \alpha_{33}^{++} & 1 & \alpha_{34}^{++} & 0 & \alpha_{35}^{++} & 0 & \cdots \\
0 & \alpha_{41}^{--} & 0 & \alpha_{42}^{--} & 0 & \alpha_{43}^{--} & 1 & \alpha_{44}^{--} & 0 & \alpha_{45}^{--} & \cdots \\
\alpha_{41}^{++} & 0 & \alpha_{42}^{++} & 0 & \alpha_{43}^{++} & 0 & \alpha_{44}^{++} & 1 & \alpha_{45}^{++} & 0 & \cdots \\
0 & \alpha_{51}^{--} & 0 & \alpha_{52}^{--} & 0 & \alpha_{53}^{--} & 0 & \alpha_{54}^{--} & 1 & \alpha_{55}^{--} & \cdots \\
\alpha_{51}^{++} & 0 & \alpha_{52}^{++} & 0 & \alpha_{53}^{++} & 0 & \alpha_{54}^{++} & 0 & \alpha_{55}^{++} & 1 & \cdots \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{vmatrix}$$

here $\alpha_{mn}^{--} = \alpha_{mn}^{(-1,+1)}$, but $\alpha_{mn}^{++} = \alpha_{mn}^{(+1,-1)}$.

If we introduce the designations /495

$$\begin{gathered}
x_{2i-1} = A_i^{+1}, \quad f_{2i-1} = f_i^{+1}, \\
x_{2i} = A_i^{-1}, \quad f_{2i} = f_i^{-1}, \qquad i, j = 1, 2, \ldots, \\
c_{2i-1,\,2j} = \alpha_{ij}^{(-1,\,+1)}, \quad c_{2i,\,2j-1} = \alpha_{ij}^{(+1,\,-1)}
\end{gathered} \tag{9.5.14}$$

nd set $c_{2i-1,2j-1} = 0$, $c_{2i,2j} = 0$, then system (9.5.11) is written n normal form:

$$x_i + \sum_{j=1}^{\infty} c_{ij} x_j = f_i, \qquad i = 1, 2, \ldots \tag{9.5.15}$$

ystem (9.5.1-) possesses the same properties as, for example, sys- ems (7.3.9), (7.3.10) when $m = 1$ in the problem for two spheres. ts matrix elements cij satisfy condition (1.9.19), and the right- and sides satisfy condition (1.9.20). System (9.5.11) is uniquely olvable under the condition that the spheroids do not touch one nother and its solution may be found by the method of truncation nd will satisfy the condition $\sum_{i=1}^{\infty} |x_i|^2 < \infty$. . By the determining the oefficients of series (9.5.7), (9.5.8), which satisfy the just-men- ioned condition, the mathematical problem formulated earlier for he function H_ϕ^1 is solved. The components E_ξ^1, E_η^1 of the electric ield may be found through H_ϕ^1 from (9.5.3), (9.5.4).

In the vicinity of the wave band,

$$E_\eta = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi = -k^2 p \frac{e^{ikR}}{R} \sin\theta\, W(\theta). \qquad (9.5.16)$$

where the function $W(\theta)$ is the damping coefficient of the dipole field which takes into account the presence in it of two coaxial ideally conducting prolate spheroids, and is given by the expression

$$W(\theta) = 1 + \frac{i}{k} e^{ikl_{0p}\cos\theta} \sum_{s=\pm 1} e^{-iskl_0\cos\theta} \times$$
$$\times \sum_{n=1}^{\infty} (-i)^n a_n^s \frac{S_{1n}(c_s, \cos\theta)}{\sin\theta}, \qquad l_0 = l/2. \qquad (9.5.17)$$

In (9.5.16), $R = r - l_{0p}\cos\theta$, where l_{0p} is the distance from point O (origin of the coordinate system $Oxyz$) to the dipole. Formulas (9.5.16), (9.5.17) are written in spherical coordinates r, θ, ϕ with the origin at point O, lying on the axis of symmetry of the spheroids midway between them (the coordinate axes of the systems $Oxyz$ and $O_s x_s y_s z_s$, $s = \pm 1$, are identically directed, and the axis /496 Oz matches the axes $O_s z_s$, $s = \pm 1$. In the neighboring band the diffraction field may be studied using the formula,

$$H_\varphi = k^2 p \sum_{n=1}^{\infty} n A_n^s \{[z_n^1(c_s, \xi_s^0)]' R_{1n}^{(3)}(c_s, \xi_s) -$$
$$- R_{1n}^{(1)}(c_s, \xi_s)[z_n^3(c_s, \xi_s^0)]'\} S_{1n}(c_s, \eta_s) \qquad (9.5.18)$$

($l > f_s \xi_s$), written in local coordinates of the s-th spheroid. Formula (9.5.18) is obtained on the basis of (9.5.6), (9.5.7), (9.5.10), (9.5.11) - (9.5.13) using the addition theorem (9.5.9).

Returning to (1.1.7) and (9.5.18) we find the density distribution of surface currents induced by the dipole field on the surface of the s-th spheroid will be given by the formula,

$$j_\eta^s = \frac{ic_0 kp}{4\pi f_s (\xi_s^{02} - 1)^{1/2}} \sum_{n=1}^{\infty} n A_n^s S_{1n}(c_s, \eta_s), \qquad (9.5.19)$$

c_0 is the speed of light.

If we substitute $\xi_{0,-1} = \xi_{-1}^0$ or $\xi_{0,+1} = \xi_{+1}^0$ into (9.5.13), then from (9.5.7) - (9.5.17) we find the solution to the problem of diffraction of the field of the emitting surface spheroidal antenna (excited by the vertical electric dipole with the moment p located at its pole) on the other spheroid which is coaxial with the antenna and which plays the role of a passive reflector.

Converting in formulas (9.5.6) - (9.5.19) from coordinates of the prolate spheroid to spherical coordinates r_s, θ_s, ϕ_s ($f_s \to 0$, $\xi_s \to \infty$ such that $f_s\xi_s \to r_s$, $\eta_s \to \cos\theta_s$, $s = \pm 1$), we find the solution to the problem on the diffraction of the field of a vertical electric dipole on two ideally conducting coaxial spheres of radii r^0_{-1} and r^0_{+1}.

If in formulas (9.5.6) - (9.5.19) we convert from coordinates of the prolate spheroid to coordinates of an oblate spheroid on the basis of formulas (9.2.16), then from them we find the solution to the problem on the diffraction of the field of a vertical electric dipole on two coaxial oblate spheroids $\xi_s = \xi^0_s$, $s = \pm 1$. Its particular case is the problem on the diffraction on two parallel coaxial infinitely thin ideally conducting circular disks of different radii. If, in such a particular problem, the dipole approaches the surface of one of the disks and we set $\xi^0_s = 0$ or $\xi^0_{-s} = 0$, then as a result we find the solution to the problem that is equivalent in the electrodynamic sense to the problem of the field of an emitting self-contained annular unit slot, located on the surface of the s-th disk (active antenna) in the presence in it of another one, i.e., the $-s$-th infinitely thin ideally conducting circular disk, /497 which is coaxial with the first and which plays the role of the passive reflector, $s = +1$, or $s = -1$.

The independent conversions from one coordinate system to another, derived in local systems, makes it possible to obtain from (9.5.6) - (9.5.19) solutions to the problems for the case when with a constant shape of one body (spheroid) the shape of the other spheroid varies in the range from a sphere to a disk and when the intermediate bodies are prolate or oblate spheroids.

With a numerical solution to the problems the approximate values of the quantities may be found from the finite system

$$x_i + \sum_{j=1}^{2N} c_{ij}x_j = f_i, \quad i = 1, 2, \ldots, 2N, \qquad (9.5.20)$$

obtained from (9.5.15) by truncation. System (9.5.20) contains $2N$ equations with $2N$ unknowns A^s_l, $s = \pm 1$, which approximate the precise values of the coefficients of the series (9.5.8). It is obvious that the accuracy of computing A^s_l will be increased with increase in N, the order of truncation of the system (9.5.15). As in the previous problems as the original value of N we can take the value $N = [c]$, where $c = \max\limits_{s=\pm 1} c_s$.

The order of the finite systems, serving as the approximate computations of the quantities A^s_n, $s = \pm 1$, may be in a number of

cases cut in half in comparison with the order of the system (9.5.20) with retention of the computation precision. Thus, if $c_1 = c_{+1} = c$ and $\xi^0_{-1} = \xi^0_{+1} = \xi^0$ (the spheroids are identical) then the matrix elements of system (9.5.11) are found to be mutually connected by the relationship,

$$\alpha_{nm} = \alpha_{nm}^{(-1,+1)} = (-1)^{n+m}\alpha_{nn}^{(+1,-1)} \tag{9.5.21}$$

(this follows from (9.5.12) and (5.5.10)). Therefore, if we write system (9.5.20) in the form,

$$\left.\begin{aligned} A_n^{+1} + \sum_{m=1}^{N} \alpha_{nm} A_m^{-1} &= f_n^{+1} \\ A_n^{-1} + \sum_{m=1}^{N} (-1)^{n+m}\alpha_{nm} A_m^{+1} &= f_n^{-1} \end{aligned}\right\}, \tag{9.5.22}$$

after term-by-term addition and subtraction of the equations in (9.5.22) we find two independent systems of N equations with N unknowns,

$$y_n + \sum_{m=1}^{N} (-1)^m \alpha_{nm} y_m = F_n, \tag{9.5.23}$$

where

$$z_n - \sum_{m=1}^{N} (-1)^m \alpha_{nm} z_m = \Phi_n, \quad n = 1, 2, \ldots, N, \tag{/498}$$

$$\begin{aligned} y_n &= A_n^{+1} + (-1)^n A_n^{-1}, \quad F_n = f_n^{+1} + (-1)^n f_n^{-1}, \\ z_n &= A_n^{+1} - (-1)^n A_n^{-1}, \quad \Phi_n = f_n^{+1} - (-1)^n f_n^{-1}. \end{aligned} \tag{9.5.24}$$

The order of system (9.5.20) may be cut in half (with retention of the accuracy of the computation) without converting to new systems such as (9.5.23) in the case when $c_1 = c_{+1} = c$, $\xi^0_{-1} = \xi^0_{+1} = \xi^0$ and $\xi_{0,-1} = \xi_{0,+1} = \xi_0$ (the dipole is located at point O which lies midway between the centers of the spheroids). Then,

$$f_n^s = -(-s)^{n+1} f_n^{-s}, \quad s = \pm 1, \tag{9.5.25}$$

and on the basis of (9.5.21) from (9.5.11) it follows that the unknown quantities A_n^{+1} and A_n^{-1} are found to be mutually connected by the relationship,

$$A_n = A_n^{+1} = -(-1)^n A_n^{-1}. \tag{9.5.26}$$

Therefore, instead of system (9.5.20) we can solve the system of N equations with N unknowns,

$$A_n - \sum_{m=1}^{N} (-1)^m \alpha_{nm} A_m = f_n, \quad f_n = f_n^{+1}, \tag{9.5.27}$$

which is obtained from this latter with the exception of the unknowns A_n^{-1} according to formula (9.5.26). In this case in (9.5.17) $l_{0_p} = 0$, and the damping coefficient itself can be written on the basis of (9.5.26), (9.5.10) in the form,

$$W(\theta) = 1 - \frac{2i}{k} \cos(kl_0 \cos\theta) \sum_{n=1,3,\ldots} i^n n A_n [(\xi^{02} - 1)^{1/2} \times$$
$$\times R_{1n}^{(1)}(c, \xi^0)]' \frac{S_{1n}(c, \cos\theta)}{\sin\theta} + \frac{2}{k} \sin(kl_0 \cos\theta) \times \tag{9.5.28}$$
$$\times \sum_{n=2,4,\ldots} i^n n A_n [(\xi^{02} - 1)^{1/2} R_{1n}^{(1)}(c, \xi^0)]' \frac{S_{1n}(c, \cos\theta)}{\sin\theta}$$

In the case of oblate spheroids,

$$W(\theta) = 1 - \frac{2i}{k} \cos(kl_0 \cos\theta) \sum_{n=1,3,\ldots} i^n n A_n [(\xi^{02} + 1)^{1/2} \times$$
$$\times R_{1n}^{(1)}(-ic, i\xi^0)]' \frac{S_{1n}(-ic, \cos\theta)}{\sin\theta} + \tag{9.5.29}$$
$$+ \frac{2}{k} \sin(kl_0 \cos\theta) \sum_{n=2,4,\ldots} i^n n A_n [(\xi^{02} + 1)^{1/2} \times$$
$$\times R_{1n}^{(1)}(-ic, i\xi^0)]' \frac{S_{1n}(-ic, \cos\theta)}{\sin\theta},$$

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and with their degeneration into disks of radius f the damping coefficient is written in the form [$R_{1n}^{(1)}(-ic, i0) = 0$, $(n-1)$ is even]

$$W(\theta) = 1 + \frac{2}{k} \sin(kl_0 \cos\theta) \sum_{n=2,4,\ldots} i^n n A_n R_{1n}^{(1)'}(-ic, i0) \times$$
$$\times \frac{S_{1n}(-ic, \cos\theta)}{\sin\theta}. \tag{9.5.30}$$

Since $S_{1n}(-ic, 0) = 0$, if $(n - 1)$ is odd, then in the plane $\theta = \pi/2$ expressions (9.5.28), (9.5.29) are simplified. Instead of them we find

$$W(\theta) = 1 - \frac{2i}{k} \sum_{n=1,3,\ldots} i^n n A_n [(\xi^{0^2} - 1)^{1/2} \times \times R_{1n}^{(1)}(c, \xi^0)]' S_{1n}(c, 0), \tag{9.5.31}$$

$$W(\theta) = 1 - \frac{2i}{k} \sum_{n=1,3,\ldots}' i^n n A_n [(\xi^{0^2} + 1)^{1/2} \times \times R_{1n}^{(1)}(-ic, i\xi^0)]' S_{1n}(-ic, 0) \tag{9.5.32}$$

and

$$W(\theta) = 1 \tag{9.5.33}$$

respectively. This latter result means that the system which consists of a vertical electric dipole and two coaxial ideally conducting circular disks of equal radii, possesses in the plane $\theta = \pi/2$ a characteristic of emission $W(\pi/2)$, identical to the characteristic of emission for a dipole in free space in the case if the dipole is located on the axis of symmetry of the disks midway between them.

Taking into account that the computation of the approximate values of the quantities A_n^s assumes a finiteness of the indices n and m in (9.5.11) - (9.5.13), in the case when the distance between the spheroids is much larger than the wave length ($kl >> 1$) and l is significantly greater than cross section dimensions of the spheroids, for A_n^s we can find approximate expressions which have a closed form. Thus, assuming in (9.5.12) that $kl >> 1$ ($kl >> n + m$) and applying the formula that is asymptotic, relative to the argument, /500 to the spherical Bessel function $h_\sigma^{(1)}(kl)$, contained in the expression Q_{1m1n}, we find on the basis of formulas (3.4.13), (5.2.2), (5.5.10), that

$$Q_{1n1m}(c_s, c_{-s}; l_0, \theta_{-s,s}) \simeq \simeq \frac{2i^{n-m-1}e^{ikl}}{(kl)N_{1n}(c_{-s})} S_{1n}(c_{-s}, \theta_{s,-s}) S_{1m}(c_s, \theta_{s,-s}),$$

where $\theta_{-s,s} = 0$ or $\theta_{-s,s} = \pi$. Since $S_{mn}(c, \pm 1) = 0$, if $m \neq 0$ we then find that in the case of widely separated spheroids, approximately all matrix elements $\alpha_{nm}^{(-s,s)}$ of the system will be equal to zero and then in such an approximation we will have

$$A_n^s = f_n^s. \tag{9.5.34}$$

From the physical point of view this result means that in the case

f sufficiently large values of kl (l is high in comparison, for xample, with the major semiaxes of the spheroids) at each point of bservation the diffraction effect is produced only by the secondary aves of the first order of scattering, in comparison with which he secondary waves of all subsequent orders of scattering and their um may be ignored.

In the general case for numerical solution of the problem the alues of the right-hand sides of system (9.5.20) are found with no pecial difficulties using the Tables of Spheroidal Wave Functions in the range allowable by the content of the tables). Difficulties ay arise in computing the matrix elements of the system, which are xpressed through the spheroidal radial functions and through the lebsch-Gordan coefficients (§4, Ch. 3), and these appear in

$$Q_{1n1m}(c_s,\ c_{-s};\ l,\ \theta_{s,-s}) = \frac{2i^{n-m}}{N_{1n}(c_{-s})} \sum_{t=0,1}^{\infty'} \sum_{r=0,1}^{\infty'} d_t^{1n}(c_{-s})\, d_r^{1m}(c_s) \times \times \sum_{\sigma=|r-t|}^{r+t+2} i^{\sigma} b_{\sigma}^{(t+1,1,\,r+1,1)} h_{\sigma}^{(1)}(kl) P_{\sigma}(\cos\theta_{s,-s}) \tag{9.5.35}$$

nder the sign of the multiple summation. The rapidity of the con- ergence of series (9.5.35) depends on the value of the parameters l, c_s, $s = \pm 1$. We can show that the compound series (9.5.35) con- erges absolutely, if

$$kl > c_{-1} + c_{+1} \quad (l > f_{-1} + f_{+1}). \tag{9.5.36}$$

rom the character of the behavior of the spherical Bessel functions $h_{\sigma}^{(1)}(kl)$ it follows that for given values of c_s, $s = \pm 1$, the con- ergence of series (9.5.35) will be improved with increase in kl. s noted above, when $kl \to \infty$ $\alpha_{nm}^{(-s,s)} \to 0$, then (9.5.34) is valid. /501

n computing (9.5.35) summation over r and t it is feasible to begin rom the numbers $r = n - 1$ and $t = m - 1$, since these values of r nd t are used to determine the dominant values of the coefficients $d_r^{1n}(c)$ and $d_t^{1m}(c)$ [for values of the parameter c of several units nd less ($c \leq 5$)]. For comparatively small values of c the coef- icients $d_r^{mn}(c)$ drop rapidly in absolute value with increase in $r > - m$ and with drop in r in the range $0 \leq r < n - m$.

As examples, Figure 61 shows normalized directional diagrams onstructed for the function $|H_{\phi}|^2$ where H_{ϕ} is the component of ne field [formula (9.5.16); $\varepsilon = \mu = 1$; the solid line], for the nction $|H_{\phi}^{1} + H_{-1}|^2$ [the broken line], where H_{-1} is the component

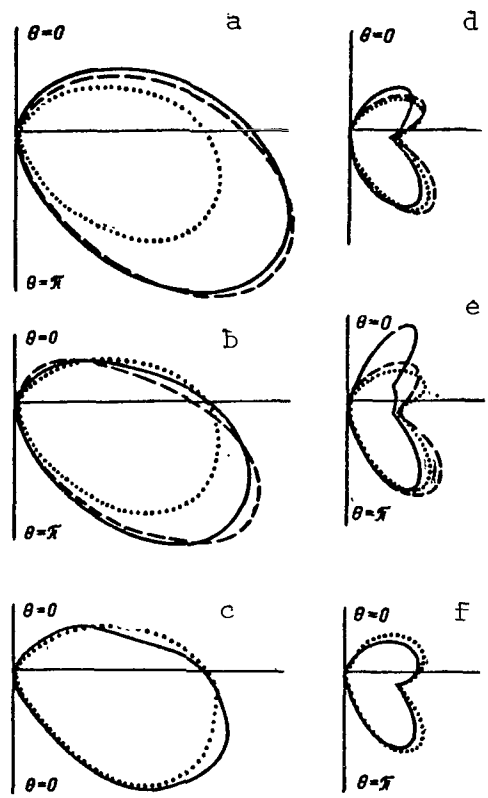

Fig. 61. The Functions $|H_\phi|^2$, $|H_\phi^1 + H_{-1}|^2$ and $H_\phi^1|^2$: a. Sphere ($kr_{-1}^0 = 1$) and Disk ($c_{+1} = 1$); b,c,d,e: Two Spheres (Respectively, $kr_{-1}^0 = 1$, $kr_{+1}^0 = 2$, $kr_{-1}^0 = kr_{-1}^0 = 1$, $kr_{-1}^0 = kr_{+1}^0 = 2$, $kr_{+1}^0 = 2$, $kr_{+1}^0 = 1$); f. Disk ($c_{+1} = 2$) and Sphere ($kr_{-1}^0 = 2$).

of the secondary field, produced /50 by the -1 body interacting with the +1 body and for the function H_ϕ^1 [its graph is shown by the dotted line). It is assumed everywhere that the dipole is located on the surface of the -1 sphere at its pole $\theta_{-1} = 0$. Computation was carried out in the coordinates of this sphere (the normalizing value is $|H_\phi^{(1)}(90°)|^2$]

On Figure 61, a is the parameter $kl = \pi/2$, and on all others $kl = 2\pi$.

§6. The Problem of the Field of Two Coaxial Spheroidal Emitters

A problem that is similar to that just studied in §5 is the problem of the field of two coaxial (prolate and oblate) spheroidal emitters (antennas), each of which represents a complete ideally conducting spheroidal surface, separated by a small azimuth gap to which an exciting voltage is supplied that is symmetric relative to the axis. The distance between the centers of the emitters is equal to l. The medium surrounding them is a uniform isotropic unbounded space with electromagnetic constants ε, μ and $\sigma \blacksquare 0$. In local spheroidal coordinates ξ_s, η_s, ϕ_s, (the numbers $s \blacksquare -1$ and $s = +1$ are attributed to the emitters, the origin of the local system of coordinates $O_s x_s y_s z_s$ coincides with the centers of the spheroid, the axes $O_s z_s$ are directed along the axis of rotation that is common for the spheroids, such that $\phi_s = \phi$, $s = \pm 1$, Fig. 62) the surfaces of the emitters are given by the equations $\xi_{-1} = \xi_{-1}^0$, $\xi_{+1} = \xi_{+1}^0$, and the position of the gap on the s-th spheroid is determined by the angular coordinate η_s^0, $s = \pm 1$. The interfocal distances of the spheroid in the general case are assumed to be different ($f_{-1} \neq f_{+1}$).

As usual, the problem consists of integrating the system of Maxwell equations (1.1.4) in the region outside the spheroid ($j^{(ext)} \equiv$

0, $\rho \equiv 0$) under the given conditions on the surface of each emitter, the conditions of radiation at infinity and on the assumption that the unknown solution to the system possesses an axial symmetry ($\partial/\partial\phi \equiv 0$). The effect of the conducting lines is not taken into account.

First let the spheroids be prolate.

In the coordinates of a prolate spheroid in the presence of axial symmetry

$$\begin{aligned} \mathbf{E} &= \{E_\xi, E_\eta, 0\}, \\ \mathbf{H} &= \{0, 0, H_\varphi\}, \end{aligned} \qquad (9.6.1)$$

where the components of the electric field E_ξ and E_η are expressed through H_ϕ by formulas (9.5.3), (9.5.4). The function H_ϕ is unknown. As follows from (9.5.3), (9.5.4), (1.1.4) this must be the solution to the Helmholtz equation (9.2.5) with the azimuth number $m = 1$. Since on the surface of each emitter the component E_η of the electric field satisfies the condition, /503

$$E_\eta = \begin{cases} 0, \text{ if } \xi_s = \xi_s^0, \text{ outside the gap;} \\ \text{the field } E_0(\eta_s) \text{ applied, when } \xi_s = \xi_s^0, \text{ to the gap} \\ \eta_s^0 - \alpha_s \leq \eta_s \leq \eta_s^0 + \alpha_s, \text{ where } 2\alpha_s \text{ determines the} \\ \text{width of the gap on the surface of the} \\ s\text{-th emitter, } s = \pm 1, \end{cases} \qquad (9.6.2)$$

then on the basis of (9.5.4) the unknown function H_ϕ must satisfy the boundary condition

$$\left.\frac{\partial}{\partial \xi_s}\left(\sqrt{\xi_s^2 - 1}\, H_\varphi\right)\right|_{\xi_s = \xi_s^0} = \qquad (9.6.3)$$

$$= \begin{cases} 0 \text{ outside the gap with } \xi = \xi_s^0; \\ ic_s \sqrt{\dfrac{\varepsilon}{\mu}} \left(\xi_s^{02} - \eta_s^2\right)^{1/2} E_0(\eta_s) \text{ in the gap with} \\ \xi_s = \xi_s^0, \ \eta_s^0 - \alpha_s \leq \eta_s \leq \eta_s^0 + \alpha_s, \ s = \pm 1. \end{cases}$$

Furthermore H_ϕ must satisfy the condition of radiation at infinity.

Let us seek the solution to the problem in the form of a sum

$$H_\varphi = \sum_{s=\pm 1} H_s, \qquad (9.6.4)$$

where

$$H_s = \sum_{n=1}^{\infty} a_n^s R_{1n}^{(3)}(c_s, \xi_s) S_{1n}(c_s, \eta_s).$$

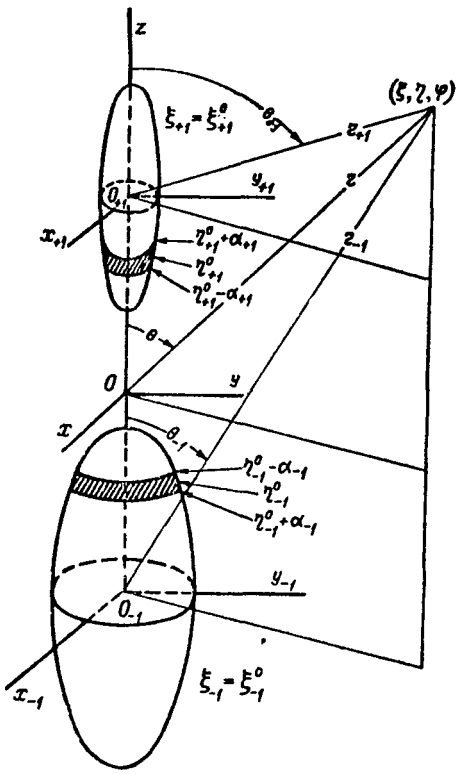

Fig. 62. The Problem of the Field of Two Coaxial Spheroidal Emitters.

The coefficients of the series a_n^s, $s = \pm 1$, are subject to determination in the future. To seek a_n^s we must first expand the right-hand side of (9.6.3) into a series of angular spheroidal functions $S_{1n}(c_s, \eta_s)$ $s = \pm 1$. We find /50

$$\frac{\partial}{\partial \xi}\left(V\overline{\xi_s^2-1}\,H_\varphi\right)\Big|_{\xi_s=\xi_s^0} = \sum_{n=1} a_n(s) S_{1n}(c_s, \eta_s), \quad s = \pm 1, \tag{9.6.5}$$

where

$$a_n(s) = \sqrt{\frac{\varepsilon}{\mu}}\,\frac{ic_s}{N_{1n}(c_s)} \int_{\eta_s^0-\alpha_s}^{\eta_s^0+\alpha_s} (\xi_s^{02}-\eta_s^2)^{1/2} E_0(\eta_s) S_{1n}(c_s, \eta_s)\, d\eta_s,$$

or after substituting the distribution of the electric field $E_0(\eta_s)$ in the gap of the s-th emitter by the distribution of voltage $V_s = E_0(\eta_s)$ $h_{\eta_s}\Delta\eta_s$, where $\Delta\eta_s = 2\alpha_s$, $h_\eta = f\sqrt{\frac{\xi^2-\eta^2}{1-\eta^2}}$ is the metrical Lame coefficient, taking into account the smallness of the width of the gap,

$$a_n(s) = ik\sqrt{\frac{\varepsilon}{\mu}}\,V_s\,\frac{(1-\eta_s^{02})^{1/2} S_{1n}(c_s, \eta_s^0)}{N_{1n}(c_s)}. \tag{9.6.6}$$

Then let us substitute into the left-hand side of (9.6.5) the series (9.6.4). As a result, using the addition theorem (9.5.8) after substituting the unknown a_n^s by new unknowns according to the formula, /50

$$a_n^s = [z_n^1(c_s, \xi_s^0)]'\, A_n^s, \quad s = \pm 1, \tag{9.6.7}$$

where $[z_n^1(c_s, \xi_s^0)]'$ has the same sense as in (9.5.10), we find for the determination of the unknown A_n^s an infinite system of linear equations,

$$A_n^s + \sum_{m=1}^{\infty} a_{nm}^{(-s,s)} A_m^{-s} = f_n^s, \quad s = \pm 1, \quad n = 1, 2, \ldots, \tag{9.6.8}$$

aving the same form as system (9.5.11) with the matrix elements,

$$\alpha_{nm}^{(-s,s)} = \frac{[z_m^1(c_{-s}, \xi_{-s}^0)]'}{[z_n^3(c_s, \xi_s^0)]'} Q_{1n1m}(c_s, c_{-s}; l, \theta_{-s,s}) \qquad (9.6.9)$$

nd with the free terms

$$f_n^s = ik \sqrt{\frac{\varepsilon}{\mu}} V_s \frac{(1-\eta_s^{02})^{1/2} S_{1n}(c_s, \eta_s^0)}{[z_n^{(1)}(c_s, \xi_s^0)]' [z_n^3(c_s, \xi_s^0)]' N_{1n}(c_s)}. \qquad (9.6.10)$$

he dimensionless determinant of system (9.6.8) may be written in he same form as in system (9.5.11). After introducing the designtion (9.5.14), system (9.6.8) assumes the normal form (9.5.15). he matrix elements of the system and its right-hand sides satisfy onditions (1.9.19) and (1.9.20), respectively. Just as in the revious problems for two bodies, system (9.6.8) is uniquely solvble. Its solution, which may be found by the method of truncation, atisfies the condition $\sum_{n=1}^{\infty} |A_n^s|^2 < \infty$, $s = \pm 1$.

By determining the coefficients A_n^s the problem for the unknown unction H_ϕ is fully solved. The components of the electric field re found through H_ϕ from (9.5.3), (9.5.4).

In the vicinity of the wave band in spherical coordinates r, , ϕ with the origin at point O lying midway between the emitters n the axis of symmetry

$$E_\eta = \sqrt{\frac{\mu}{\varepsilon}} H_\varphi = \sqrt{\frac{\mu}{\varepsilon}} \frac{e^{ikr}}{ikr} W(\theta), \qquad (9.6.11)$$

here

$$W(\theta) = \sum_{s=\pm 1}' e^{-iskl_0 \cos\theta} \sum_{n=1}^{\infty} (-i)^n [(\xi_s^{02} - 1)^{1/2} R_{1n}^{(1)}(c_s, \xi_s^0)]' \times \times A_n^s S_{1n}(c_s, \cos\theta), \quad l_0 = l/2. \qquad (9.6.12)$$

he coefficient of the directional effect of the linear system of wo coaxial spheroidal emitters, which are independent in this case f the azimuth angle ϕ, may be computed according to formula (1.4.18). n the particular case of two identical spheroidal emitters ($g(\theta,\phi) = (\theta)$), from (1.4.18)

$$g(\theta) = \frac{|W(\theta)|^2}{W_0}, \qquad (9.6.13)$$

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where $W(\theta)$ is determined by formula (9.6.12) and

$$W_0 = \frac{1}{2}\sum_{n=1}^{\infty}\left|\left[(\xi^{02}-1)^{1/2}\, R_{1n}^{(1)}(c, \xi^0)\right]'\right|^2\left(|A_n^{+1}|^2 + |A_n^{-1}|^2\right) N_{1n}(c) + \\ + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} i^{n-m} A_n^{-1} A_m^{+1*} \tau_{nm} + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} i^{n-m} A_n^{+1} A_m^{-1*} \sigma_{nm}, \tag{9.6.14}$$

where in turn $c = c_s$, $\xi^0 = \xi_s^0$, $s = \pm 1$ and

$$\tau_{nm} = \left[(\xi^{02}-1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' \left[(\xi^{02}-1)^{1/2} R_{1m}^{(1)}(c, \xi^0)\right]' \times \\ \times \sum_{r=0,1}^{\infty}{}' \sum_{t=0,1}^{\infty}{}' d_r^{1n}(c)\, d_t^{1m}(c) \sum_{\sigma=|r-t|}^{r+t+2} i^{-\sigma} b_\sigma^{(1,r+1,1,t+1)} j_\sigma(kl),$$

$$\sigma_{nm} = \left[(\xi^{02}-1)^{1/2} R_{1n}^{(1)}(c, \xi_0)\right]' \left[(\xi^{02}-1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' \times \\ \times \sum_{r=0,1}^{\infty}{}' \sum_{t=0,1}^{\infty}{}' d_r^{1n}(c)\, d_t^{1m}(c) \sum_{\sigma=|r-t|}^{r+t+2} i^{\sigma} b_\sigma^{(1,r+1,1,t+1)} j_\sigma(kl). \tag{9.6.15}$$

On the basis of the addition theorem (9.5.9) and (9.6.8) - (9.6.10) for the function H_ϕ (9.6.4) we find when $l > f_s \xi_s$ the expression

$$H_\varphi = \sum_{n=1}^{\infty} A_n^s \left\{\left[(\xi_s^{02}-1)^{1/2} R_{1n}^{(1)}(c_s, \xi_s^0)\right]' R_{1n}^{(3)}(c_s, \xi_s) - \right. \\ \left. - R_{1n}^{(1)}(c_s, \xi_s)\left[(\xi_s^{02}-1)^{1/2} R_{1n}^{(3)}(c_s, \xi_s^0)\right]'\right\} \times \\ \times S_{1n}(c_s, \eta_s) + \sum_{n=1}^{\infty} \frac{\alpha_n(s) R_{1n}^{(1)}(c_s, \xi_s)}{\left[(\xi_s^{02}-1)^{1/2} R_{1n}^{(1)}(c_s, \xi_s^0)\right]'} S_{1n}(c_s, \eta_s), \tag{9.6.16}$$

where $\alpha_\eta(s)$ is determined by formula (9.6.6). Using (9.6.6) we can study the diffraction field near the emitters (in the neighboring band). Formula (9.6.16) is written in local coordinates of the s-th emitter, $s = \pm 1$.

From (1.1.7) and (9.6.16) we find the law of the density distribution of surface currents excited on the surface of the s-th emitter /5

$$j_\eta^s = -\frac{c_0}{4\pi}\left\{\sum_{n=1}^{\infty} \frac{A_n^s\, i}{c\,(\xi_s^{02}-1)^{1/2}} S_{1n}(c_s, \eta_s) + \right. \\ \left. + \sum_{n=1}^{\infty} \frac{\alpha_n(s) R_{1n}^{(1)}(c_s, \xi_s^0) S_{1n}(c_s, \eta_s)}{\left[(\xi_s^{02}-1)^{1/2} R_{1n}^{(1)}(c_s, \xi_s^0)\right]'}\right\}. \tag{9.6.17}$$

With a numerical solution to the problem, the approximate values of the quantities A_n^s are found from a truncated system such as (9.5.20) if we apply to (9.6.8) the definitions in (9.5.14). In the general case, in seeking approximate values of A_n^s, $s = \pm 1$, it is fitting to solve the finite systems of $2N$ equations with $2N$ unknowns, A_n^s, $s = \pm 1$, $n = 1, 2, \ldots, N$. In the particular case of two identical emitters and a symmetric position of the gap relative to point O we can reduce the order of the solvable finite system by half and still retain the same degree of computational precision. Let $c_{-1} = c_{+1} = c$, $\xi_{-1}^0 = \xi_{+1}^0 = \xi^0$. Then, just as in the previous problem

$$\alpha_{nm} = \alpha_{nm}^{(-1,+1)} = (-1)^{n+m} \alpha_{nm}^{(+1,-1)}. \tag{9.6.18}$$

Furthermore, let the values η_{-1}^0 and η_{+1}^0, by which we determine the positions of the gaps on the surfaces of the emitters, be such that

$$\eta_s^0 = -\eta_{-s}^0, \quad s = \pm 1. \tag{9.6.19}$$

Then in (9.6.10) $S_{1n}(c, \eta_{-s}^0) = (-1)^{n+1} S_{1n}(c, \eta_s^0)$ and if the ratio of voltages in the gaps of the emitters is equal to

$$V_s = e^{i\alpha} V_{-s}, \quad s = \pm 1, \ \alpha = 0; \ \pi, \tag{9.6.20}$$

where α is the phase difference of supply currents from the emitters the right-hand sides of system (9.6.8) will be mutually related by the relationship

$$f_n^{-s} = (-1)^{n+1} e^{-i\alpha} f_n^s, \ s = \pm 1. \tag{9.6.21}$$

In this case the unknowns A_n^s, A_n^{-s} of system (9.6.8) are found to be proportional quantities, and

$$A_n = A_n^{+1} = (-1)^{n+1} e^{i\alpha} A_n^{-1}. \tag{9.6.22}$$

Using (9.6.22) from system (9.6.8) we can exclude one series of unknowns, for example A_n^{-1}, $n = 1, 2, \ldots$, by reducing it to a system relative to the unknowns A_n. As a result, seeking approximate values of A_n leads to the truncated system

$$A_n - \sum_{m=1}^{N} (-1)^m \alpha_{nm} e^{i\alpha} A_m = f_n, \quad f_n = f_n^{+1}, \tag{9.6.23}$$

consisting of N equations with N unknowns A_n^{+1}, $n = 1, 2, \ldots, N$. /50
In this case the function $W(\theta)$ will be given by the expression

$$W(\theta) = \sum_{n=1}^{\infty} (-i)^n \left[(\xi^{02} - 1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' \times \times A_n \beta_n(\alpha, \theta) S_{1n}(c, \cos\theta), \tag{9.6.24}$$

where

$$\beta_n(\alpha, \theta) = e^{-ikl_0 \cos\theta} - (-1)^n e^{-i\alpha + ikl_0 \cos\theta}. \tag{9.6.25}$$

If the emitters are supplied by cophasal currents, when $\alpha = 0$, then

$$\beta_n(0, \theta) = \begin{cases} 2\cos(kl_0 \cos\theta), & n \text{ is odd}, \\ -2i \sin(kl_0 \cos\theta), & n \text{ is even}, \end{cases} \tag{9.6.26}$$

and then

$$W(\theta) = -2\Big\{\cos(kl_0 \cos\theta) \sum_{n=1,3,\ldots} i^n \left[(\xi^{02} - 1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' \times \times A_n S_{1n}(c, \cos\theta) + i \sin(kl_0 \cos\theta) \times \times \sum_{n=2,4,\ldots} i^n \left[(\xi^{02} - 1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' A_n S_{1n}(c, \cos\theta)\Big\}, \tag{9.6.27}$$

but in the equatorial plane $\theta = \pi/2$

$$W\left(\frac{\pi}{2}\right) = -2 \sum_{n=1,3,\ldots} i^n \left[(\xi^{02} - 1)^{1/2} \times \times R_{1n}^{(1)}(c, \xi)\right]' A_n S_{1n}(c, 0). \tag{9.6.28}$$

In the case of an antiphase supply from the emitters when $\alpha = \pi$,

$$\beta_n(\pi, \theta) = \begin{cases} 2\cos(kl_0 \cos\theta), & n \text{ is even}, \\ -2i \sin(kl_0 \cos\theta), & n \text{ is odd} \end{cases} \tag{9.6.29}$$

and then

$$W(\theta) = 2\Big\{\cos(kl_0 \cos\theta) \sum_{n=2,4,\ldots} i^n \left[(\xi^{02} - 1)^{1/2} R_{1n}^{(1)}(c, \xi^0)\right]' \times \times A_n S_{1n}(c, \cos\theta) + i \sin(kl_0 \cos\theta) \sum_{n=1,3,\ldots} i^n \left[(\xi^{02} - 1)^{1/2} \times \times R_{1n}^{(1)}(c, \xi^0)\right]' A_n S_{1n}(c, \cos\theta)\Big\}, \tag{9.6.30}$$

and when $\theta = \pi/2$

$$W\left(\frac{\pi}{2}\right) = 0$$

that each point of the equatorial plane in its wave band a damping of the diffraction waves occurs for all orders of scattering, arriving from the points of the surfaces of the -1 and the +1 emitters which are symmetrically located relative to the plane $\theta = \pi/2$). /509

In the case when $\eta_s^0 = 0$, $s = \pm 1$ (the gaps are arranged in local equatorial planes),

$$S_{1n}(c, \eta_s^0) = S_{1n}(c, 0) = \begin{cases} 0, & n \text{ is even}, \\ \dfrac{(-1)^{\frac{n-1}{2}}(n+1)!}{2^n\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!}, & n \text{ is odd}, \end{cases}$$

and then $f_n^s = 0$ for even values of n for any α, $s = \pm 1$. If $c_s = c$ $\xi_s^0 = \xi^0$, $\eta_s^0 = 0$, $s = \pm 1$ and furthermore $c << 1$, then the function H_ϕ will reach the dominant value because of the first two terms in (9.6.4) with summation indices $n = 1$. In this case with cophasal supply from the emitters in the wave band in the spherical coordinates r, θ, ϕ with the origin at point O

$$H_\varphi = A\frac{e^{ikr}}{r}\sin\theta\cos(kl_0\cos\theta), \qquad (9.6.31)$$

and with antiphase supply

$$H_\varphi = B\frac{e^{ikr}}{r}\cdot\sin\theta\sin(kl_0\cos\theta), \qquad (9.6.32)$$

where A and B are certain constants. The expressions (just as in the case of two spherical emitters in a similar situation) are identical to the expression for the field of a linear system of two unit dipoles (symmetric vibrator) with respective values of their moments and located at a distance l from one another.

In the general case when the distance between the emitters l is much greater than the wavelength and the cross section dimensions of the emitters, we will have approximately $A_n^s = f_n^s$, $s = \pm 1$.

If $c_s = c$, $\xi_s^0 = \xi^0$, $s = \pm 1$, but $\eta^0{}_1 \neq \eta^0{}_1$, then by using (9.6.18) the solution to a finite system of equations such as (9.5.20)

of $2N$ equations with $2N$ unknowns A_n^{+1} A_n^{-1}, $n = 1, 2, \ldots, N$, we can substitute (and still retain the same degree of computational accurary) the solution to two independent systems of equations such as (9.5.23) - (9.5.24) each of which is a system of N equation with N unknown.

With a conversion in formulas (9.6.3) - (9.6.32) from the coordinate of the prolate spheroid ξ_n, η_s, ϕ to spherical coordinates r_s, θs, ϕ, $s = \pm 1$, from them we find the solution to the problem studied in §5, Chapter 7. /510

After subsituting the coordinates of the prolate spheroid in formulas (9.6.3) - (9.6.32) with the coordinates of the oblate spheroid, on the basis of (9.2.16) we find the solution to the problem of the field of two prolate coaxial spheroidal emitters arranged at a distance l from one another with a given voltage distribution in the gap. By virtue of the independence of the local coordinate systems from one another we can, by retaining the shape of one of the emitters constant and varying the shape of the other emitter, obtain from (9.6.3) - (9.6.32) with the proper choices of coordinate systems the solution to the problem on the field of a linear system of two coaxial emitters with surfaces of different shapes. If the supply is carried out only to one of the emitters, for example, to the s-th ($s = -1$ or $s = +1$) but not to the $-s$-th (in this case the $-s$-th spheroid plays the role of a passive antenna) and if the $-s$-th body is an infinitely thin ideally conducting circular disc of radius f_{-s}, then after conversion in the $-s$-th local coordinate from the coordinate of the prolate spheroid (with a focal distance f_{-s}) to coordinates of the oblate spheroid in which the surface of the disc is determined by the value $\xi_{-s} = \xi_{-s}^0 = 0$, from (9.6.3) - (9.6.17) we find the solution to the problem on the field of a spheroidal antenna in the presence of a passive reflector that is coaxial with the antenna, i.e., an ideally conducting circular disc of radius f_{-s}. For such a case in system (9.6.8) we will have $f_n^{-s} = 0$ ($s = -1$ or $s = +1$) but

$$W(\theta) = e^{-iskl_0\cos\theta}\sum_{n=1}^{\infty}(-i)^n\left[(\xi_s^{02}-1)^{1/2}R_{1n}^{(1)}(c_s,\ \xi_s^0)\right]'\times$$
$$\times A_n^s S_{1n}(c_s,\ \cos\theta) + e^{iskl_0\cos\theta}\sum_{n=2,4,\ldots} i^n R_{1n}^{(1)'}(-ic_{-s},\ i0)\times \qquad (9.6.33)$$
$$\times A_n^{-s} S_{1n}(-ic_{-s},\ \cos\theta).$$

In the equatorial plane when $\theta = \pi/2$

$$W\left(\frac{\pi}{2}\right) = -\sum_{n=1,3,\ldots} i^n\left[(\xi_s^{02}-1)^{1/2}R_{1n}^{(1)}(c_s,\ \xi_s^0)\right]\times \qquad (9.6.34)$$
$$\times A_n^s S_{1n}(s_s,\ 0)$$

$s = +1$ or $s = -1$).

With values of ξ_s^0 near 1, the s-th spheroidal emitter may be ssumed as a thin wire antenna of length $2f_s$ excited at the point $s = f_s\ \xi_s^0\ \eta_s^0$.

§7. The Field of a Magnetic (Electric) Dipole, Arranged Coaxially to Two Disks

In an unbounded uniform and isotropic space with electromag- /511
etic constants ε, μ, σ ($\sigma = 0$) at a distance l from one another, here are located ideally conducting infinitely thin circular disks f radii $\rho_{-1} = a_{-1}$ and $\rho_{+1} - a_{+1}$ (in local cylindrical coordinates s, ϕ_s, z_s with origins at the center of each disk and with axes sz_s, directed along the axis of rotation (axis of symmetry) that s common for the disk; the numbers $s = -1$ and $s = +1$ are attributed o the disk. In the general case $a_{-1} \neq a_{+1}$. The disks are excited y the unit electric or magnetic dipole located at point P on the xis of rotation of the disk and directed such that its dipole oment (p or m) forms the angle α with the Oz axis (Fig. 63). The oint P with the dipole may be found outside the gap between the isks, and the gap between the disks, and also on the surface of ne of the disks at its center. We posed the problem on seeking he diffraction field produced by the presence of disks in the ipole field. In the case when the dipole is found on the surface f one of the disks this will be a problem that is equivalent in he electrodynamic sense to the problem on the field of emission rom a unit self-contained slot located on the surface of one of he disks (active antenna) in the presence of another disk that is oaxial to the first and which plays the role of the reflector passive antenna). The local coordinate systems are introduced ich that the dipole moment (p or m) lies in the plane $O_s x_s z_s$, $s = 1$, and is directed toward the positive values of x_s, $\phi_s = \phi$, $s = 1$.

The problems are solved in local coordinates of the oblate pheroid ξ_s, η_s, ϕ_s ($\phi_s = \phi$) with a focal distance $f_s = a_s$, in which ie surfaces of the disk are given by the equation $\xi_s = 0$, $s = \pm 1$, id the position of point P with the dipole is given by the values $= \xi_{s0}$, $\eta_s = \eta_{s0}$, $s = \pm 1$.

Magnetic Dipole. The field of the magnetic dipole in free pace is determined by the magnetic Hertz vector (9.3.1) where $\mathrm{m} = \{|\mathrm{m}| \sin\alpha, 0, |\mathrm{m}| \cos\alpha\}$. It is obvious that if the Hertz vector ^{*0} of the primary dipole field is represented in the form of a sum two vectors

$$\vec{\Pi}^{*0} = \vec{\Pi}_1^{*0} + \vec{\Pi}_2^{*0},$$

where

$$\vec{\Pi}_1^{*0} = \{0,\ 0,\ m_0 \cos\alpha\, e^{ikR}/R\},$$

$$\vec{\Pi}_2^{*0} = \{m_0 \sin\alpha\, e^{ikR}/R,\ 0,\ 0\},\ m_0 = |\mathbf{m}|, \qquad (9.7.1)$$

then the secondary electromagnetic field E^1, H^1 may be assumed as the result of the superposition of two particular fields, one of which corresponds to the case of excitation of the disk by the vertical component of the magnetic dipole and the second to the case of excitation of the disk by the horizontal component of the dipole. In both cases the fields are found through the respective magnetic Hertz vector from the relation in (9.3.4). Solution to the problem for the general case of an arbitrarily oriented dipole may be found from the solution to two particular problems: (1) problem on the diffraction of the field of a vertical magnetic dipole on two disks and (2) the problem on diffraction of the field of the horizontal magnetic dipole on two disks. /51

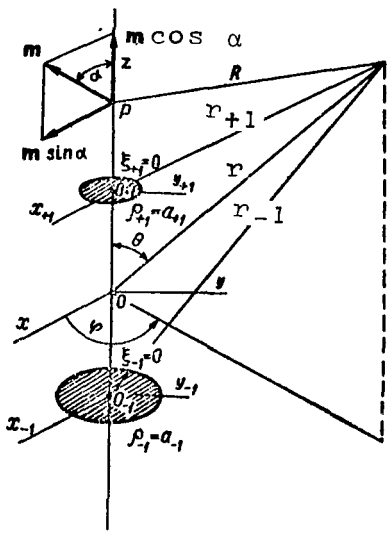

Fig. 63. Excitation of Two Coaxial Circular Disks by a Dipole.

The problem for the vertical magnetic dipole is solved according to the scheme in §5 of this chapter. The unknown function is the only non-zero azimuth component E_ϕ^1 of the vector E^1 of the secondary field. The components of the complete magnetic field H are found through $E_\phi = E_\phi^0 + E_\phi^1$ according to the formulas

$$H_\xi = \frac{-i}{k_0 \mu f (\xi^2+\eta^2)^{1/2}} \cdot \frac{\partial}{\partial \eta} (\sqrt{1-\eta^2}\, E_\varphi),$$

$$H_\eta = \frac{i}{k_0 \mu f (\xi^2+\eta^2)^{1/2}} \cdot \frac{\partial}{\partial \xi} (\sqrt{\xi^2+1}\, E_\varphi), \qquad (9.7.2)$$

where the component E_ϕ^0 of the primary dipole field in local coordinates of the s-th sphere may be written in the form of a series /51

$$E_\varphi^0 = -\frac{4k^2 m_0 \cos\alpha}{f_s (\xi_{s0}^2+1)^{1/2}} \sum_{n=1}^{\infty} \varepsilon_{ns} \frac{\sigma_{1n}(-ic_s)}{N_{1n}(-ic_s)} S_{1n}(-ic_s,\ \dot{\eta}_s) \times$$

$$\times \begin{cases} R_{1n}^{(1)}(-ic_s,\ i\xi_{s0})\, R_{1n}^{(3)}(-ic_s,\ i\xi_s), & \xi_s > \xi_{s0}, \\ R_{1n}^{(1)}(-ic_s,\ i\xi_s)\, R_{1n}^{(3)}(-ic_s,\ i\xi_{s0}), & \xi_s < \xi_{s0}, \quad s = \pm 1. \end{cases} \qquad (9.7.3)$$

Here $\varepsilon_{ns} = (-s)^{n+1}$ if the dipole is found in the gap between the disks; $\varepsilon_{ns} = 1$ if the dipole is found "over" the disk and $\varepsilon_{ns} = (-1)^{n+1}$ if the dipole is found outside the gap "under" the disk, $s = \pm 1$. The unknown function E^1_ϕ must be the solution to the Helmholtz equation (9.2.5) with the azimuth number $m = 1$, which satisfies (on the surface of each disk) the boundary condition

$$E^0_\varphi + E^1_\varphi = 0, \quad \xi_s = \xi^0_s = 0, \quad s = \pm 1. \tag{9.7.4}$$

At infinity E^1_ϕ must satisfy the condition of radiation. Analogous conditions will also be satisfied by the function E^1_ϕ in the case when instead of the disks we look at oblate spheroids $\xi_s = \xi^0_s \neq 0$, $s = \pm 1$. Therefore we can first solve the problem for the spheroids and then from the solution to this problem by a limited conversion over ξ^0_s when $\xi^0_s \to 0$ we obtain the solution to the problem for the disk.

If we seek the function E^1_ϕ in the form

$$E^1_\varphi = k^2 m_1 \sum_{s=\pm 1} E_s, \quad m_1 = m_0 \cos\alpha, \tag{9.7.5}$$

where

$$E_s = \sum_{n=1}^{\infty} a^s_n R^{(3)}_{1n}(-ic_s, i\,\xi_s) S_{1n}(-ic_s, \eta_s), \quad s = \pm 1, \tag{9.7.6}$$

then by using the addition theorem (9.5.9), written in the coordinates of the oblate spheroid after substitution of the unknowns A^s_n by the new unknowns a^s_n according to the formula,

$$a^s_n = R^{(1)}_{1n}(-ic_s, i\,\xi^0_s) A^s_n, \quad s = \pm 1, \tag{9.7.7}$$

from the boundary conditions from E_ϕ we obtain an infinite system of linear equations

$$A^s_n + \sum_{m=1}^{\infty} \alpha^{(-s,s)}_{nm} A^{-s}_m = f^s_n, \quad s = \pm 1, \quad n = 1, 2, \ldots, \tag{9.7.8}$$

with the matrix elements /514

$$\alpha^{(-s,s)}_{nm} = \frac{R^{(1)}_{1m}(-ic_{-s}, i\,\xi^0_{-s})}{R^{(3)}_{1n}(-ic_s, i\,\xi^0_s)} Q_{1n1m}(c_s, c_{-s}; l, \theta_{-s,s}) \tag{9.7.9}$$

and with the free terms

$$f_n^s = \frac{4\sigma_{1n}(-ic_s)\,R_{1n}^{(3)}(-ic_s,\ i\xi_{s0})\,\varepsilon_{n_s}}{f_s(\xi_{s0}^2+1)^{1/2}\,N_{1n}(-ic_s)\,R_{1n}^{(3)}(-ic_s,\ i\xi_s^0)}. \qquad (9.7.10)$$

The properties of system (9.7.8) do not differ from the properties of system (9.6.8) or system (9.5.11). Its determinant may be written in the same form as system (9.5.11) and the system itself after the introduction of the designation in (9.5.14) may be reduced to normal form (9.5.15). The only solution to (9.7.8) which satisfies the condition $\sum_{n=1}^{\infty}|A_n^s|^2<\infty$, $s=\pm 1$, may be found by the method of truncation.

If we convert in formulas (9.7.7) - (9.7.10) to the limit over ξ_s^0 and ξ_s^0 tends to zero, then from them we obtain the solution to the problem for the disk. Since the function $R_{1n}^{(1)}(-ic_s,\ i0)$ is nonzero only for even values $(n-1)$ and is equal to zero for odd values $(n-1)$, then when $\xi_s^0=0$, $s=\pm 1$, $A_n^s=0$, $n=2,4,\ldots$ In this case summation in (9.7.6) is carried out in fact only over odd numbers of n.

Using the addition theorem (9.5.9) applied to (9.7.6) and formulas (9.7.7) - (9.7.10) for the electric field near the disk (neighboring band) we obtain the expression (in coordinates of the s-th disk)

$$E_\varphi = k^2 m_1 \sum_{m=1,3,\ldots} A_n^s\{R_{1n}^{(3)}(-ic_s,\ i\xi_s)\,R_{1n}^{(1)}(-ic_s,\ i0)- \\ -R_{1n}^{(1)}(-ic_s,\ i\xi_s)\,R_{1n}^{(3)}(-ic_s,\ i0)\}S_{1n}(-ic_s,\eta_s),\ s=\pm 1. \qquad (9.7.11)$$

The density distribution of surface currents induced on the disk by the field of the vertical magnetic dipole may be studied by using the formula

$$j_\varphi^s = \frac{-c_0\sqrt{\frac{\varepsilon}{\mu}}\,k^2 m_1}{4\pi c_s^2\,\eta_s}\sum_{n=1,3,\ldots} A_n^s S_{1n}(-ic_s,\eta_s),\ s=\pm 1, \qquad (9.7.12)$$
$$j_\xi^s = j_\eta^s = 0.$$

From (9.7.12) it is obvious that the solution obtained to the problem automatically satisfies the Meixner condition at the edge of the disk ($\xi_s = \eta_s = 0$).

In the vicinity of the wave band in spherical coordinates r, θ, ϕ with the origin at O lying midway on the line of the disk /515

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}} H_\eta = k^2 m_0 \cos\alpha \sin\theta \frac{e^{ikR}}{R} W(\theta), \tag{9.7.13}$$

where the damping coefficient $W(\theta)$ is determined by the expression

$$W(\theta) = 1 - \frac{e^{sikz_{s0}\cos\theta}}{ik\sin\theta} \sum_{s=\pm 1} e^{-iskl_0\cos\theta} \sum_{n=1,3,\ldots} i^n A_n^s \times \times R_{1n}^{(1)}(-ic_s, i0) S_{1n}(-ic_s, \cos\theta) \tag{9.7.14}$$

($R = r - z_{s0}\cos\theta$).

In the case of two equal disks and $\xi_{-1,0} = \xi_{+1,0}$ (the dipole is located midway in the dap between the disks) we will have $z_{s0} = 0$ and

$$\alpha_{nm}^{(-1,+1)} = \alpha_{nm}^{(+1,-1)}, \tag{9.7.15}$$

$$f_n^{+1} = f_n^{-1}. \tag{9.7.16}$$

Therefore, in system (9.7.11)

$$A_n^{-1} = A_n^{+1} = A_n. \tag{9.7.17}$$

For such a case the damping coefficient is

$$W(\theta) = 1 - \frac{2\cos(kl_0\cos\theta)}{k\sin\theta} \sum_{n=1,3,\ldots} i^n A_n \times \times R_{1n}^{(1)}(-ic, i0) S_{1n}(-ic, \cos\theta), \tag{9.7.18}$$

and in the equatorial plane when $\theta = \pi/2$

$$W\left(\frac{\pi}{2}\right) = 1 - \frac{2}{k} \sum_{n=1,3,\ldots} i^n A_n R_{1n}^{(1)}(-ic, i0) S_{1n}(-ic, 0). \tag{9.7.19}$$

With a numerical solution to the problem the approximate values of the quantities are found according to the same scheme as in the previous problem.

In the case of a horizontal magnetic dipole the Hertz vector of the complete field is sought in the form of (9.3.5) where just as before $\Phi^1(\xi, \eta)$ is the potential of the primary field and $\Phi^1(\xi, \eta)$

and $\Psi(\xi, \eta)$ are the potentials of secondary field. In local coordinates of the s-th disk

$$\Phi^0(\xi_s, \eta_s) = 2ikm_2 \sum_{n=0}^{\infty} \frac{S_{0n}(-ic_s, \eta_{s0})}{N_{0n}(-c_s)} \times$$

$$\times \begin{Bmatrix} R_{0n}^{(3)}(-ic_s, i\xi_{s0}) R_{0n}^{(1)}(-ic_s, i\xi_s) \\ R_{0n}^{(3)}(-ic_s, i\xi_s) R_{0n}^{(1)}(-ic_s, i\xi_{s0}) \end{Bmatrix} \times \qquad (9.7.20)$$

$$\times S_{0n}(-ic_s, \eta_s), \quad \begin{matrix} \xi_{s0} > \xi_s, \\ \xi_{s0} < \xi_s, \end{matrix}$$

$$m_2 = m_0 \sin \alpha.$$

Let us set /516

$$\Phi^1(\xi, \eta) = 2ikm_2 \sum_{s=\pm 1} \Phi_s(\xi_s, \eta_s), \quad \Psi(\xi, \eta) = \qquad (9.7.21)$$

$$= 2ikm_2 \sum_{s=\pm 1} \Psi_s(\xi_s, \eta_s),$$

where

$$\Phi_s(\xi_s, \eta_s) = \sum_{n=0}^{\infty} a_n^s R_{0n}^{(3)}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_s). \qquad (9.7.22)$$

$$\Psi(\xi_s, \eta_s) = \sum_{n=1}^{\infty} b_n^s R_{1n}^{(3)}(-ic_s, i\xi_s) S_{1n}(-ic_s, \eta_s). \qquad (9.7.23)$$

The functions $\Phi(\xi, \eta)$ and $\Psi(\xi, \eta)$ on the surface of each disk must satisfy the boundary conditions (9.3.31)

$$\frac{\partial \Phi}{\partial \xi_s} = \eta_s C_s, \qquad \xi_s = 0, \quad s = \pm 1, \qquad (9.7.24)$$

$$\Psi = (1 - \eta_s^2)^{1/2} C_s,$$

with the integration constants $C_{\pm 1}$. These constants will be determined in the future either from the Neixner conditions at the edges of the disk (if the dipole does not lie on the surface of one of the disks)

$$j_\rho^s = \frac{c_0}{4\pi} \left\{ H_{\varphi|_{\substack{\xi_s=0 \\ \eta_s=-0}}} - H_{\varphi|_{\substack{\xi_s=0 \\ \eta_s=+0}}} \right\} = 0, \quad s = \pm 1, \qquad (9.7.25)$$

where j_ρ^s is the radial component of the density vector of the complete surface current induced on the s-th disk, $s = \pm 1$, or from (9.7.25) and the reciprocity theorem for two magnetic dipoles (1.3.14) in the case when the dipole is located on the surface of one of the disks.

Through the potential $\Phi(\xi, \eta)$ and $\Psi(\xi, \eta)$ the electromagnetic ield is found from the formulas (9.3.18), (9.3.19) written in one f the coordinate systems.

If we introduce the substitution of unknowns a_n^s, b_n^s from 9.7.22), (9.7.23) by the new unknowns A_n^s and B_n^s after setting

$$a_n^s = nR_{0n}^{(1)'}(-ic_s, i0)\, A_n^s, \quad n = 1, 3, \ldots,$$
$$b_n^s = R_{1n}^{(1)}(-ic_s, i0)\, B_n^s, \quad n = 1, 3, \ldots, \tag{9.7.26}$$

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hen from (9.7.24) after applying to (9.7.22), (9.7.23) the addition heorem (5.5.18) for determination of the unknowns A_n^s, B_n^s we find he infinite system of linear equations

$$A_n^s + \sum_{m=1,3,\ldots} \alpha_{nm}^{(-s,s)} A_m^{-s} = f_n^s, \tag{9.7.27}$$
$$B_n^s + \sum_{m=1,3,\ldots} \beta_{nm}^{(-s,s)} B_m^{-s} = \varphi_n^s \tag{9.7.28}$$

ith the matrix elements

$$\alpha_{nm}^{(-s,s)} = \frac{mR_{0m}^{(1)'}(-ic_{-s}, i0)}{nR_{0n}^{(3)'}(-ic_s, i0)} Q_{0n0m}(c_s, c_{-s}; l, \theta_{-s,s}), \tag{9.7.29}$$

$$\beta_{nm}^{(-s,s)} = \frac{R_{1m}^{(1)}(-ic_{-s}, i0)}{R_{1n}^{(3)}(-ic_s, i0)} Q_{1n1m}(c_s, c_{-s}; l, \theta_{-s,s}) \tag{9.7.30}$$

nd with the free terms

$$f_n^s = -\frac{S_{0n}(-ic_s, \eta_{s0})\, R_{0n}^{(3)}(-ic_s, i\xi_{s0})}{nN_{0n}(-ic_s)\, R_{0n}^{(3)'}(-ic_s, i0)} + \frac{C_s d_1^{0n}(-ic_s)}{3ikm_2 nN_{0n}(-ic_s)\, R_{0n}^{(3)'}(-ic_s, i0)\, R_{0n}^{(1)'}(-ic_s, i0)}, \tag{9.7.31}$$

$$\varphi_n^s = \frac{2d_0^{1n}(-ic_s) C_s}{3ikm_2 N_{1n}(-ic_s)\, R_{1n}^{(1)}(-ic_s, i0)\, R_{1n}^{(3)}(-ic_s\, i0)}. \tag{9.7.32}$$

ubstitution of the unknowns according to the formulas in (9.7.26) hen n assumes only odd values it is possible on the strength of he fact that for even numbers of n $a_n^s = b_n^s = 0$, $s = \pm 1$, [to estalish this fact it is sufficient to look at the system obtained rom (9.7.24) for a_n^s and b_n^s without substituting (9.7.26)]. Just

as in all previous cases here through all $R'_{mn}(-ic, i0)$ we denote value of the derivative $\frac{d}{d\xi}[R_{mn}(-ic, i\xi)]$ when $\xi = 0$. The expression $S'_{mn}(-ic, \eta)$ has the same sense.

The determinants of systems (9.7.27) and (9.7.28) may be written in the same form as system (9.5.11). The systems (9.7.27), (9.7.28) themselves after the introduction of the designations in (9.5.14) for their matrix elements and right-hand sides assume normal from (9.5.15). Just as in the previous problems systems (9.7.27) and (9.7.28) are uniquely solvable [86]. Their solutions, which satisfy the condition $\sum_{n=1,3,\ldots} |A_n^s|^2, |B_n^s|^2 < \infty$, may be found by the method of truncation [86-88].

For complete determination of the unknown quantities A_n^s and B_n^s it is necessary to find the constant $C_{\pm 1}$. We shall assume first that $\xi_{s0} \neq 0$ when $s = -1$ or $s = +1$ (the dipole does not lie on the surface of one of the disks). Then using the addition theorem (5.5.18) and formulas (9.3.19), (9.7.21) - (9.7.23) after arguments which are similar to those given in deriving (9.3.35), from (9.7.25) we obtain the relationship /51

$$\sum_{n=1,3,\ldots} nA_n^s R_{0n}^{(1)'}(-ic_s, i0) R_{0n}^{(3)}(-ic_s, i0) S_{0n}(-ic_s, 0) =$$
$$= \sum_{n=1,3,\ldots} B_n^s R_{1n}^{(1)}(-ic_s, i0) R_{1n}^{(3)'}(-ic_s, i0) S_{1n}(-ic_s, 0)$$

or

$$\sum_{n=1,3,\ldots} nA_n^s S'_{0n}(-ic_s, 0) = -\sum_{n=1,3,\ldots} B_n^s S_{1n}(-ic_s, 0), \qquad (9.7.33)$$

if we use the value of the Wronskian determinant for the radial spheroidal functions. After substituting into (9.7.33) the coefficients A_n^s and B_n^s expressed through $C_{\pm 1}$, from systems (9.7.27) and (9.7.28) we obtain a linear system of two equations with two unknowns $C_{\pm 1}$. It is obvious that the solutions to systems (9.7.27) and (9.7.28) will have the form

$$A_n^s = \alpha_n^s + \beta_n^s C_s + \gamma_n^s C_{-s},$$
$$B_n^s = \delta_n^s C_s + \tau_n^s C_{-s}, \qquad (9.7.34)$$

where α_n^s, β_n^s, γ_n^s, δ_n^s, τ_n^s are quantities which depend only on c_s and l, n, s, but are independent of $C_{\pm 1}$. Therefore after substituting (9.7.34) into (9.7.33) we obtain

$$\alpha_s C_s + \beta_s C_{-s} = \gamma_s, \quad s = \pm 1, \tag{9.7.35}$$

where

$$\begin{aligned}
\alpha_s &= \sum_{n=1,3,\ldots} n\,\beta_n^s S_{0n}'(-ic_s, 0) + \sum_{n=1,3,\ldots} \delta_n^s S_{1n}(-ic_s, 0),\\
\beta_s &= \sum_{n=1,3,\ldots} n\,\gamma_n^s S_{0n}'(-ic_s, 0) + \sum_{n=1,3,\ldots} \tau_n^s S_{1n}(-ic_s, 0),\\
\gamma_s &= -\sum_{n=1,3,\ldots} n\,\alpha_n^s S_{0n}'(-ic_s, 0).
\end{aligned} \tag{9.7.36}$$

Solving system (9.7.35) relative to C_s, we will have

$$c_{-s} = \frac{\alpha_s\gamma_{-s} - \beta_{-s}\gamma_s}{\alpha_s\alpha_{-s} - \beta_s\beta_{-s}}, \quad s = \pm 1. \tag{9.7.37}$$

The series which appear in (9.7.37) converge for all values of $\xi_{s0} \neq 0$, $s = \pm 1$. By determining the integration constants $C_{\pm 1}$, and through them the quantities A_n^s, B_n^s also from the formulas in (9.7.34) the problem for the potentials $\Phi^1(\xi, \eta)$, $\Psi(\xi, \eta)$ of the secondary field is completely solved. /519

In the case when the dipole is found on the surface of one of the disks, for example on the surface of the s-th disk, where $s = +1$ or only $s = -1$, the coordinate ξ_{s0} with the same value of s will vanish. Here in (9.7.36) the series which correspond to the number s, attributed to the disk with the dipole on it, are found to be divergent. For this reason in system (9.7.35), only one equation is suitable for seeking the constants $C_{\pm 1}$,

$$\alpha_{-s} C_{-s} + \beta_{-s} C_{+s} = \gamma_{-s}, \tag{9.7.38}$$

which corresponds to the number $-s$ attributed to the disk which is free of the dipole. The second equation relative to C_{-1} and C_{+1} is obtained from the reciprocity theorem (1.3.14) for the magnetic dipole. In order to use it let us introduce into study the second magnetic dipole with the moment m', assuming it to be located at point O' on the axis Oz such that $\mathrm{m}' = \mathrm{m}$. Then

$$\mathbf{m}'\mathbf{H}(O') = \mathbf{m}\mathbf{H}(O_s), \tag{9.7.39}$$

where m, just as before, is the moment of the given dipole located at the center of the s-th disk O_s ($s = +1$ or $s = -1$). Carrying the point O' along the axis Oz to infinity (and along with the dipole with the moment m' as well), we find on the basis of (9.3.20) that

$$\Phi|_{\eta_s=1,\ \xi_s=\infty,\ s=\pm 1} = \Phi^{\mathrm{pl}}_{|\eta_s=1,\ \xi_s=0,\ s=-1 \text{ or } s=+1}, \tag{9.7.40}$$

where through Φ^{pl} we denote the potential of the field, which arises in the problem of diffraction of a plane electromagnetic wave

$$E^0_{y_s} = \sqrt{\frac{\mu}{\varepsilon}} H^0_{x_s} = \sqrt{\frac{\mu}{\varepsilon}} k^2 \Pi^{*0}_{x_s} = \sqrt{\frac{\mu}{\varepsilon}} E_s e^{-ikz_s} \tag{9.7.41}$$

with the amplitude

$$E_s = k^2 m_0 \sin\alpha \frac{e^{ikz_{s0}}}{z_{s0}}, \quad z_{-1,0} - z_{+1,0} = l, z_{s0} = f_s \xi_{s0}, \tag{9.7.42}$$

on two disks for the same data as that in the problem on the field of the dipole source. Solution to the problem on diffraction of a plane wave may be found from (9.7.20)-(9.7.37) after substituting in formulas (9.7.20) and (9.7.31) the functions $R^{(3)}_{0n}(-ic_s, i\xi_{s0})$ by its expression which is asymptotic relative to the arguments

$$R^{(3)}_{0n}(-ic_s, i\xi_{s0}) = (-i)^{n+1} \frac{e^{ikz_{s0}}}{kz_{s0}}.$$

Then by taking (9.7.42) into account instead of, for example, (9.7.31) in system (9.7.27) the free terms are written in the form /520

$$f^s_n = -(-i)^{n+1} E_s \frac{S_{0n}(-ic_s, 1)}{k^3 m_0 \sin\alpha\, n N_{0n}(-ic_s)} + \\ + \frac{d^{0n}_1(-ic_s) C^{\mathrm{pl}}_s}{3ikm_0 \sin\alpha\, n N_{0n}(-ic_s) R^{(3)'}_{0n}(-ic_s, i0) R^{(1)'}_{0n}(-ic_s, i0)}, \tag{9.7.43}$$

where the integration constants C^{pl}_s from (9.7.24) corresponding to the case of excitation of disks by the field of the plane wave (9.7.41) are found from the system of equations (9.7.35)

$$\alpha^{\mathrm{pl}}_s C^{\mathrm{pl}}_s + \beta^{\mathrm{pl}}_s C^{\mathrm{pl}}_{-s} = \gamma^{\mathrm{pl}}_s, \quad s = \pm 1, \tag{9.7.44}$$

with the expressions α, β, γ that are the same in form as in (9.7.36) if we represent the solution to the problem of diffraction of a

lane wave in the form of (9.7.34):

$$A_n^{s,\text{pl}} = \alpha_n^{s,\text{pl}} + \beta_n^{s,\text{pl}} C_s^{\text{pl}} + \gamma_n^{s,\text{pl}} C_{-s}^{\text{pl}},$$
$$B_n^{s,\text{pl}} = \delta_n^{s,\text{pl}} C_s^{\text{pl}} + \tau_n^{s,\text{pl}} C_{-s}^{\text{pl}}. \qquad (9.7.45)$$

y determining through (9.7.34) and (9.7.45) the potentials Φ^{pl} and $^{\text{pl}}$, respectively, [since (1.3.14) is valid also for the primary ields then by H and H′ in (9.7.39) we mean only the secondary ields] and substituting their expressions into (9.7.40), we find he second equation for the constant $C_{\pm 1}$:

$$N_s C_s + M_s C_{-s} = N_s^{\text{pl}} C_s^{\text{pl}} + M_s^{\text{pl}} C_{-s}^{\text{pl}} + K_s^{\text{pl}} - K_s, \qquad (9.7.46)$$

here s = -1 or only s = +1. Here

$$K_s = F_s + F_{-s},\ K_s^{\text{pl}} = F_s^{\text{pl}} + F_{-s}^{\text{pl}},\ M_s = G_{-s} + E_s^0,$$
$$M_s^{\text{pl}} = G_{-s}^{\text{pl}} + E_s^{\text{pl}},\ N_s = M_{-s},\ N_s^{\text{pl}} = M_{-s}^{\text{pl}},$$

here in turn

$$F_s = \frac{iE_s}{k^3 m_2} \sum_{n=1,3,\ldots} i^n \alpha_n^s\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, S_{0n}(-ic_s, 1),$$

$$F_s^{\text{pl}} = \sum_{n=1,3,\ldots} \alpha_n^{s,\text{pl}}\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, R_{0n}^{(3)}(-ic_s, i\xi_s)\, S_{0n}(-ic_s, 1),$$
$$\xi_s = 0,\ \xi_{-s} = l/f_{-s},$$

$$G_s = \frac{iE_s}{k^3 m_2} \sum_{n=1,3,\ldots} i^n \beta_n^s\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, S_{0n}(-ic_s, 1),$$

$$G_s^{\text{пл}} = \sum_{n=1,3,\ldots} \beta_n^{s,\text{пл}}\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, R_{0n}^{(3)}(-ic_s, i\xi_s)\, S_{0n}(-ic_s, 1),$$
$$\xi_s = 0,\ \xi_{-s} = l/f_{-s},$$
$$E_s = \frac{iE_s}{k^3 m_2} \sum_{n=1,3,\ldots} i^n \gamma_n^s\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, S_{0n}(-ic_s, 1), \qquad (9.7.47)$$

$$E_s^{\text{пл}} = \sum_{n=1,3,\ldots} \gamma_n^{s,\text{пл}}\, nR_{0n}^{(1)\prime}(-ic_s, i0)\, R_{0n}^{(3)}(-ic_s, i\xi_s)\, S_{0n}(-ic_s, 1),$$
$$\xi_s = 0,\ \xi_{-s} = l/f_{-s}.$$

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quation (9.7.46) along with (9.7.38) forms the system of two equaions for $C_{\pm 1}$, by the solution of which we find for the computation f $C_{\pm 1}$ the formulas [in (9.7.46) the number s corresponds to a disk ith a dipole, whereas in (9.7.38) we take the number $-s$ correspond-

ing to a disk without a dipole],

$$C_s = \frac{\alpha_{-s}D_s - \gamma_{-s}M_s}{\alpha_{-s}N_s - \beta_{-s}M_s}, \quad C_{-s} = \frac{\gamma_{-s}N_s - \beta_{-s}D_s}{\alpha_{-s}N_s - \beta_{-s}M_s}, \tag{9.7.48}$$

where

$$D_s = N_s^{p1}C_s^{p1} + M_s^{p1}C_{-s}^{p1} + K_s^{p1} - K_s.$$

By determining the constants $C_{\pm 1}$ we have fully solved the problem also in the case when the horizontal magnetic dipole is located on the surface of one of the disks.

In the vicinity of the wave band in spherical coordinates r, θ, ϕ with the origin at the center of the s-th disk (s = -1 or s = +1) the components of the total field are given by the expression

$$E_\varphi = -\sqrt{\frac{\mu}{\varepsilon}}H_\eta = -k^2\sqrt{\frac{\mu}{\varepsilon}}m_2\frac{e^{ikR}}{R}\cos\theta\cos\varphi\, W_1(\theta),$$
$$E_\eta = \sqrt{\frac{\mu}{\varepsilon}}H_\varphi = -k^2\sqrt{\frac{\mu}{\varepsilon}}m_2\frac{e^{ikR}}{R}\sin\varphi\, W_2(\theta), \tag{9.7.49}$$

in which

$$W_2(\theta) = 1 - 2e^{\pm ika\cos\theta}\Bigg\{\sum_{n=1,3,\ldots} ni^n A_n^s R_{0n}^{(1)'}(-ic_s, i0)\times$$

$$\times S_{0n}(-ic_s, \cos\theta) + e^{-iskl\cos\theta}\sum_{n=1,3,\ldots} ni^n A_n^{-s} R_{0n}^{(1)'}(-ic_{-s}, i0)\times$$

$$\times S_{0n}(-ic_{-s}, \cos\theta)\Bigg\},$$

$$W_1(\theta) = 1 - 2e^{\pm ika\cos\theta}\Bigg\{\sum_{n=1,3,\ldots} ni^n A_n^s R_{0n}^{(1)'}(-ic_s, i0)\times$$

$$\times S_{0n}(-ic_s, \cos\theta) + e^{-iskl\cos\theta}\sum_{n=1,3,\ldots} ni^n A_n^{-s} R_{0n}^{(1)'}(-ic_{-s}, i0)\times \tag{9.7.50}$$

$$\times S_{0n}(-ic_{-s}, \cos\theta) - \operatorname{tg}\theta\sum_{n=1,3,\ldots} i^n B_n^s R_{1n}^{(1)}(-ic_s, i0)\times$$

$$\times S_{1n}(-ic_s, \cos\theta) - e^{-iskl\cos\theta}\operatorname{tg}\theta\sum_{n=1,3,\ldots} i^n B_n^{-s} R_{1n}^{(1)}(-ic_{-s}, i0)\times$$

$$\times S_{1n}(-ic_{-s}, \cos\theta)\Bigg\}.$$

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ere through a we denote the distance from O_s to the dipole. The ign "plus" is taken in the case when the dipole is located "over" ie point, but the sign "minus" in the case when it is located nder" O_s. If the dipole is located at the center of the s-th isk then we will have $a = 0$ and $R = r_s = r$. For disks of equal idii, when $c_1 = c_{+1} = c$, in (9.7.49)

$$W_2(\theta) = 1 - 2e^{\pm ika\cos\theta}\left\{\sum_{n=1,3,\ldots} ni^n R_{0n}^{(1)'}(-ic, i0)\times\right.$$
$$\left.\times\left[A_n^s + e^{-iskl\cos\theta}A_n^{-s}\right]S_{0n}(-ic,\cos\theta)\right\},$$

$$W_1(\theta) = 1 - 2e^{\pm ika\cos\theta}\left\{\sum_{n=1,3,\ldots} ni^n R_{0n}^{(1)'}(-ic, i0)\left[A_n^s + \right.\right. \qquad (9.7.51)$$
$$+ e^{-iskl\cos\theta}A_n^{-s}\left]S_{0n}(-ic,\cos\theta) - \operatorname{tg}\theta\sum_{n=1,3,\ldots} i^n R_{1n}^{(1)}(-ic, i0)\times\right.$$
$$\left.\times\left[B_n^s + e^{-iskl\cos\theta}B_n^{-s}\right]S_{1n}(-ic,\cos\theta)\right\}.$$

f, when $c_{-1} = c_{+1} = c$, the dipole is located in the gap midway etween the disks ($\xi_{-1,0} = \xi_{+1,0}$) then $A_n^s = -A_n^{-s}$, $B_n^s = -B_n^{-s}$ ($C_s = C_{-s}$) as a result of which the functions W_1 and W_2 are written as /523 he expression (in spherical coordinates with the origin at the oint of the dipole location)

$$W_2(\theta) = 1 + 4i\sin(kl_0\cos\theta)\times$$
$$\times\sum_{n=1,3,\ldots} ni^n A_n R_{0n}^{(1)'}(-ic, i0)S_{0n}(-ic,\cos\theta),$$
$$W_1(\theta) = 1 + 4i\sin(kl_0\cos\theta)\times \qquad (9.7.52)$$
$$\times\left\{\sum_{n=1,3,\ldots} ni^n A_n R_{0n}^{(1)'}(-ic, i0)S_{0n}(-ic,\cos\theta) - \right.$$
$$\left. - \operatorname{tg}\theta\sum_{n=1,3,\ldots} i^n B_n R_{1n}^{(1)}(-ic, i0)S_{1n}(-ic,\cos\theta)\right\},$$

here $A_n = A_n^{+1}$, $B_n = B_n^{+1}$ and $l_0 = l/2$.

On the basis of (9.7.49) the total power emitted by the system, onsisting of a horizontal magnetic dipole in two ideally conducting ircular disks with a common axis of rotation (the dipole is located t a certain point on the axis of rotation of the disks) and deter- ined by the formula

$$\overline{P} = \frac{c_0}{8\pi}\operatorname{Re}\int_0^{2\pi}\int_0^{\pi}([\mathbf{EH}^*]\mathbf{r})r^2\sin\theta\,d\theta\,d\varphi,$$

if the integration is carried out over the surface of a sphere of an infinitely large radius with the center at the origin of the spherical system of coordinates r, θ, ϕ, may be computed from the expression

$$\bar{P} = \frac{c_0}{8}\sqrt{\frac{\mu}{\varepsilon}}\,k^4 m_2^2 \int_0^{\pi} \{|W_2(\theta)|^2 + \cos^2\theta\,|W_1(\theta)|^2\}\sin\theta\,d\theta, \tag{9.7.53}$$

where c_0 is the speed of light in a vacuum and the functions $W_1(\theta)$ and $W_2(\theta)$ have the form of (9.7.50) or (9.7.51), (9.7.52).

If the irradiating system from the dipole and the two disks is assumed to be an antenna system, then for computation of the coordination of the directional effect of such an antenna we find on the basis of (1.4.18) and (9.7.53) the formula

$$g(\theta,\varphi) = 4\frac{\cos^2\theta\cos^2\varphi\,|W_1(\theta)|^2 + \sin^2\varphi\,|W_2(\theta)|^2}{\int_0^{\pi}\{\cos^2\theta\,|W_1(\theta)|^2 + |W_2(\theta)|^2\}\sin\theta\,d\theta}, \tag{9.7.54}$$

where W_1 and W_2 are determined just as before.

The electromagnetic field and the neighboring band may be studied on the basis of the expressions /52

$$\begin{aligned}\Phi(\xi_s,\eta_s) = 2ikm_2 \sum_{n=1,3,\ldots} nA_n^s\,[&R_{0n}^{(1)'}(-ic_s, i0)\,R_{0n}^{(3)}(-ic_s, i\xi_s) - \\ &- R_{0n}^{(1)}(-ic_s, i\xi_s)\,R_{0n}^{(3)'}(-ic_s, i0)]\,S_{0n}(-ic_s,\eta_s) + \\ &+ \frac{2C_s}{3}\sum_{n=1,3,\ldots}\frac{d_1^{0n}(-ic_s)}{N_{0n}(-ic_s)\,R_{0n}^{(1)'}(-ic_s, i0)}\times \\ &\times R_{0n}^{(1)}(-ic_s, i\xi_s)\,S_{0n}(-ic_s,\eta_s),\end{aligned} \tag{9.7.55}$$

$$\begin{aligned}\Psi(\xi_s,\eta_s) = 2ikm_2 \sum_{n=1,3,\ldots} B_n^s\,[&R_{1n}^{(1)}(-ic_s, i0)\,R_{1n}^{(3)}(-ic_s, i\xi_s) - \\ &- R_{1n}^{(1)}(-ic_s, i\xi_s)\,R_{1n}^{(3)}(-ic_s, i0)]\,S_{1n}(-ic_s,\eta_s) + \\ &+ \frac{2C_s}{3}\sum_{n=1,3,\ldots}\frac{d_0^{1n}(-ic_s)}{N_{1n}(-ic_s)\,R_{1n}^{(1)}(-ic_s, i0)}\times \\ &\times R_{1n}^{(1)}(-ic_s, i\xi_s)\,S_{1n}(-ic_s,\eta_s),\end{aligned} \tag{9.7.56}$$

by which the potentials of the total field $\Phi(\xi, \eta)$ and $\Psi(\xi, \eta)$ are represented in local coordinates of the s-th disk after application to (9.7.21)-(9.7.23) of the addition theorem (5.5.18), taking into account (9.7.27)-(9.7.32) in the region for the points of which $\zeta > f_s \xi_s$. The law of the density distribution of surface currents induced on the surface of the s-th disk may be studied by using the formulas

$$j_\eta = -\frac{c}{4\pi} H_\varphi, \quad j_\varphi = \frac{c}{4\pi} H_\eta, \tag{9.7.57}$$

in which in coordinates of the s-th disk when $\xi_s = 0$ ($s = -1$ or $s = +1$)

$$\begin{aligned} H_\varphi &= \frac{\sin\varphi}{f_s^2 \eta_s^2}\left\{\eta_s \frac{\partial \Phi}{\partial \eta_s} - \frac{\eta_s}{\sqrt{1-\eta_s^2}} \cdot \frac{\partial \Psi}{\partial \xi_s} - k^2 \Phi\right\}, \\ H_\eta &= \frac{\cos\varphi}{f_s \eta_s}\sqrt{1-\eta_s^2}\left\{\frac{-\partial}{\partial \eta_s}\left(\sqrt{1-\eta_s^2} \cdot \frac{\partial \Phi}{\partial \eta_s}\right) + \right. \\ &\left. + \frac{\partial}{\partial \eta_s}\left(\frac{1}{\eta_s} \cdot \frac{\partial \Psi}{\partial \xi_s}\right) - \frac{k^2 \eta_s}{\sqrt{1-\eta_s^2}} \Phi\right\}. \end{aligned} \tag{9.7.58}$$

The functions Φ, Ψ which appear in (9.7.58) are determined by expressions (9.7.55), and (9.7.56). In particular when $\xi_s = 0$ /525

$$\begin{aligned} \Phi|_{\xi_s=0} &= \frac{2m_2}{a_s} \sum_{n=1,3,\ldots} n A_n^s S_{0n}(-ic_s, \eta_s), \\ \left.\frac{\partial \Phi}{\partial \eta_s}\right|_{\xi_s=0} &= \frac{2m_2}{a_s} \sum_{n=1,3,\ldots} n A_n^s S'_{0n}(-ic_s, \eta_s), \\ \left.\frac{\partial \Psi}{\partial \xi_s}\right|_{\xi_s=0} &= -\frac{2m_2}{a_s} \sum_{n=1,3,\ldots} B_n^s S_{1n}(-ic_s, \eta_s). \end{aligned} \tag{9.7.59}$$

For a numerical solution to the problem the approximate values of the quantities A_n^s and B_n^s ($s = \pm 1$) are found from the truncated systems of normal type obtained from (9.7.27)-(9.7.32) after introduction for the matrix elements of the systems and their right-hand sides the respective designations. Since the determinants of the systems (9.7.27) and (9.7.28) are equal

$$\begin{vmatrix} 1 & \alpha_{11}^- & 0 & \alpha_{13}^- & 0 & \alpha_{15}^- & 0 & \alpha_{17}^- & 0 & \ldots \\ \alpha_{11}^+ & 1 & \alpha_{13}^+ & 0 & \alpha_{15}^+ & 0 & \alpha_{17}^+ & 0 & \alpha_{19}^+ & \ldots \\ 0 & \alpha_{31}^- & 1 & \alpha_{33}^- & 0 & \alpha_{35}^- & 0 & \alpha_{37}^- & 0 & \ldots \end{vmatrix}$$

$$\begin{vmatrix}
\alpha_{31}^{+} & 0 & \alpha_{33}^{+} & 1 & \alpha_{35}^{+} & 0 & \alpha_{37}^{+} & 0 & \alpha_{39}^{+} & \dots \\
0 & \alpha_{51}^{-} & 0 & \alpha_{53}^{-} & 1 & \alpha_{55}^{-} & 0 & \alpha_{57}^{-} & 0 & \dots \\
\alpha_{51}^{+} & 0 & \alpha_{53}^{+} & 0 & \alpha_{55}^{+} & 1 & \alpha_{57}^{+} & 0 & \alpha_{59}^{+} & \dots \\
0 & \alpha_{71}^{-} & 0 & \alpha_{73}^{-} & 0 & \alpha_{75}^{-} & 1 & \alpha_{77}^{-} & 0 & \dots \\
\alpha_{71}^{+} & 0 & \alpha_{73}^{+} & 0 & \alpha_{75}^{+} & 0 & \alpha_{77}^{+} & 1 & \alpha_{79}^{+} & \dots \\
0 & \alpha_{91}^{-} & 0 & \alpha_{93}^{-} & 0 & \alpha_{95}^{-} & 0 & \alpha_{97}^{-} & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots
\end{vmatrix} \qquad (9.7.60)$$

where $\alpha_{nm}^{-} = \alpha_{nm}^{(-1,+1)}$ and $\alpha_{nm}^{+} = \alpha_{nm}^{(+1,-1)}$ or respectively $\alpha_{nm}^{-} = \beta_{nm}^{(-1,+1)}$ and $\alpha_{nm}^{+} = \beta_{nm}^{(-1,+1)}$, then after the introduction of the designations

$$x_{2i-1} = A_{2i-1}^{+1},\ x_{2i} = A_{2i-1}^{-1},$$
$$f_{2i-1} = f_{2i-1}^{+1},\ f_{2i} = f_{2i-1}^{-1},$$
$$c_{2i-1,2j} = \alpha_{2i-1,2j-1}^{(-1,+1)},\quad c_{2i,2j-1} = \alpha_{2i-1,2j-1}^{(+1,-1)}, \qquad (9.7.61)$$
$$c_{2i-1,2j-1} = c_{2i,2j} = 0,\ i,\ j = 1, 2,\dots,$$

and the analogous designations for system (9.7.28) /526

$$x_{2i-1} = B_{2i-1}^{+1},\quad x_{2i} = B_{2i-1}^{-1},$$
$$f_{2i-1} = \varphi_{2i-1}^{+1},\quad f_{2i} = \varphi_{2i-1}^{-1}, \qquad (9.7.62)$$
$$c_{2i-1,2j} = \beta_{2i-1,2j-1}^{(-1,+1)},\quad c_{2i,2j-1} = \alpha_{2i-1,2j-1}^{(+1,-1)},$$
$$c_{2i-1,2j-1} = c_{2i,2j} = 0,\ i, j = 1, 2,\dots,$$

the normal form for systems (9.7.27) and (9.7.28) will be

$$x_i + \sum_{j=1}^{\infty} c_{ij}x_j = f_i,\ i = 1,2,\dots \qquad (9.7.63)$$

By truncation of (9.7.63) the final system is obtained

$$x_i + \sum_{j=1}^{2N} c_{ij}x_j = f_i,\ i = 1, 2,\dots,2N, \qquad (9.7.64)$$

consisting of $2N$ equations with $2N$ unknowns $A_n^{\pm 1}$, n = 1, 3, ..., $2N$ - 1, or $B_n^{\pm 1}$, n = 1, 3, ..., $2N$ - 1. As the original value of N for computing $A_n^{\pm 1}$ (or $B_n^{\pm 1}$) with a given degree of accuracy we may take $N = [c] + 1$; $c = \max\limits_{s=\pm 1} c_s$ where just as before, the brackets denote

he integral part of c. If we exclude from (9.7.64), one series of nknowns $A_n^{+1}(A_n^{-1})$ or respectively $B_n^{+1}(B_n^{-1})$, we may obtain the system consisting of N equations with N unknowns $A_n^{-1}(A_n^{+1})$ or respectively $B_n^{-1}(B_n^{+1})$, $n = 1, 3, \ldots, 2N - 1$. However as a result of the peration of excluding these series of unknowns, the structure of he matrix elements and the right-hand sides of the systems thus btained is significantly complicated.

In the case of disks of equal radii when $c_{+1} = c_{-1} = c$, in ystems (9.7.27), (9.7.28)

$$\alpha_{nm} = \alpha_{nm}^{(-1,+1)} = \alpha_{nm}^{(+1,-1)},$$
$$\alpha_{nm} = \beta_{nm}^{(-1,+1)} = \beta_{nm}^{(+1,-1)}, \qquad (9.7.65)$$

n the basis of which, seeking approximate values of $A_n^{\pm 1}$ and $B_n^{\pm 1}$ ay be reduced to solving the independent systems of N equations ith N unknowns

$$y_n + \sum_{\substack{m=1,3,\ldots 2N-1 \\ n=1,3,\ldots,2N-1,}} \alpha_{nm} y_m = d_n, \qquad (9.7.66)$$

$$z_n - \sum_{m=1,3,\ldots,2N-1} \alpha_{nm} z_m = e_n,$$

f we set /527

$$y_n = A_n^{+1} + A_n^{-1}, \quad d_n = f_n^{+1} + f_n^{-1}$$
$$z_n = A_n^{+1} - A_n^{-1}, \quad e_n = f_n^{+1} - f_n^{-1} \qquad (9.7.67)$$

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$$y_n = B_n^{+1} + B_n^{-1}, \quad d_n = \varphi_n^{+1} + \varphi_n^{-1},$$
$$z_n = B_n^{+1} - B_n^{-1}, \quad e_n = \varphi_n^{+1} - \varphi_n^{-1} \qquad (9.7.68)$$

espectively.

When $c_{+1} = c_{-1} = c$ if the dipole is located at point O on the xis of rotation of the disk in the gap midway between them, then $_{-1,0} = \xi_{+1,0}$ and on the strength of the oddness of the index n $_{0n}(-ic, \eta_{-1,0}) = -S_{0n}(-ic, \eta_{+1,0})$. In this case we can extablish hat the integration constants C_{+1} and C_{-1} will differ from one

another only in their signs, i.e., $C_{+1} = -C_{-1}$. From (9.7.27) - (9.7.32) it follows that then

$$A_n = A_n^{+1} = -A_n^{-1}. \quad B_n = B_n^{+1} = -B_n^{-1} \quad n = 1, 3,. \tag{9.7.69}$$

As a result seeking the approximate values of A_n and B_n leads to solution of the systems

$$A_n - \sum_{m=1,3,\ldots,2N-1} \alpha_{nm} A_m = f_n; \quad f_n = f_n^{+1} = -f_n^{-1},$$

$$B_n - \sum_{m=1,3,\ldots,2N-1} \alpha_{nm} B_m = \varphi_n; \quad \varphi_n = \varphi_n^{+1} = -\varphi_n^{-1}, \tag{9.7.70}$$

obtained from (9.7.64) with the exception of the series of unknowns A_n^{-1} or B_n^{-1} respectively, $n = 1, 3, \ldots, 2N - 1$.

The comments noted in §5 of this chapter concerning computation of the right-hand sides of the matrix elements of system (9.5.11) are completely applicable to (9.7.29)-(9.7.32) as well.

Now let us apply to systems (9.7.27) and (9.7.28) the method of successive approximations, taking as the zero approximations the values ${}^{s}A_n^0 = 0$, ${}^{s}B_n^0 = 0$, $s = \pm 1$. Then, for example, from (9.7.27) we find

$${}^{s}A_n^1 = f_n^s,$$

$${}^{s}A_n^m = -\sum_{p=1,3,\ldots} \alpha_{np}^{(-s,s)} {}^{s}A_p^{m-1}, \quad m = 2, 3,\ldots, \tag{9.7.71}$$

and on the assumption that the iteration process is a converging one /52

$$A_n^s = \sum_{m=1}^{\infty} {}^{s}A_n^m. \tag{9.7.72}$$

As the result the potential $\Phi^1(\xi, \eta)$ of the secondary field is written in the form

$$\Phi^1_{(\xi,\eta)} = 2ikm_2 \sum_{s=\pm 1} \sum_{n=1,3,\ldots} n \left(\sum_{m=1}^{\infty} {}^{s}A_n^m \right) R_{0n}^{(1)'}(-ic_s i\,0) \times$$
$$\times R_{0n}^{(3)}(-ic_s, i\,\xi_s) S_{0n}(-ic_s, \eta_s). \tag{9.7.73}$$

In a similar manner, from system (9.7.28) we find

$$ {}^{s}B_n^1 = \varphi_n^s, $$

$$ {}^{s}B_n^m = -\sum_{p=1,3,\ldots} \beta_{np}^{(-s,s)-s} B_n^{m-1}, \quad m = 2, 3, \ldots, \tag{9.7.74} $$

and

$$ B_n^s = \sum_{m=1}^{\infty} {}^{s}B_n^m, \tag{9.7.75} $$

and the potential $\Psi(\xi, \eta)$ is written in the form

$$ \Psi(\xi, \eta) = 2ikm_2 \sum_{s=\pm 1} \sum_{n=1,3,\ldots} \left(\sum_{m=1}^{\infty} {}^{s}B_n^m \right) R_{1n}^{(1)}(-ic_s, i0) \times \\ \times R_{1n}^{(1)}(-ic_s, i\xi_s) S_{1n}(-ic_s, \eta_s). \tag{9.7.76} $$

Let us establish the physical sense of the quantities (9.7.71), and (9.7.74). Let us denote through Φ_s^1 and Ψ_s^1 the potential functions by which we determine the field E_s^1, H_s^1 excited by the s-th disk ($s = \pm 1$) acted on only by the field E^0, H^0 of the dipole, without taking into account the influence of the field E_s^1, H_s^1, scattered by the $-s$-th disk ($s = \pm 1$), and let us denote them (§9, Chapter 1) as functions of the first order of scattering (the diffraction field E_s^1, H_s^1 of the first order of scattering corresponds to them). Acted on by the field E_{-s}^1, H_{-s}^1 excited by the $-s$-th disk and determined by the function Φ_{-s}^1, Ψ_{-s}^1, the s-th disk excites the field E_s^2, H_s^2 of the second order of scattering, whose value will be determined by the function Φ_s^2, Ψ_s^2, of the second order of scattering. As a result we find that under the action of the field E_{-s}^{n-1}, H_{-s}^{n-1} $(n-1)$-th order of scattering produced by the $-s$-th disk, the s-th disk will excite the stray field E_s^n, H_s^n of the n-th order of scattering, determined by the functions Φ_s^n, Ψ_s^n, of the n-th order of scattering, etc. The complete field E_s, H_s, scattered by the s-th disk to which the function Φ_s, Ψ_s corresponds may be found as the sum of the field of all successive orders of scattering. For it

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$$ \mathbf{E}_s = \sum_{n=1}^{\infty} \mathbf{E}_s^n, \quad \mathbf{H}_s = \sum_{n=1}^{\infty} \mathbf{H}_s^n, $$

$$ \Phi_s = \sum_{n=1}^{\infty} \Phi_s^n, \quad \Psi_s = \sum_{n=1}^{\infty} \Psi_s^n. $$

The complete field E^1, H^1, scattered by both disks, taking into account their interaction, will be equal to

$$\mathbf{E}^1 = \sum_{s=\pm 1} \mathbf{E}_s, \quad \mathbf{H}^1 = \sum_{s=\pm 1} \mathbf{H}_s$$

and will be determined through the potential function

$$\Phi^1(\xi, \eta) = \sum_{s=\pm 1} \Phi_s(\xi, \eta), \quad \Psi(\xi, \eta) = \sum_{s=\pm 1} \Psi_s(\xi, \eta).$$

If we now require that on the surface of the s-th disk ($s = \pm 1$) that the tangential components of the vectors $E^0 + E_s^1$, $E_s^2 + E_{-s}^1$,..., $E_s^n + E_s^{n-1}$, simultaneously vanish (here the tangential components of the vector E of the electric intensity of the total field also vanish on the surface of the s-th disk, $s = \pm 1$), we then find that all functions $\Phi^0 + \Phi_s^1$, Ψ_s^1,..., $\Phi_s^n + \Phi_{-s}^{n-1}$, $\Psi_s^n + \Psi_{-s}^{n-1}$ corresponding to them will satisfy boundary conditions (9.3.31) on the surface of the disk. Therefore, if we represent the potential functions Φ_s^m and Ψ_s^m of the m-the order of scattering in the form of the series

$$\Phi_s^m(\xi, \eta) = 2ikm_2 \sum_{n=1,3,\ldots} n^s A_n^m R_{0n}^{(1)'}(-ic_s, i0) \times \times R_{0n}^{(3)}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_s), \qquad (9.7.77)$$

$$\Psi_s^m(\xi, \eta) = 2ikm_2 \sum_{n=1,3,\ldots} {}^sB_n^m R_{1n}^{(1)}(-ic_s, i0) \times \times R_{1n}^{(3)}(-ic_s, i\xi_s) S_{1n}(-ic_s, \eta_s),$$

then from (9.3.31) for the coefficients of the series ${}^sA_n^m$ and ${}^sB_n^m$,, we find expressions which are identically equal to (9.7.71) and (9.7.74), respectively. However, it follows that the quantities ${}^sA_n^m$ and ${}^sB_n^m$ from (9.7.71) and (9.7.74) obtained on the basis of the formal application of the method of successive approximations to the infinite systems (9.7.27) and (9.7.28) are simply coefficients of scattering of the m-th order, by the use of which we determine the potentials of the secondary electromagnetic field E_s^m, H_s^m of the m-th order of scattering ($s = \pm 1$, $m = 1, 2,\ldots$) which arises as a result of the multiple successive secondary diffractions of waves on the disks. Formulas (9.7.71) and (9.7.74) may be assumed as recurrent relationships from which the coefficients of scattering of

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arbitrary order are found through all of the previous ones and finally through ${}^{s}A_n^1 = f_n^s$, ${}^{s}B_n^1 = \phi_n^s$.

In the case when the distance between the centers of the disks is much greater than the wavelength ($kl >> 1$) and is high in comparison with the radii of the disks, finding the approximate values of A_n^s and B_n^s [i.e., the coefficients of the total secondary field, produced by the s-th disk ($s = \pm 1$)] may be reduced to summation of the geometrically decreasing progression obtained from (9.7.72) or from (9.7.75) after making the substitution in the matrix elements $\alpha_{nm}^{(-s,s)}$ and $\beta_{nm}^{(-s,s)}$ of the spherical Bessel function $h_\sigma^{(1)}(kl)$ by its expression which is asymptotic relative to the argument. Based on the same scheme as in §3, Chapter 6 in a similar situation we find approximately that [87]

$$A_n^s = A_{ns} - \frac{P_{\cdot}^{-s} - LQ(-s)P^s}{1 - [L^2Q(-s)Q(s)]} La_{ns} - \\ - \frac{q^{-s}La_{ns}C_{-s}}{1 - [L^2Q(-s)Q(s)]} + \left[B_{ns} + \frac{q^s a_{ns}L^2Q(-s)}{1 - [L^2Q(-s)Q(s)]}\right]C_s, \tag{9.7.78}$$

where $Q(s) = \sum_{n=1,3,\ldots} a_{ns}b_{ns}$, $p^s = \sum_{n=1,3,\ldots} A_{ns}b_{ns}$, $q^s = \sum_{n=1,3,\ldots} B_{ns}b_{ns}$,

$$L = -2ie^{ikl}/kl,\ a_{ns} = \frac{i^n S_{0n}(-ic_s, 1)}{nR_{0n}^{(3)'}(-ic_s, i0)N_{0n}(-ic_s)},\ b_{ns} = -i^n n \times \\ \times R_{0n}^{(1)'}(-ic_s, i0)S_{0n}(-ic_s, 1),$$

$$A_{ns} = -\frac{S_{0n}(-ic_s, \eta_{s0})R_{0n}^{(3)}(-ic_s, i\xi_{s0})}{nN_{0n}(-ic_s)R_{0n}^{(3)'}(-ic_s, i0)},$$

$$B_{ns} = \frac{d_1^{0n}(-ic_s)}{3ikm_2nN_{0n}(-ic_s)R_{0n}^{(1)'}(-ic_s, i0)R_{0n}^{(3)'}(-ic_s, i0)}$$

on the assumption that $|L^2Q(-s)Q(s)| < 1$;

$$B_n^s = \varphi_n^s,\ s = \pm 1. \tag{9.7.79}$$

The integration constants C_s which appear in (9.7.78) and (9.7.79) are found according to the previous scheme. In the corresponding problem of the diffraction of a plane electromagnetic wave (9.7.41) we will have ($kl >> 1$) /531

$$A_n^{s,пл} = A_{ns}^{пл} - \frac{P_{пл}^{-s} - LQ(-s)P_{пл}^s}{1 - [L^2Q(-s)Q(s)]} La_{ns} -$$

$$-\frac{q^{-s}La_{ns}C_{-s}^{\mathrm{pl}}}{1-[L^2Q(-s)Q(s)]}+ \qquad (9.7.80)$$

$$+\left[B_{ns}+\frac{q^s a_{ns}L^2Q(-s)}{1-[L^2Q(-s)Q(s)]}\right]C_s^{\mathrm{pl}},$$

$$B_n^{s,\mathrm{pl}}=\varphi_n^{s,\mathrm{pl}},\ s=\pm 1, \qquad (9.7.81)$$

where L, $Q(s)$, B_{ns}, a_{ns}, b_{ns}, q^s are determined as above but

$$A_{ns}^{\mathrm{pl}}=-E_s\frac{i^{n+1}S_{0n}(-ic_s,1)}{k^3m_2nN_{0n}(-ic_s)R_{0n}^{(3)'}(-ic_s,i0)},$$

$$E_s=E_{-s}e^{-iskl},\ s=\pm 1,\ P_{\mathrm{pl}}^s=\sum_{n=1,3,\ldots}A_{ns}^{\mathrm{pl}}b_{ns},$$

$$\varphi_n^{s,\mathrm{pl}}=D_{ns}C_s^{\mathrm{pl}},$$

$$D_{ns}=\frac{2d_0^{1n}(-ic_s)}{3ikm_2N_{1n}(-ic_s)R_{1n}^{(1)}(-i\varsigma_s,i0)R_{1n}^{(3)}(-ic_s,i0)}.$$

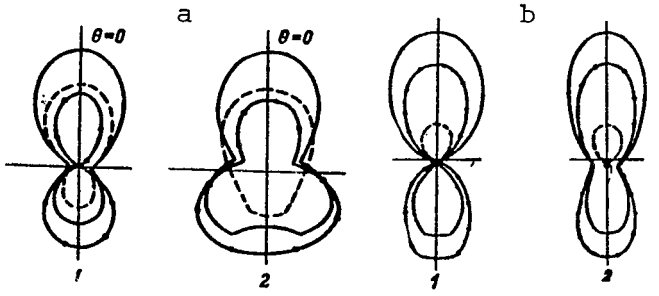
Formulas (9.7.78)-(9.7.81) are suitable for numerical computation and analysis of the diffraction field when $kl >> l$ (l is also high in comparison with the radii of the disk) since their use does not involve preliminary computation of the matrix elements of the systems. This case has been studied more fully in [87].

As examples, on Figure 64 we find the directional diagram constructed for the components H_ϕ and E_ϕ of the total field when the horizontal magnetic dipole is located on the surface of the -1 disk at its center [formulas (9.7.27) - (9.7.32) and (9.7.49), $\varepsilon = \mu = 1$; the plane Oxz, Oxy; the angular coordinate of the dipole $\eta_{-1} = 1$). The broken line denotes the directional diagram of the field of the emitting self-contained unit slot [-1 disk of the dipole at its center, $\eta_{-1} = 1$] in the absence in space of the +1 disk; the dotted lines show the diagram of the functions H_ϕ and E_ϕ of the secondary field scattered on both disks [the dipole is at the center of the -1 disk, $\eta_{-1} = 1$]. Change in the directional diagrams for the total field with change in the distance between the disks is shown on Figure 65 where the broken line means the same as on Figure 64.

Electric Dipole. In this case the problem of diffraction on /532
two coaxial infinitely thin circular disks may be solved according to the same scheme as for the magnetic dipole. The electromagnetic field is found for the electric Hertz vector $\vec{\Pi}$ from the relationship

$$\mathbf{E}=\operatorname{grad}\operatorname{div}\vec{\Pi}+k^2\vec{\Pi},$$

$$\mathbf{H}=-ik_0\varepsilon\operatorname{rot}\vec{\Pi},\ k_0=\frac{2\pi}{\lambda_0}. \qquad (9.7.82)$$



ig. 64. The Functions $|H_\phi|^2$ and $|H_\phi^2|^2$ in the Plane Oyz (1) With alues of $c_{+1} = c_{-1} = 1$, $kl = 3$ and the Functions $|E_\phi|^2$, $|E_\phi^2|^2$ in he Plane Oxz (2) With the Same Values of the Parameters (a) and espectively for the Values $c_{+1} = 1$, $c_{-1} = 2$ and $kl = 3$ (b).

If the Hertz vector of the primary field is represented in the orm of a sum of two vectors

$$\vec{\Pi}^0 = \vec{\Pi}_1 + \vec{\Pi}_2 \quad \left(\vec{\Pi}^0 = \mathbf{p}\frac{e^{ikR}}{R}, \quad \mathbf{p} = \{|\mathbf{p}|\sin\alpha, 0, |\mathbf{p}|\cos\alpha\}\right),$$
$$\vec{\Pi}_1 = \{|\mathbf{p}|\sin\alpha\, e^{ikR}/R, 0, 0\}, \qquad (9.7.83)$$
$$\vec{\Pi}_2 = \{0, 0, |\mathbf{p}|\cos\alpha\, e^{ikR}/R\},$$

hen the secondary field may be assumed as the result of a super- osition of two particular electromagnetic fields, one of which orresponds to the case of excitation of the disk by a vertical lectric dipole with the moment $\{0, 0, |p| \cos\alpha\}$, and the second o the case of excitation of the disk by the horizontal electric ipole with the moment $\{|p| \sin\alpha\ 0, 0\}$ under the conditions that oth dipoles are found at one and the same point P on the axis of otation of the disks. Therefore, solution to the problem for an rbitraryily oriented electric dipole may be obtained from the olutions to the problems of diffraction of the field of the ver- ical and horizontal dipoles on two disks. The first of these roblems was solved in §5 of this chapter. /533

In the case of a horizontal electric dipole the Hertz vector of the total field is sought in the form

$$\vec{\Pi} = \{\Pi_x, 0, \Pi_z\},$$

here

$$\Pi_x = \Pi_x^0 + \Pi_x^1 = \Phi^0(\xi, \eta) + \Phi^1(\xi, \eta) = \Phi(\xi, \eta),$$
$$\Pi_z = \Pi_z^1 = \Psi(\xi, \eta)\cos\varphi. \qquad (9.7.84)$$

On the basis of the relationship in (9.3.8) the components of the electromagnetic field will be expressed through Φ and Ψ in one of the coordinate systems of the oblate spheroids by the formulas in (9.3.18) if in (9.3.18) we first substitute E_η and E_ϕ by H_η and H_ϕ, respectively, and substitute $-\varepsilon$ for μ.

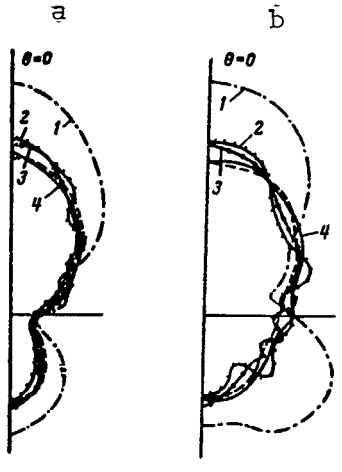


Fig. 65. The Functions $|H_\phi|^2$ in the Plane Oyz (a) and $|E_\phi|^2$ in the Plane Oxz (b) as a Function of kl When $c_{+1} = c_{-1}$: $kl = 3(1)$, 3.5π (2), 5π (3); 50π (4).

The functions Φ and Ψ do not depend on the azimuth angles ϕ. Just as in the case of a magnetic dipole they must be the solutions to the Helmholtz equation with the azimuth number $m = 0$ or $m = 1$, respectively, satisfying the condition of radiation at infinity and the given boundary conditions on the surface of each disk. In local coordinates of the s-th disk these conditions have the form

$$\left.\begin{aligned} \Phi(\xi, \eta) &= -C_s \\ \frac{\partial \Psi(\xi, \eta)}{\partial \xi_s} &= C_s c_s^2 \eta_s \sqrt{1-\eta_s^2} \end{aligned}\right\}, \quad \xi_s = 0, \quad s = \pm 1, \tag{9.7.85}$$

where C_s represents constants requiring determination in the future. At the edges of the disks the unknown field must satisfy the Meixner condition which also in this case may be described by formula (9.7.25). /53

In local coordinates of the s-th disk the potential $\Phi(\xi, \eta)$ of the primary dipole field can be represented by the series

$$\Phi^0 = 2ikp \sum_{n=0}^{\infty} \frac{S_{0n}(-ic_s, \eta_{s0})}{N_{0n}(-ic_s)} S_{0n}(-ic_s, \eta_s) \times \tag{9.7.86}$$
$$\times \begin{cases} R_{0n}^{(1)}(-ic_s, i\xi_s) R_{0n}^{(3)}(-ic_s, i\xi_{s0}), & \xi_{s0} > \xi_s, \\ R_{0n}^{(1)}(-ic_s, i\xi_{s0}) R_{0n}^{(3)}(-ic_s, i\xi_s), & \xi_{s0} < \xi_s, \end{cases}$$

and the potentials Φ and Ψ of the secondary field are sought in the form

$$\Phi^1(\xi, \eta) = 2ipk \sum_{s=\pm 1} \Phi_s(\xi_s, \eta_s),$$

$$\Psi(\xi, \eta) = 2ipk \sum_{s=\pm 1} \Psi_s(\xi_s, \eta_s), \tag{9.7.87}$$

ıere

$$\Phi_s = \sum_{n=0}^{\infty} a_n^s R_{0n}^{(3)}(-ic_s, i\xi_s) S_{0n}(-ic_s, \eta_s),$$
$$\Psi_s = \sum_{n=0}^{\infty} b_n^s R_{1n}^{(3)}(-ic_s, i\xi_s) S_{1n}(-ic_s, \eta_s). \qquad (9.7.88)$$

ɜing the addition theorem (5.5.18) from (9.7.85) for the coeffi-
ients of the series a_n^s and b_n^s we find the infinite systems of
inear equations from which we immediately find that

$$a_n^s = b_n^s = 0 \text{ with odd values of } n. \qquad (9.7.89)$$

ter substitution of the unknowns a_n^s and b_n^s with the new unknowns
A_n^s and B_n^s according to the formulas

$$a_n^s = R_{0n}^{(1)}(-ic_s, i0) A_n^s, \quad n = 0, 2, \ldots,$$
$$b_n^s = R_{1n}^{(1)\prime}(-ic_s, i0) B_n^s, \quad n = 2, 4, \ldots, \qquad (9.7.90)$$

or the unknowns A_n^s, B_n^s we find the systems /535

$$A_n^s + \sum_{m=0,2,\ldots} \alpha_{nm}^{(-s,s)} A_m^{-s} = f_n^s, \quad n = 0, 2, \ldots; \; s = \pm 1, \qquad (9.7.91)$$

$$B_n^s + \sum_{m=2,4,\ldots} \beta_{nm}^{(-s,s)} B_m^{-s} = \varphi_n^s, \quad n = 2, 4, \ldots; \; s = \pm 1, \qquad (9.7.92)$$

ith the matrix elements

$$\alpha_{nm}^{(-s,s)} = -\frac{R_{0m}^{(1)}(-ic_{-s} i0)}{R_{0n}^{(3)}(-ic_s, i0)} Q_{0n0m}(c_s, c_{-s}; l, \theta_{-s,s}), \qquad (9.7.93)$$

$$\beta_{nm}^{(-s,s)} = -\frac{R_{1m}^{(1)\prime}(-ic_{-s}, i0)}{R_{1n}^{(3)\prime}(-ic_s, i0)} Q_{1n1m}(c_s, c_{-s}; l, \theta_{-s,s}) \qquad (9.7.94)$$

nd with right-hand sides

$$f_n^s = -\frac{S_{0n}(-ic_s, \eta_{s0}) R_{0n}^{(3)}(-ic_s, i\xi_{s0})}{N_{0n}(-ic_s) R_{0n}^{(3)}(-ic_s, i0)} +$$
$$+ \frac{d_0^{0n}(-ic_s) C_s}{ikpN_{0n}(-ic_s) R_{0n}^{(1)}(-ic_s, i0) R_{0n}^{(3)}(-ic_s, i0)}, \qquad (9.7.95)$$

$$\varphi_n^s = -\frac{2c_s^2\, d_1^{1n}(-ic_s)\, C_s}{5ikpN_{1n}(-ic_s)\, R_{1n}^{(1)'}(-ic_s, i0)\, R_{1n}^{(3)'}(-ic_s, i0)}. \qquad (9.7.96)$$

The determinants of the systems (9.7.91) and (9.7.92) may be written in the form

$$\begin{vmatrix}
1 & \alpha_{00}^- & 0 & \alpha_{02}^- & 0 & \alpha_{04}^- & 0 & \alpha_{06}^- & 0 & \alpha_{08}^- & \dots \\
\alpha_{00}^+ & 1 & \alpha_{02}^+ & 0 & \alpha_{04}^+ & 0 & \alpha_{06}^+ & 0 & \alpha_{08}^+ & 0 & \dots \\
0 & \alpha_{20}^- & 1 & \alpha_{22}^- & 0 & \alpha_{24}^- & 0 & \alpha_{26}^- & 0 & \alpha_{28}^- & \dots \\
\alpha_{20}^+ & 0 & \alpha_{22}^+ & 1 & \alpha_{24}^+ & 0 & \alpha_{26}^+ & 0 & \alpha_{28}^+ & 0 & \dots \\
0 & \alpha_{40}^- & 0 & \alpha_{42}^- & 1 & \alpha_{44}^- & 0 & \alpha_{46}^- & 0 & \alpha_{48}^- & \dots \\
\alpha_{40}^+ & 0 & \alpha_{42}^+ & 0 & \alpha_{44}^+ & 1 & \alpha_{46}^+ & 0 & \alpha_{48}^+ & 0 & \dots \\
0 & \alpha_{60}^- & 0 & \alpha_{62}^- & 0 & \alpha_{64}^- & 1 & \alpha_{66}^- & 0 & \alpha_{68}^- & \dots \\
\alpha_{60}^+ & 0 & \alpha_{62}^+ & 0 & \alpha_{64}^+ & 0 & \alpha_{66}^+ & 1 & \alpha_{68}^+ & 0 & \dots \\
0 & \alpha_{80}^- & 0 & \alpha_{82}^- & 0 & \alpha_{84}^- & 0 & \alpha_{86}^- & 1 & \alpha_{88}^- & \dots \\
\alpha_{80}^+ & 0 & \alpha_{82}^+ & 0 & \alpha_{84}^+ & 0 & \alpha_{86}^+ & 0 & \alpha_{88}^+ & 1 & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots &
\end{vmatrix} \qquad (9.7.97)$$

where $\alpha_{nm}^- = \alpha_{nm}^{(-1,+1)}$, $\alpha_{nm}^+ = \alpha_{nm}^{(+1,-1)}$ or $\alpha_{nm}^- = \beta_{n+2,m+2}^{(-1,+1)}$, $\alpha_{nm}^+ = \beta_{n+2,m+2}^{(+1,-1)}$; $n, m = 0, 2, \dots$ /53

After introduction of the designations

$$\begin{gathered}
x_{2l+1} = A_{2l}^{+1}, \quad f_{2l+1} = f_{2l}^{+1}, \\
x_{2l+2} = A_{2l}^{-1}, \quad f_{2l+2} = f_{2l}^{-1}, \\
c_{2l+1,2j+2} = \alpha_{2l,2j}^{(-1,+1)}, \quad c_{2l+2,2j+1} = \alpha_{2l,2j}^{(+1,-1)}, \qquad (9.7.98) \\
c_{2l+1,2j+1} = c_{2l+2,2j+2} = 0; \; i, j = 0, 1, \dots,
\end{gathered}$$

in system (9.7.91) and the designations

$$\begin{gathered}
x_{2l+1} = B_{2l+2}^{+1}, \quad f_{2l+1} = \varphi_{2l+2}^{+1}, \\
x_{2l+2} = B_{2l+2}^{-1}, \quad f_{2l+2} = \varphi_{2l+2}^{-1}, \\
c_{2l+1,2j+2} = \beta_{2l+2,2j+2}^{(-1,+1)}, \quad c_{2l+2,2j+1} = \beta_{2l+2,2j+2}^{(+1,-1)}, \qquad (9.7.99) \\
c_{2l+1,2j+1} = c_{2l+2,2j+1} = 0; \; i,j = 0, 1, 2, \dots,
\end{gathered}$$

in system (9.7.92), both systems are written in normal form

$$x_l + \sum_{j=0}^{\infty} c_{lj} x_j = f_l, \quad i = 0, 1, \dots \qquad (9.7.100)$$

ystems (9.7.91) and (9.7.92) which are similar to systems (9.7.27) id (9.7.28) have a unique solution which satisfies the condition $\sum_{n=0}^{\infty} |x_n|^2 < \infty$. It may be found by the method of truncation.

For a numerical solution to the problem the approximate values $A_n^{\pm 1}$ and $B_n^{\pm 1}$ are found from the truncated systems

$$x_i + \sum_{j=0}^{2N} c_{ij} x_j = f_i, \quad i = 0, 1, \ldots, 2N, \tag{9.7.101}$$

btained from (9.7.100), where as the original value of N we can ake $N = [c] + 1$, $c = \max_{s=\pm 1} c_s$.

The cases which lead to a decrease in the order of system 9.7.101) and to a simplification of the computational formulas ay be studied as before.

§8. Scattering of an Obliquely Incident Electromagnetic Wave on Two Coaxial Circular Disks.

The problem on scattering of a plane electromagnetic wave bliquely incident on two coaxial ideally conducting infinitely hin circular disks of radii $\rho_{-1} = a_{-1}$ and $\rho_{+1} = a_{+1}$ can be conveni- ntly solved in local coordinates of the oblate spheroid ξ_s, η_s, /537 s with origins at the centers of the disks and with focal dis- nces $f_s = a_s$, $s = \pm 1$.

If the phase of the wave (9.4.1) is determined at any point elative to its value of point O lying on the axis of the disks idway between the disks (point O is the origin of the coordinate ystem $Oxyz$, Fig. 66), then in local coordinates of the s-th disk he equation of the incident wave (9.4.1) is written in the form

$$\mathbf{E}^0 = \mathbf{E}' \Psi, \quad \Psi = e^{ik(\mathbf{n}\mathbf{r}_{0s}) + ik(\mathbf{n}\mathbf{r}_s)}, \quad s = \pm 1, \tag{9.8.1}$$

here, as before, $E = |\mathbf{E}'| = \text{const}$, and $\mathbf{r}_{0_s}$ is the radius vector onnecting the center of the s-th disk O_s with point O; $\mathbf{r}_s$ is the adius vector of the observation point. The disks are located at distance l from one another in a uniform isotropic unbounded space ith electric and magnetic constants ε, μ and conductivity $\sigma = 0$ $k = k_0\sqrt{\varepsilon\mu}$, $k_0 = \omega/c_0$). The vector $\mathbf{n}$ forms an angle α with the ositive direction of the axis Oz (the axes Oz of the coordinate ystems coincide with the axis of symmetry of the disk $\phi_s = \phi$, $s = 1$) and is parallel to the plane Oxz. Therefore in (9.8.1) $\mathbf{n}\mathbf{r}_{0_s}) = s l_0 \cos\alpha$, $s = \pm 1$, $l_0 = l/2$, but $(\mathbf{n}\mathbf{r}_s) = x_s \sin\alpha + z_s \cos\alpha$.

Just as in §4 of this chapter we can look at two cases of polarization of the wave (9.8.1): the case when $E^0 \perp$ pl. Oxz and the case when $E^0 \parallel$ pl. Oxz. To these there correspond the electric Hertz vectors (9.4.5) and (9.4.6), respectively, with functions Ψ taken from (9.8.1).

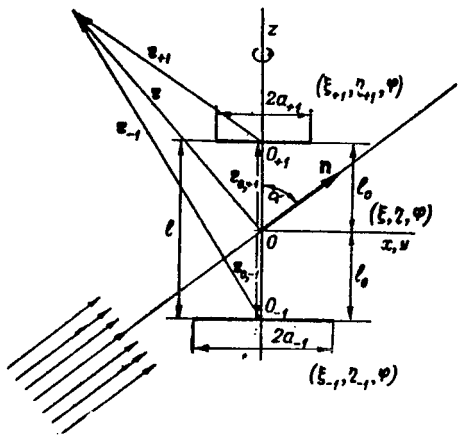


Fig. 66. Diffraction of a Plane Wave With Oblique Incidence on Two Coaxial Circular Disks.

The total field is determined by the formulas in (9.4.7) written in one of the coordinate systems ($\vec{\Pi} = \vec{\Pi}^0 + \vec{\Pi}^1$). /5

It is assumed that the field scattered on the disk is determined through the vector $\vec{\Pi}^1 = \{\Pi^1_x, \Pi^1_y, 0\}$ (it is parallel to the surfaces of the disks and $\Pi^1_x \equiv 0$ ($\Pi^1_y \not\equiv 0$) in the case of polarization such as (9.4.5) or $\Pi^1 \equiv 0$ ($\Pi^1 \not\equiv 0$) in the case of polarization of the field such as (9.4.6)).

In mathmetical formulation the diffraction problem consists of seeking the potentials Π^1_x, Π^1_y of the stray field which satisfy, in the region outside the disk, the homogeneous scalar wave Helmholtz equation; on the surface of each of the disks ($\xi_s = 0$, $s = \pm 1$) they satisfy the boundary conditions (9.4.10), and at their edges ($\xi_s = 0$, $\eta_s = 0$, $s = \pm 1$), they satisfy the Meixner conditions (9.4.11) and the supplementary condition of radiation at infinity.

Just as in the case of one disk the stray field is divided into two parts, one of which is determined by the vector $\vec{\Pi}_1 = \{\Pi_{1,x}, \Pi_{1,y}, 0\}$ and the second part by the vector $\vec{\Pi}_2 = \{\Pi_{2x}, \Pi_{2y}, 0\}$ such that

$$\vec{\Pi}^1 = \vec{\Pi}_1 + \vec{\Pi}_2.$$

Of the vector $\vec{\Pi}_1$ we require that on the surface of the disk (or together with the vector $\vec{\Pi}^0$ of the primary field) that it satisfy condition (9.4.13)

$$(\vec{\Pi}^0 + \vec{\Pi}_1)|_{\xi_s=0} = 0, \quad s = \pm 1. \tag{9.8.2}$$

Then condition (9.4.10), when $\xi_s = 0$, $s = \pm 1$, will be satisfied only by the second part of the stray field $\vec{\Pi}_2$.

When $E^0 \perp$ pl. Oxz, then in local coordinates of the s-th disk or the primary field we find the expansion (9.4.14) to be valid

$$\Pi_y^0 = \frac{2Ee^{ikl_0\cos\alpha}}{k^2} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{i^n S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)} \times \\ \times R_{|m|n}^{(1)}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{9.8.3}$$

$\Pi_x^0 = 0$). If also $E^0 \parallel$ pl. Oxz, then in coordinates of the s-th disk

$$\Pi_x^0 = \frac{2Ee^{ikl_0\cos\alpha}}{k^2\cos\alpha} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{i^n S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)} \times \\ \times R_{|m|n}^{(1)}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{9.8.4}$$

$(\Pi_y^0 = 0)$.

Let us first find the first part of the stray field $\vec{\Pi}_1$. /539

Depending on the type of polarization of the incident wave he vector $\vec{\Pi}_1$ is sought in the form

$$\Pi_{1.y} = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} a_{mn}^s \times \\ \times R_{|m|n}^{(3)}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{9.8.5}$$

$\Pi_{1,x} = 0$) or in the form

$$\Pi_{1.x} = \frac{2E}{k^2\cos\alpha} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} b_{mn}^s \times \\ \times R_{|m|n}^{(3)}(-ic_s, i\xi_s) S_{mn}(-ic_s, \eta_s) e^{im\varphi} \tag{9.8.6}$$

$\Pi_{1,y} = 0$). Using addition theorem (5.5.18) from boundary condition 9.8.2) for the coefficients a_{mn}^s and b_{mn}^s of series (9.8.5) and 9.8.6), we find the infinite systems of linear equations of normal ype whose right hand sides and matrix elements (other than the di- gonal ones, equal to unity) will contain as the coefficient the unction $R_{|m|n}^{(1)}(-ic_s, i0)$, equal to zero for odd values of $(n \pm m)$. herefore from the system we immediately find that for odd values of $(n \pm m)$

$$a_{mn}^s = b_{mn}^s = 0. \tag{9.8.7}$$

If in the systems for a_{mn}^{s} and b_{mn}^{s} we make the substitution of unknowns according to the formulas

$$\begin{aligned} a_{mn}^{s} &= R_{|m|n}^{(1)}(-ic_s,\ i0)\,A_{mn}^{s}, \\ &(n \pm m) — \text{четное},\ s = \pm 1, \\ b_{mn}^{s} &= R_{|m|n}^{(1)}(-ic_s, i0)\,B_{mn}^{s}, \end{aligned} \tag{9.8.8}$$

then from them for the new unknown quantities A_{mn}^{s} and B_{mn}^{s} we find the infinite systems

$$A_{mn}^{s} + \sum_{q=|m|}^{\infty} \alpha_{mnq}^{(-s,s)} A_{mq}^{-s} = f_{mn}^{s}, \tag{9.8.9}$$

$$s = \pm 1,\ |m| \leqslant n,\ n = 0, 1, \ldots,$$

$$B_{mn}^{s} + \sum_{q=|m|}^{\infty} \beta_{mnq}^{(-s,s)} B_{mq}^{-s} = \varphi_{mn}^{s}, \tag{9.8.10}$$

with the matrix elements /54

$$\begin{aligned} \alpha_{mnq}^{(-s,s)} = \beta_{mnq}^{(-s,s)} &= \frac{R_{|m|n}^{(1)}(-ic_s,\ i0)}{R_{|m|n}^{(3)}(-ic_s,\ i0)} \times \\ &\times Q_{mnmq}(c_s,\ c_{-s};\ l,\ \theta_{-s,s}) \end{aligned} \tag{9.8.11}$$

and with the free terms

$$f_{mn}^{s} = -\frac{i^n e^{iskl_0\cos\alpha} S_{mn}(-ic_s, \cos\alpha)}{N_{mn}(-ic_s)\, R_{|m|n}^{(3)}(-ic_s,\ i0)}, \tag{9.8.12}$$

$$\varphi_{mn}^{s} = f_{mn}^{s}.$$

After introduction of the designations similar to those used in §, Chapter 7 for systems (7.3.9) and (7.3.10), system (9.8.9) which is identical to system (9.8.10), may be reduced to the normal form of (7.3.17) with a determinant such as (7.3.14). The values of the unknowns $A_{mn}^{s} \equiv B_{mn}^{s}$ for each m are found by the method of truncation.

In the case of normal incidence of the plane wave (9.8.1) on the disks when $\alpha = 0$ (or $\alpha = \Pi$), $f_{mn}^{s} = \phi_{mn}^{s} = 0$, if $m \neq 0$. Then

$$\begin{aligned} A_{mn}^{s} = B_{mn}^{s} &= 0, \quad \text{if } m \neq 0, \\ A_{0n}^{s} = B_{0n}^{s} &= A_{n}^{s}. \end{aligned} \tag{9.8.13}$$

n this case

$$\Pi_{1,x} = \Pi_{1,y} = \frac{E}{k^2} \sum_{s=\pm 1} \sum_{n=0,2,\ldots} A_n^s \times$$
$$\times R_{0n}^{(1)}(-ic_s,\ i0)\, R_{0n}^{(3)}(-ic_s,\ i\xi_s)\, S_{0n}(-ic_s,\ \eta_s). \qquad (9.8.14)$$

or the primary field

$$\Pi_x^0 = \Pi_y^0 = \frac{2Ee^{iskl_0}}{k^2} \sum_{n=0}^{\infty} \frac{i^n S_{0n}(-ic_s,\ 1)}{N_{0n}(-ic_s)} \times \qquad (9.8.15)$$
$$\times R_{0n}^{(1)}(-ic_s,\ i\xi_s)\, S_{0n}(-ic_s,\ \eta_s).$$

he infinite system obtained here from (9.8.9) [or (9.8.10)] for he unknowns A_n^s is identical to that studied in [86, 87].

By determining A_{mn}^s and B_{mn}^s we have completely solved the problem on seeking the first part of the stray field.

The second part of the stray field $\vec{\Pi}_2$ is sought as a function /541 f the type of polarization of the incident wave in the form

$$\Pi_{2,y} = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{a}_{mn}^s \times \qquad (9.8.16)$$
$$\times R_{|m|n}^{(3)}(-ic_s,\ i\xi_s)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi}$$

$\Pi_{2,x} = 0$) or in the form

$$\Pi_{2,x} = \frac{2E}{k^2 \cos\alpha} \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-1}^{n} \bar{b}_{mn}^s \times$$
$$\times R_{|m|n}^{(3)}(-ic_s,\ i\xi_s)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi} \qquad (9.8.17)$$

$\Pi_{2,y} = 0$).

The coefficients a_{mn}^{-s} and b_{mn}^{-s} of the series are found in the ollowing manner. Just as in the case of one disk the function Φ rom (9.4.10) ($\xi_s = 0$, $s = \pm 1$) is first expanded on the surface of the s-th disk into a series of cylindrical wave functions

$$\Phi(\rho_s,\ \varphi) = \sum_{m=-\infty}^{\infty} c_m^s J_m(k\rho_s)\, e^{im\varphi} \qquad (9.8.18)$$

(in local polar coordinates ρ_s, $\phi_s = \phi$ with the center at O_s, $s = \pm 1$). Then (9.8.18) is substituted into boundary conditions (9.4.10) (when $\xi_s = 0$, $s = \pm 1$). As a result we obtain the relationship

$$\Pi_{2,y} = \frac{ik}{2} \sum_{m=-\infty}^{\infty} x_m^s J_m(k\rho_s) e^{im\varphi} \qquad (9.8.19)$$

($\Pi_{2,x} = 0$), or

$$\Pi_{2,x} = \frac{k}{2} \sum_{m=-\infty}^{\infty} y_m^s J_m(k\rho_s) e^{im\varphi} \qquad (9.8.20)$$

($\Pi_{2,y} = 0$), respectively, where

$$\begin{aligned} x_m^s &= c_{m+1}^s + c_{m-1}^s, \\ y_m^s &= c_{m+1}^s - c_{m-1}^s. \end{aligned} \qquad (9.8.21)$$

After substitution into the left-hand sides of (9.8.19) and (9.8.20) the potentials $\Pi_{2,y}$ and $\Pi_{2,x}$ by their expansions (9.8.16) and (9.8.17) and applying the addition theorem (5.5.18) for the coefficients $\bar{a}_{mn}^s$ and $\bar{b}_{mn}^s$ we obtain infinite systems of linear equations /54 from which, just as in seeking $\vec{\Pi}_1$, we immediately find that

$$\bar{a}_{mn}^s = \bar{b}_{mn}^s = 0, \quad \text{if } (n \pm m) \text{ is odd} \qquad (9.8.22)$$

If in these systems we make the substitution of unknowns according to the formulas

$$\begin{aligned} \bar{a}_{mn}^s &= R_{|m|n}^{(1)}(-ic_s, i0)\, \bar{A}_{mn}^s, \\ &\qquad (n \pm m) \text{ is even} \\ \bar{b}_{mn}^s &= R_{|m|n}^{(1)}(-ic_s, i0)\, \bar{B}_{mn}^s, \end{aligned} \qquad (9.8.23)$$

then for the new unknown $\bar{A}_{mn}^s$ and $\bar{B}_{mn}^s$ we find the infinite systems

$$\bar{A}_{mn}^s + \sum_{q=|m|}^{\infty} \alpha_{mnq}^{(-s,s)} \bar{A}_{mq}^{-s} = F_{mn}^s. \qquad (9.8.24)$$

$$s = \pm 1, \ |m| \leqslant n; \ n = 0, 1, \ldots,$$

$$\bar{B}_{mn}^s + \sum_{q=|m|}^{\infty} \alpha_{mnq}^{(-s,s)} \bar{B}_{mn}^{-s} = \Phi_{mn}^s, \qquad (9.8.25)$$

with the same values of the matrix elements $\alpha_{mnq}^{(-s,s)}$ as in (9.8.9) and (9.8.10) and with the right-hand sides

$$F_{mn}^{s} = -ix_m^s P_{mn}^s,$$
$$\Phi_{mn}^{s} = -\cos\alpha\, y_m^s P_{mn}^s, \tag{9.8.26}$$

where

$$P_{mn}^{s} = \frac{k^3}{2E} i^{n-m} \frac{S_{mn}(-ic_s,\ 0)}{N_{mn}(-ic_s) R_{|m|n}^{(3)}(-ic_s,\ i0)}. \tag{9.8.27}$$

The right-hand sides of (9.8.26) contain the still undetermined quantities x_m^s and y_m^s, $s = \pm 1$. These may be determined from the Meixner condition (9.4.11), $\xi_s = \eta_s = 0$, $s = \pm 1$, written for the total stray field $\vec{\Pi}^1$. In the first of the cases examined in the problem of polarization of an incident field after substitution into (9.4.11) ($\xi_s = \eta_s = 0$, $s = \pm 1$) of expression (9.8.5) and (9.8.16), on the basis of addition theorem (5.5.18) we find the relationship

$$\sum_{n=|m|}^{\infty} (A_{mn}^s + \bar{A}_{mn}^s) S_{mn}(-ic_s,\ 0) = 0, \quad s = \pm 1, \tag{9.8.28}$$

and in the second case, after substitution into (9.4.11) of expressions (9.8.6) and (9.8.17), we find the relationship /543

$$\sum_{n=|m|}^{\infty} (B_{mn}^s + \bar{B}_{mn}^s) S_{mn}(-ic_s,\ 0) = 0. \tag{9.8.29}$$

It is obvious that the solution to systems (9.8.24) and (9.8.25), having the same structure as (9.8.9) and (9.8.10), may be written in the form

$$\bar{A}_{mn}^s = \alpha_{mn}^s x_m^s + \beta_{mn}^s x_m^{-s}.$$
$$s = \pm 1,$$
$$\bar{B}_{mn}^s = \gamma_{mn}^s y_m^s + \delta_{mn}^s y_m^{-s}, \tag{9.8.30}$$

where α_{mn}^s, β_{mn}^s, γ_{mn}^s and δ_{mn}^s are expressions that are independent of x_m^s and y_m^s (in practice the values of these expressions for each m may be found with any degree of accuracy from (9.8.24) and (9.8.25) by solving the truncated systems obtained from them). After substitution of (9.8.30) into (9.8.28) or, respectively, into (9.8.29) for determining x_m^s and y_m^s, we obtain the systems of equations

$$\alpha_m^s x_m^s + \beta_m^s x_m^{-s} = \tau_m^s, \quad s = \pm 1, \tag{9.8.31}$$

or

$$\gamma_m^s y_m^s + \delta_m^s y_m^{-s} = \sigma_m^s, \quad s = \pm 1, \tag{9.8.32}$$

where

$$\tau_m^s = -\sum_{n=|m|}^{\infty} A_{mn}^s S_{mn}(-ic_s,0),$$

$$\alpha_m^s = \sum_{n=|m|}^{\infty} \alpha_{mn}^s S_{mn}(-ic_s,\ 0).$$

$$\beta_m^s = \sum_{n=|m|}^{\infty} \beta_{mn}^s S_{mn}(-ic_s,0), \qquad (9.8.33)$$

$$\sigma_m^s = \tau_m^s,$$

$$\gamma_m^s = -i\cos\alpha\ \alpha_m^s,$$

$$\delta_m^s = -i\cos\alpha\ \beta_m^s,$$

$$\gamma_{mn}^s = -i\cos\alpha\ \alpha_{mn}^s,\ \delta_{mn}^s = -i\cos\alpha\ \beta_{mn}^s,$$

by the solution of which we find

$$x_m^s = \frac{\tau_m^s \alpha_m^{-s} - \tau_m^{-s}\beta_m^s}{\alpha_m^s \alpha_m^{-s} - \beta_m^s \beta_m^{-s}},\ s = \pm 1, \qquad (9.8.34)$$

$$y_m^s = \frac{i}{\cos\alpha} x_m^s,\ s = \pm 1. \qquad (9.8.35)$$

In the same manner we fully determine the values of the unknowns /544
$\overline{A}_{mn}^s$ and $\overline{B}_{mn}^s$ and through them the second part of the stray field $\vec{\Pi}_2$ as well for both cases of polarization of the wave (9.8.1).

With normal incidence of the wave (9.8.1) on the disks when $\alpha = 0$ ($\alpha = \pi$), $x_m^s = y_m^s = 0$, if $m \neq 0$, $x_0^s = -iy_0^s$, $s = \pm 1$. Therefore

$$\overline{A}_{0n}^s = \overline{B}_{0n}^s = \overline{A}_n^s,\ s = \pm 1,$$

and

$$\Pi_{2,x} = \Pi_{2,y} = \frac{2E}{k^2} \sum_{s=\pm 1} \sum_{n=0,2,\ldots}' \overline{A}_n^s \times$$
$$\times R_{0n}^{(1)}(-ic_s,\ i0)\, R_{0n}^{(3)}(-ic_s,\ i\xi_s)\, S_{0n}(-ic_s,\ \eta_s). \qquad (9.8.36)$$

If $\alpha \neq 0$ ($\alpha \neq \pi$), then for the total stray field

$$\Pi_x^1 = \Pi_{1,x} + \Pi_{2,x} = \frac{2E}{k^2\cos\alpha} \times$$
$$\times \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (B_{mn}^s + \overline{B}_{mn}^s) \times \qquad (9.8.37)$$
$$\times R_{|m|n}^{(1)}(-ic_s,\ i0)\ R_{|m|n}^{(3)}(-ic_s,\ i\xi_s)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi},$$

when $E^0 \parallel$ pl. Oxz ($\Pi_y^1 = \Pi_{1,y} + \Pi_{2,y} \equiv 0$), and

$$\Pi_y^1 = \Pi_{1,y} + \Pi_{2,y} = \frac{2E}{k^2} \times$$

$$\times \sum_{s=\pm 1} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (A_{mn}^s + \bar{A}_{mn}^s) \times \qquad (9.8.38)$$

$$\times R_{|m|n}^{(1)}(-ic_s,\ i0)\, R_{|m|n}^{(3)}(-ic_s,\ i\xi_s)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi},$$

when $E^0 \perp$ pl. Oxz ($\Pi_x^1 = \Pi_{1,x} + \Pi_{2,x} \equiv 0$).

If $\alpha = 0 (\alpha = \pi)$, then $\Pi_x^1 \equiv \Pi_y^1$.

The components of the vectors E^1, H^1 of the electric field cattered on the disks at any point outside the disks are found rom the relationships in (9.4.7) which are written in one of the oordinate systems.

In the case when $E^0 \parallel$ pl. Oxz, in the vicinity of the wave and and the spherical coordinates r, θ, ϕ with the origin at point

$$E_\eta^1 = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = -E\, \frac{e^{ikr}}{ikr}\, S_1(\theta,\ \varphi;\ \alpha),$$

$$E_\varphi^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\eta^1 = -E\, \frac{e^{ikr}}{ikr}\, S_2(\theta,\ \varphi;\ \alpha), \qquad (9.8.39)$$

here the amplitude functions are equal to /545

$$S_1(\theta,\ \varphi;\ \alpha) = 2\, \frac{\cos\theta \cos\varphi}{\cos\alpha} \sum_{s=\pm 1} e^{-iskl_0 \cos\theta} \times \qquad (9.8.40)$$

$$\times \sum_{n,m} (-i)^n (B_{mn}^s + \bar{B}_{mn}^s)\, R_{|m|n}^{(1)}(-ic_s,\ i0)\ S_{mn}(-ic_s,\ \cos\theta)\, e^{im\varphi},$$

$$S_2(\theta,\ \varphi;\ \alpha) = \frac{\sin\varphi}{\cos\theta\ \cos\varphi}\, S_1(\theta,\ \varphi;\ \alpha). \qquad (9.8.41)$$

The value of the back scattering cross section (in the direc- ion of the source: $\theta = \pi - \alpha$, $\phi = 0$) for two disks may be computed rom the formula

$$\sigma_B = \frac{4\pi}{k^2} |S_1(\pi - \alpha,\ 0;\ \alpha)|^2, \qquad (9.8.42)$$

where

$$S_1(\pi - a,\ 0;\ \alpha) = -2 \sum_{s=\pm 1} e^{iskl_0\cos\alpha} \sum_{n,m} (-i)^n \times$$

$$\times (B^s_{mn} + \bar{B}^s_{mn})\, R^{(1)}_{|m|n}(-ic_s,\ i0)\, S_{mn}(-ic_s,\ \cos\alpha). \qquad (9.8.43)$$

When $\alpha = 0$

$$S_1(\pi,\ 0;\ 0) = -2 \sum_{s=\pm 1} e^{iskl_0} \sum_{n=0,2,\ldots} i^n \times$$

$$\times (A^s_n + \bar{A}^s_n)\, R^{(1)}_{0n}(-ic_s,\ i0)\, S_{0n}(-ic_s,\ 1). \qquad (9.8.44)$$

Using the addition theorem and the relationships (9.8.10) - (9.8.12), (9.8.25) - (9.8.27) we can find the expression of the potential Π_x of the total electromagnetic field near the s-th disk

$$\Pi_x = \frac{2E}{k^2\cos\alpha} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (B^s_{mn} + \bar{B}^s_{mn}) \times$$

$$\times \{R^{(1)}_{|m|n}(-ic_s,\ i0)\, R^{(3)}_{|m|n}(-ic_s,\ i\xi_s) -$$

$$- R^{(1)}_{|m|n}(-ic_s,\ i\xi_s)\, R^{(3)}_{|m|n}(-ic_s,\ i0)\}\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi} - \qquad (9.8.45)$$

$$- k \sum_{n=0}^{\infty} \sum_{m=-n}^{n} i^{n-m}\, y^s_m \frac{S_{mn}(-ic_s,\ 0)}{N_{mn}(-ic_s)} \times$$

$$\times R^{(1)}_{|m|n}(-ic_s,\ i\xi_s)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi}$$

($l > f_s\xi_s$). Using (9.8.45) from (9.4.34) and (1.1.7) we find the law for the density distribution of surface currents induced on the s-th disk: /54

$$j^s_\eta = \frac{iEc_0\sqrt{\frac{\varepsilon}{\mu}}\cos\varphi}{2\pi\eta_s c_s^2\cos\alpha}\, \Psi_1(\eta_s,\ \varphi),$$

$$j^s_\varphi = -\frac{iEc_0\sqrt{\frac{\varepsilon}{\mu}}\sin\varphi}{2\pi\eta_s c_s^2\cos\alpha}\, \Psi_1(\eta_s,\ \varphi), \qquad (9.8.46)$$

where

$$\Psi_1(\eta_s,\varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (B^s_{mn} + \bar{B}^s_{mn})\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi}. \qquad (9.8.47)$$

In the case when $E^0 \perp$ pl. Oxz, in the vicinity fo the wave band in the spherical coordinates r, θ, ϕ with the origin at point O

$$E_\eta^1 = \sqrt{\frac{\mu}{\varepsilon}}\, H_\varphi^1 = -E\,\frac{e^{ikr}}{ikr}\, S_3(\theta,\ \varphi;\ \alpha),$$

$$E_\varphi^1 = -\sqrt{\frac{\mu}{\varepsilon}}\, H_\eta^1 = E\,\frac{e^{ikr}}{ikr}\, S_4(\theta,\ \varphi;\ \alpha), \qquad (9.8.48)$$

where

$$S_3(\theta,\ \varphi;\ \alpha) = 2\cos\theta\sin\varphi \sum_{s=\pm 1} e^{-iskl_0\cos\theta} \sum_{n,m} (-i)^n \times$$

$$\times (A_{mn}^s + \bar{A}_{mn}^s)\, R_{|m|n}^{(1)}(-ic_s,\ i0)\, S_{mn}(-ic_s,\ \eta_s)\, e^{im\varphi}, \qquad (9.8.49)$$

$$S_4(\theta,\ \varphi;\ \alpha) = \frac{\cos\varphi}{\cos\theta\sin\varphi}\, S_3(\theta,\ \varphi;\ \alpha). \qquad (9.8.50)$$

The value of the back scattering cross section ($\theta = \pi - \alpha$, $\phi = 0$) is found according to the formula

$$\sigma_B = \frac{4\pi}{k^2}\, |S_4(\pi - \alpha,\ 0;\ \alpha)|^2, \qquad (9.8.51)$$

where

$$S_4(\pi - \alpha,\ 0;\ \alpha) = 2 \sum_{s=\pm 1} e^{iskl_0\cos\alpha} \sum_{n,m} (-i)^n \times$$

$$\times (A_{mn}^s + \bar{A}_{mn}^s)\, R_{|m|n}^{(1)}(-ic_s,\ i0)\, S_{mn}(-ic_s,\ \cos\alpha). \qquad (9.8.52)$$

When $\alpha = 0$ ($\alpha = \pi$) /547

$$S_1(\pi,\ 0;\ 0) = 2 \sum_{s=\pm 1} e^{iskl_0} \sum_{n=0,2,\ldots} i^n \times$$

$$\times (A_n^s + \bar{A}_n^s)\, R_{0n}^{(1)}(-ic_s,\ i0)\, S_{0n}(-ic_s,\ 1). \qquad (9.8.53)$$

Just as in the previous case we find the expression for the potential Π_y of the total field near the s-th disk

$$\Pi_y = \frac{2E}{k^2} \sum_{n=0}^{\infty} \sum_{m=-1}^{n} (A_{mn}^s + \bar{A}_{mn}^s)\, \{R_{|m|n}^{(1)}(-ic_s,\ i0) \times$$

$$\times R_{|m|n}^{(3)}(-ic_s,\ i\xi_s) - R_{|m|n}^{(1)}(-ic_s,\ i\xi_s) R_{|m|n}^{(3)}(-ic_s,\ i0)\} \times$$

$$\times S_{mn}(-ic_s,\ \eta_s)\,e^{im\varphi} - ki\sum_{n=0}^{\infty}\sum_{m=-n}^{n} i^{n-m}x_m^s\,\frac{S_{mn}(-ic_s,\ 0)}{N_{mn}(-ic_s)}\times \tag{9.8.54}$$

$$\times R^{(1)}_{|m|n}(-ic_s,\ i\xi_s)\,S_{mn}(-ic_s,\ \eta_s)\,e^{im\varphi},$$

by the use of which we find, in turn, from (9.4.35) and (1.1.7) the law of density distribution of surface currents on the s-th disk:

$$j_\eta^s = \frac{iEc_0\sqrt{\frac{\varepsilon}{\mu}}\sin\varphi}{2\pi\eta_s c_s^2}\,\Psi_2(\eta_s,\ \varphi),$$

$$j_\varphi^s = -\frac{iEc_0\sqrt{\frac{\varepsilon}{\mu}}\cos\varphi}{2\pi\eta_s c_s^2}\,\Psi_2(\eta_s,\ \varphi), \tag{9.8.55}$$

where

$$\Psi_2(\eta_s,\ \varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(A_{mn}^s + \bar{A}_{mn}^s)\,S_{mn}(-ic_s,\ \eta_s)\,e^{im\varphi}. \tag{9.8.56}$$

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Translated for the National Aeronautics and Space Administration by:
Aztec School of Languages, Inc.,
Research Translation Division (359)
Maynard, Massachusetts
NASw-1692.

APPENDIX

TABLES OF VALUES FOR THE COEFFICIENTS $b_n^{(n_1 0 n_2 0)}$ AND $b_n^{(n_1 1 n_2 1)}$ FROM (3.4.13). /557

n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
1	1	0	0.3333333333 †	0.6666666667		13	12	0.4814814815	6.7407407407
	1	2	6666666667	—0.6666666667		13	14	5185185185	—6.7407407407
	2	1	4000000000	1.2000000000		14	13	4827586207	7.2413793103
	2	3	6000000000	—1.2000000000		14	15	5172413793	—7.2413793103
	3	2	4285714286	1.7142857143		15	14	4838709677	7.7419354839
	3	4	5714285714	—1.7142857143		15	16	5161290323	—7.7419354839
	4	3	4444444444	2.2222222222		16	15	4848484848	8.2424242424
	4	5	5555555556	—2.2222222222		16	17	5151515152	—8.2424242424
	5	4	4545454545	2.7272727273		17	16	4857142857	8.7428571429
	5	6	5454545455	—2.7272727273		17	18	5142857143	—8.7428571429
	6	5	4615384615	3.2307692308		18	17	4864864865	9.2432432432
	6	7	5384615385	—3.2307692308		18	19	5135135135	—9.2432432432
	7	6	4666666667	3.7333333333		19	18	4871794872	9.7435897436
	7	8	5333333333	—3.7333333333		19	20	5128205128	—9.7435897436
	8	7	4705882353	4.2352941176		20	19	4878048780	10.2439024390
	8	9	5294117647	—4.2352941176		20	21	5121951220	—10.2439024390
	9	8	4736842105	4.7368421053					
	9	10	5263157895	—4.7368421053	2	2	0	0.2000000000	1.2000000000
	10	9	4761904762	5.2380952381		2	2	2857142857	0.8571428571
	10	11	5238095238	—5.2380952381		2	4	5142857143	— 2.0571428571
	11	10	4782608696	5.7391304348		3	1	2571428571	2.0571428571
	11	12	5217391304	—5.7391304348		3	3	2666666667	0.8000000000
	12	11	4800000000	6.2400000000		3	5	4761904762	— 2.8571428571
	12	13	5200000000	—6.2400000000		4	2	2857142857	2.8571428571

†We have not included the values of the coefficients $b_n^{(00n_2 0)}$ and $b_n^{(01n_2 1)}$ in the Table, since for all values of n_2, $b_n^{(00n_2 0)} = 1$, and $b_n^{(01n_2 0)} = 0$. Let us remember that $b_n^{(n_1 0 n_2 0)} = b_n^{(n_2 0 n_1 0)}$ and $b_n^{(n_1 1 n_2 1)} = b_n^{(n_2 1 n_1 1)}$. Since the sign of the Clebsch-Gordan coefficient $(n_1 n_2 00|n0)$ is determined by the coefficient $(-1)^{n+L/2}$, where $L = n_1+n_2+n$, from the table then we can find both $(n_1 n_2 00|n0)$ and $(n_1 n_2 1,-1|n0)$. V.V. Lugin and G.M. Koblova participated in programming and computing the Tables of Coefficients.

n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
4	4	0,2597402597	0,7792207792	2	14	16	0,4004449388	—11,2124582870
4	6	4545454545	— 3,6363636364		15	13	3503893215	11,2124582870
5	3	3030303030	3,6363636364		15	15	2507836991	0,7523510972
5	5	2564102564	0,7692307692		15	17	3988269794	—11,9648093842
5	7	4405594406	— 4,4055944056		16	14	3519061584	11,9648093842
6	4	3146853147	4,4055944056		16	16	2506912442	0,7520737327
6	6	2545454545	0,7636363636		16	18	3974025974	—12,7168831169
6	8	4307692308	— 5,1692307692		17	15	3532467532	12,7168831169
7	5	3230769231	5,1692307692		17	17	2506142506	0,7518427518
7	7	2533936652	0,7601809955		17	19	3961389961	—13,4687258687
7	9	4235294118	— 5,9294117647		18	16	3544401544	13,4687258687
8	6	3294117647	5,9294117645		18	18	2505494506	0,7516483516
8	8	2526315789	0,7578947368		18	20	3950103950	—14,2203742203
8	10	4179566563	— 6,6873065016		19	17	3555093555	14,2203742204
9	7	3343653251	6,6873065015		19	19	2504943968	0,7514831905
9	9	2521008403	0,7563025210		19	21	3939962477	—14,9718574109
9	11	4135338346	— 7,4436090226		20	18	3564727955	14,9718574109
10	8	3383458647	7,4436090226		20	20	2504472272	0,7513416816
10	10	2517162471	0,7551487414		20	22	3930799773	—15,7231990924
10	12	4099378882	— 8,1987577640					
11	9	3416149068	8,1987577640	3	3	0	0,1428571429	1,7142857143
11	11	2514285714	0,7542857143		3	2	1904761905	1,7142857143
11	13	4069565217	— 8,9530434783		3	4	2337662338	0,4675324675
12	10	3443478261	8,9530434783		3	6	4329004329	— 3,8961038961
12	12	2512077295	0,7536231884		4	1	1904761905	2,8571428571
12	14	4044444444	— 9,7066666667		4	3	1818181818	1,8181818182
13	11	3466666667	9,7066666667		4	5	2197802198	0,2197802198
13	13	2510344828	0,7531034183		4	7	4079254079	— 4,8951048951
13	15	4022988506	—10,4597701149		5	2	2164502165	3,8961038961
14	12	3486580038	10,4597701149		5	4	1798201798	1,9780219780
14	14	2508960573	0,7526881720		5	6	2121212121	— 0,0000000000
5	8	0,3916083916	—5,8741258741		14	15	0,1950539882	— 1,7554858931
6	3	2331002331	4,8951048951		14	17	3438163616	—14,4402871878
6	5	1794871795	2,1538461538		15	12	2811766160	13,4964775677
6	7	2073220897	— 0,2073220897		15	14	1824698599	3,8318670577
6	9	3800904977	— 6,8416289593		15	16	1945018274	— 1,9450182743
7	4	2447552448	5,8741258741		15	18	3418516967	—15,3833263511
7	6	1796791444	2,3358288770		16	13	2831428860	14,4402871878
7	8	2040485830	— 0,4080971660		16	15	1827138379	4,0197044335
7	10	3715170279	— 7,8018575851		16	17	1940239360	— 2,1342632956
8	5	2533936652	6,8416289593		16	19	3401193401	—16,3257283257
8	7	1800128673	2,5206001429		17	14	2848764139	15,3833263511
8	9	2016806723	— 0,6050420168		17	16	1829368539	4,2075476398
8	11	3648827952	— 8,7571870854		17	18	1936063936	— 2,3232767233
9	6	2600619195	7,8018575851		17	20	3385803386	—17,2675972676
9	8	1804511278	2,7067669173		18	15	2864162864	16,3257283257
9	10	1998923139	— 0,7995692556		18	17	1831411831	4,3953883954
9	12	3595946388	— 9,7090552468		18	19	1932385347	— 2,5121009511
10	7	2653693056	8,7571870854		18	21	3372039957	—18,2090157700
10	9	1808549507	2,8936792108		19	16	2877932878	17,2675972676
10	11	1984962406	— 0,9924812030		19	18	1833288663	4,5832216564
10	13	3552795031	—10,6583850932		19	20	1929120534	— 2,7007687473
11	8	2696959791	9,7090552468		19	22	3359657926	—19,1500501767
11	10	1812356979	3,0810068650		20	17	2890319963	18,2090157700
11	12	1973775017	— 1,1842650104		20	19	1835017093	4,7710444421
11	14	3516908213	—11,6057971014		20	21	1926203877	— 2,8893058161
12	9	2732919255	10,6583850932		20	23	3348459066	—20,0907543959
12	11	1815873016	3,2685714286					
12	13	1964617691	— 1,3752323838	4	4	0	0,1111111111	2,2222222222
12	15	3486590038	—12,5517241379		4	2	1443001443	2,4531024531
13	10	2763285024	11,6057971014		4	4	1618381618	1,6183816184
13	12	1819090455	3,4562718641		4	6	2020202020	— 0,2020202020
13	14	1956989247	— 1,5655913978		4	8	3807303807	— 6,0916860917
13	16	3460635274	—13,4964775677		5	1	151515151 5	3,6363636364
14	11	2789272031	12,5517241379		5	3	1398601399	2,6573426573
14	13	1822024472	3,6440489433		5	5	1538461538	1,5384615385

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n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
4	5	7	0,1919648978	— 0,5758946935	4	12	8	0,2265446224	11,7803203661
	5	9	3628136569	— 7,2562731386		12	10	1454360539	4,7993897788
	6	2	1748251748	4,8951048951		12	12	1429285357	1,4292853573
	6	4	1398601399	2,9370629371		12	14	1701729780	— 2,8929406265
	6	6	1497326203	1,4973262032		12	16	3149178099	—15,1160548758
	6	8	1854987118	— 0,9274935591		13	9	2302737520	12,8953301127
	6	10	3500833532	— 8,4020004763		13	11	1461047254	5,1136653895
	7	3	1903651904	6,0916860917		13	13	1425932195	1,4259321952
	7	5	1407742584	3,2378079437		13	15	1690467897	— 3,2118890050
	7	7	1473078006	1,4730780056		13	17	3119815133	—16,2230386924
	7	9	1809954751	— 1,2669683258		14	10	2334499417	14,0069965017
	7	11	3405572755	— 9,5356037152		14	12	1467008431	5,4279311957
	8	4	2015631427	7,2562731386		14	14	1423264907	1,4232649071
	8	6	1418519561	3,5462989023		14	16	1680879990	— 3,5298479792
	8	8	1457489879	1,4574898785		14	18	3094347255	—17,3283446253
	8	10	1776820568	— 1,5991385112		15	11	2361883574	15,1160548758
	8	12	3331538565	—10,6609234083		15	13	1472343007	5,7421377288
	9	5	2100500119	8,4020004763		15	15	1421107628	1,4211076280
	9	7	1428911646	3,8580614432		15	17	1672620138	— 3,8470263164
	9	9	1446839605	1,4468396054		15	19	3072045653	—18,4322739161
	9	11	1751437417	— 1,9265811588		16	12	2385740984	16,2230386924
	9	13	3272311213	—11,7803203661		16	14	1477136961	6,0562615401
	10	6	2167182663	9,5356037152		16	16	1419337660	1,4193376596
	10	8	1438378555	4,1712978097		16	18	1665431343	— 4,1635783571
	10	10	1439224660	1,4392246601		16	20	3052353052	—19,5350595351
	10	12	1731381594	— 2,2507960723		17	13	2406714531	17,3283446253
	10	14	3223832528	—12,8953301127		17	15	1481463550	6,3702932668
	11	7	2221025710	10,6609234083		17	17	1417867224	1,4178672243
	11	9	1446839605	4,4852027768		17	19	1659118732	— 4,4796205772
	11	11	1433583960	1,4335839599		17	21	3034835962	—20,6368845393
	11	13	1715142429	— 2,5727136432		18	14	2425299199	18,4322739161
	11	15	3183408296	—14,0069965017		18	16	1485384711	6,6842312004
4	18	18	0,1416632148	1,4166321483	5	8	9	0,1298445800	0,7790674798
	18	20	1653531886	— 4,7952424697		8	11	1634674923	— 2,4520123839
	18	22	3019152055	—21,7378947952		8	13	3118320097	—12,4732803877
	19	15	2441882442	19,5350595351		9	4	1718591007	8,5929550326
	19	17	1488952708	6,9980777298		9	6	1182099634	4,6101885730
	19	19	1415584615	1,4155846147		9	8	1161767294	2,7882415068
	19	21	1648552868	— 5,1105138910		9	10	1279310809	0,6396554045
	19	23	3005027367	—22,8382079885		9	12	1604074124	— 2,8873334231
	20	16	2456771969	20,6368845393		9	14	3054157132	—13,7437070938
	20	18	1492211702	7,3118373402		10	5	1800428673	9,9023577042
	20	20	1414688391	1,4146883914		10	7	1195936921	5,0229350674
	20	22	1644087921	— 5,4254901396		10	9	1157471684	2,8936792108
	20	24	2992240016	—23,9379201313		10	11	1264927023	0,5059708094
						10	13	1579736448	— 3,3174465399
5	5	0	0,0909090909	2,7272727273		10	15	3001499250	—15,0074962519
	5	2	1165501166	3,1468531469		11	6	1865661596	11,1939695787
	5	4	1258741259	2,5174825175		11	8	1208237986	5,4370709382
	5	6	1426024955	1,2834224599		11	10	1154933369	3,0028267600
	5	8	1803459698	— 1,0820758189		11	12	1253759085	0,3761277256
	5	10	3437182013	— 8,5929550326		11	14	1559918965	— 3,7438055166
	6	1	1258741259	4,4055944056		11	16	2957488997	—16,2661894859
	6	3	1142191142	3,4265734266		12	7	1918966213	12,4732803877
	6	5	1206636501	2,5339366516		12	9	1218096334	5,8516624041
	6	7	1363961116	1,0911688930		12	11	1153458359	3,1143375681
	6	9	1727684081	— 1,5549156726		12	13	1244861440	0,2489722880
	6	11	3300785901	— 9,9023577042		12	15	1543470689	— 4,1673708600
	7	2	1468531469	5,8741258741		12	17	2920146965	—17,5208817878
	7	4	1151789387	3,8009049774		13	8	1963386728	13,7437070938
	7	6	1182099634	2,6006191950		13	10	1228683904	6,2662879087
	7	8	1324990799	0,9274935591		13	12	1152649482	3,2274185488
	7	10	1674311689	— 2,0091740269		13	14	1237621658	0,1237621658
	7	12	3198277022	—11,1939695787		13	16	1529600791	— 4,5888023730
	8	3	1612505142	7,2562731386		13	18	2888057438	—18,7723733141
	8	5	1166944511	4,2010002382		14	9	2000999500	15,0074962519
	8	7	1169109528	2,6889519150		14	11	1237177110	6,6807563960

n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
14	13	0,1152268141	3,3415784776	5	19	22	0,1477457389	— 7,0917954651
14	15	1231626611	0,0000000000		19	24	2762067707	—26,2396432209
14	17	1517747903	— 5,0085680786		20	15	2144091900	22,5129649520
14	19	2860180435	—20,0212630468		20	17	1272535164	9,1622531781
15	10	2033273686	16,2661894859		20	19	1153439316	4,0370376048
15	12	1244734427	7,0949862312		20	21	1210107956	— 0,7260647738
15	14	1152166830	3,4565004888		20	23	1471850139	— 7,5064357084
15	16	1226588101	— 0,1226588101		20	25	2747975525	—27,4797552528
15	18	1507502509	— 5,4270090310					
15	20	2835734449	—21,2680083648	6	6	0	0,0769230769	3,2307692308
16	11	2061280210	17,5208817878		6	2	0979020979	3,8181818182
16	13	1251491556	7,5089493376		6	4	1036610448	3,3171534348
16	15	1152249428	3,5719732271		6	6	1125809175	2,3641992682
16	17	1222299331	— 0,2444598663		6	8	1288185499	0,7729112992
16	19	1498558855	— 5,8443795344		6	10	1643869658	— 2,1370305559
16	21	2814120619	—22,5129649520		6	12	3157273471	—11,3661844953
17	12	2085819260	18,7723733441		7	1	1076923077	5,1692307692
17	14	1257562548	7,9226440517		7	3	0967503085	4,1602632661
17	16	1152453655	3,6878516968		7	5	1000238152	3,4008097166
17	18	1218608300	— 0,3655824899		7	7	1082508822	2,2732685271
17	20	1490684049	— 6,2608730051		7	9	1239425536	0,4957702144
17	22	2794872188	—23,7564135976		7	11	1584377233	— 2,6934412955
18	13	2107501373	20,0212630468		7	13	3049024095	—12,8059011980
18	15	1263042642	8,3360814395		8	2	1266968326	6,8416289593
18	17	1152737581	3,8040340165		8	4	0982052004	4,6156444175
18	19	1215400932	— 0,4861603727		8	6	0985083028	3,5462989023
18	21	1483697581	— 6,6766391157		8	8	1056152086	2,2179193804
18	23	2777619890	—24,9985790145		8	10	1205504416	0,2411003832
19	14	2126800836	21,2680083648		8	12	1539911159	— 3,2338134338
19	16	1268011339	8,7492782379		8	14	2964328981	—14,2287791089
19	18	1153072679	3,9204471082		9	3	1400333413	8,4020004763
19	20	1212590050	— 0,6062950249		9	5	1000238152	5,1012145749
9	7	0,0978493844	3,7182766083	6	14	12	0,0978005621	4,6944269801
9	9	1038756640	2,1813889435		14	14	1001884200	2,1039568192
9	11	1180598555	0,0000000000		14	16	1116195172	— 1,1161951718
9	13	1505395909	— 3,7634897721		14	18	1407002341	— 6,3315105362
9	15	2896183487	—15,6393908309		14	20	2689058529	—22,5880916426
10	4	1500357228	9,90235770 42		15	9	1775080202	17,0407699376
10	6	1017633598	5,5969847893		15	11	1079718205	8,0978865406
10	8	0975884527	3,9035381095		15	13	0979428174	4,8971408724
10	10	1026607439	2,1558756226		15	15	0998616171	2,0970939592
10	12	1161570917	— 0,2323141835		15	17	1109123467	— 1,3309481608
10	14	1477817967	— 4,2856721045		15	19	1395209968	— 6,8365288453
10	16	2840128323	—17,0407699376		15	21	2662823812	—23,9654143037
11	5	1578636735	11,3661844953		16	10	1807354387	18,4350147507
11	7	1033289500	6,0964080475		16	12	1088253527	8,5972028648
11	9	0975277067	4,0961636829		16	14	0980898787	5,1006736942
11	11	1017757375	2,1372904879		16	16	0995947603	2,0914899669
11	13	1146582905	— 0,4586331622		16	18	1103050616	— 1,5442708625
11	15	1455272364	— 4,8023988006		16	20	1384893568	— 7,3399359101
11	17	2793184053	—18,4350147507		16	22	2639601511	—25,3401745041
12	6	1641782205	12,8059011980		17	11	1835520949	19,8236262514
12	8	1047139588	6,5969794050		17	13	1095900714	9,0959759274
12	10	0975719571	4,2931661104		17	15	0982366500	5,3047790979
12	12	1011096037	2,1233016768		17	17	0993739294	2,0868525169
12	14	1134486520	— 0,6806919121		17	19	1097781487	— 1,7564503788
12	16	1436494656	— 5,3150302268		17	21	1375792303	— 7,8420161249
12	18	2753281424	—19,8236262514		17	23	2618898754	—26,7127672897
13	7	1693902275	14,2287791089		18	12	1860325286	21,2077082644
13	9	1059352677	7,0976629332		18	14	1102785619	9,5942348846
13	11	0976718771	4,4929063482		18	16	0983801901	5,5092906445
13	13	1005948639	2,1124921410		18	18	0991890476	2,0829700006
13	15	1124528645	— 0,8996229158		18	20	1093168302	— 1,9677029444
13	17	1420612037	— 5,8245093511		18	22	1367703411	— 8,3429908079
13	19	2718935957	—21,2077082644		18	24	2600325004	—28,0835100418
14	8	1737710092	15,6393908309		19	13	1882340970	22,5880916426
14	10	1070144045	7,5980227204		19	15	1109013052	10,0920187716

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n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
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	19	21	1089097161	— 2,1781943214
	19	23	1360466885	— 8,8430347534
	19	25	2583566733	—29,4526607581
	20	14	1902017008	23,9654143037
	20	16	1114670433	10,5893691110
	20	18	0986517736	5,9191064183
	20	20	0988991885	2,0768829576
	20	22	1085478898	— 2,3880535750
	20	24	1353954759	— 9,3422878340
	20	26	2568369282	—30,8204313815
7	7	0	0,0666666667	3,7333333333
	7	2	0844645551	4,4766214178
	7	4	0883846803	4,0656952954
	7	6	0938174313	3,2836100948
	7	8	1026814528	2,0536290559
	7	10	1183586154	0,1183586154
	7	12	1520168708	— 3,3443711581
	7	14	2936097276	—14,3868766546
	8	1	0941176471	5,9294117647
	8	3	0840200048	4,8731602763
	8	5	0857346987	4,2010002382
	8	7	0906012819	3,2616461476
	8	9	0991540429	1,8839268149
	8	11	1144272446	— 0,2288544892
	8	13	1471942666	— 3,9742451994
	8	15	2847508134	—15,9460455531
	9	2	1114551084	7,8018575851
	9	4	0857346987	5,4012860205
	9	6	0848027998	4,4097455916
7	13	12	0,0857899667	3,5173886364
	13	14	0918393850	1,2857513895
	13	16	1048252857	— 1,7820298569
	13	18	1341342232	— 6,9749796070
	13	20	2589463769	—23,5641202938
	14	7	1518671005	15,9460455531
	14	9	0939748342	8,2697854109
	14	11	0852409110	5,7111410339
	14	13	0855056343	3,5912366397
	14	15	0911779982	1,1853139769
	14	17	1038139565	— 2,0762791308
	14	19	1326310711	— 7,5599710511
	14	21	2557884942	—25,0672724326
	15	8	1561536719	17,4892112518
	15	10	0951239151	8,8465241064
	15	12	0855056343	5,9853943996
	15	14	0852955467	3,6677085092
	15	16	0906312319	1,0875747828
	15	18	1029513908	— 2,3678819891
	15	20	1313261142	— 8,1422190805
	15	22	2530124950	—26,5663119801
	16	9	1598340615	19,0202533142
	16	11	0961463354	9,4223408726
	16	13	0857661428	6,2609284278
	16	15	0851384300	3,7460909185
	16	17	0901726396	0,9918990358
	16	19	1022072419	— 2,6573882887
	16	21	1301824975	— 8,7222273293
	16	23	2505526513	—28,0618969508
	17	10	1630307427	20,5418735793
	17	12	0970604497	9,9972263215
	17	14	0860172783	6,5373131489
	17	16	0850199174	3,8258962809
	17	18	0897831897	0,8978318968
	17	20	1015588616	— 2,9452069878
7	9	8	0,0887167752	3,2825206830
	9	10	0967380087	1,7412841567
	9	12	1115108081	— 0,5575540403
	9	14	1434352733	— 4,5899287448
	9	16	2776065278	—17,4892112518
	10	3	1238390093	9,5356037152
	10	5	0877020409	5,9637387784
	10	7	0845418681	4,6498027481
	10	9	0875248650	3,3259418707
	10	11	0949906884	1,6148417020
	10	13	1092626063	— 0,8741008503
	10	15	1404210176	— 5,1955776498
	10	17	2717179045	—19,0202533142
	11	4	1332615426	11,1939695787
	11	6	0895517566	6,5372782339
	11	8	0845766590	4,9054462243
	11	10	0867306285	3,3824945113
	11	12	0936750740	1,4988011836
	11	14	1074776703	— 1,1822543737
	11	16	1379490500	— 5,7938617788
	11	18	2667775790	—20,5418735793
	12	5	1407241890	12,8059011980
	12	7	0912101225	7,1143895544
	12	9	0847482141	5,1696410618
	12	11	0861810681	3,4472427224
	12	13	092653641	1,3897974612
	12	15	1060269865	— 1,4843778111
	12	17	1358846296	— 6,3865775918
	12	19	2625716261	—22,0560165950
	13	6	1468048638	14,3868766546
	13	8	0926778716	7,6922633420
	13	10	0849820271	5,4388197352
7	17	22	0,1291719888	— 9,3003831957
	17	24	2483575718	—29,5545510440
	18	11	1658347112	22,0560165950
	18	13	0978817305	10,5712268887
	18	15	0862565707	6,8142690851
	18	17	0849300443	3,9067820374
	18	19	0894488619	0,8050397570
	18	21	1009890094	— 3,2316483023
	18	23	1282725920	— 9,8769895861
	18	25	2463864800	—31,0446964748
	19	12	1683151450	23,5641202938
	19	14	0986231041	11,1444107668
	19	16	0864830508	7,0916101669
	19	18	0848617408	3,9885018159
	19	20	0891591169	0,7132729349
	19	22	1004843322	— 3,5169516286
	19	24	1274669120	—10,4522867803
	19	26	2446065983	—32,5326775694
	20	13	1705256628	25,0672724326
	20	15	0992953546	11,7168518475
	20	17	0866965892	7,3692100831
	20	19	0848098917	4,0708747994
	20	21	0889058907	0,6223412347
	20	23	1000343298	— 3,8013045320
	20	25	1267410096	—11,0264678310
	20	27	2429912717	—34,0187780343
8	8	0	0,0588235294	4,2352941176
	8	2	0743034056	5,1269349845
	8	4	0771612289	4,7839961896
	8	6	0807645713	4,1189931350
	8	8	0862524203	3,1050871325
	8	10	0949791358	1,6146453089
	8	12	1100811823	— 0,6604870939
	8	14	1420692231	— 4,6882843608

n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 0 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 0 n_2 1)}$
8	16	0,2755653033	—17,6361794135	8	12	12	0,0794818810	2,8613477142
9	1	0835913313	6,6873065015		12	14	0870057331	0,7830515981
9	3	0743034056	5,5727554180		12	16	1006655521	— 2,2146421471
9	5	0751731779	4,9614297400		12	18	1299685641	— 7,4082081540
9	7	0782795075	4,1488138998		12	20	2524727174	—24,2373808736
9	9	0835464621	3,0076726343		13	5	1270426706	—14,2287791089
9	11	0920678979	1,3810184691		13	7	0817745926	8,0956846654
9	13	1068345484	— 1,0683454837		13	9	0751798674	6,1647491245
9	15	1380610005	— 5,3843790179		13	11	0752125685	4,5879666778
9	17	2681426689	—19,3062721610		13	13	0787551895	2,8351868209
10	2	0995134896	8,7571870854		13	15	0859678269	0,6017747883
10	4	0761494529	6,1681056862		13	17	0993003063	— 2,5818079626
10	6	0746264639	5,2238524700		13	19	1280837201	— 8,0692743640
10	8	0768878719	4,2288329519		13	21	2486832583	—25,8630588591
10	10	0817269384	2,9421697820		14	6	1328837129	15,9460455531
10	12	0899280710	1,1690649232		14	8	0832819583	8,7446056259
10	14	1043165624	— 1,4604318733		14	10	0755395797	6,4964038504
10	16	1348374564	— 6,0676855363		14	12	0750049424	4,7253113681
10	18	2620136936	—20,9610954891		14	14	0781875845	2,8147530420
11	3	1110512855	10,6609234083		14	16	0851145482	0,4255727411
11	5	0781801050	6,8016691345		14	18	0981469926	— 2,9444097778
11	7	0746264639	5,5223583255		14	20	1264621840	— 8,7258906991
11	9	0760560896	4,3351971073		14	22	2453784974	—27,4823917036
11	11	0804356635	2,8956838868		15	7	1377826517	17,6361794135
11	13	0882930152	0,9712231670		15	9	0846180325	9,3926016119
11	15	1023067414	— 1,8415213446		15	11	0759050017	6,8314501493
11	17	1321870887	— 6,7415415222		15	13	0748752041	4,8668882647
11	19	2568635473	—22,6039921626		15	15	0777350882	2,7984631763
12	4	1199353883	12,4732803877		15	17	0844015907	0,2532047720
12	6	0800753803	7,4470103648		15	19	0971599707	— 3,3034390028
12	8	0748552040	5,8387059102		15	21	1250521527	— 9,3789114544
12	10	0755395797	4,4568351997		15	23	2424703078	—29,0961369306
16	8	0,1419578835	19,3062721610	8	19	27	0,2336454535	—35,5141089370
16	10	0858056540	10,0392615237		20	12	1539467789	25,8630588591
16	12	0762617819	7,1686075010		20	14	0894488619	12,6122895259
16	14	0747976333	5,0114414300		20	16	0775144381	8,5265881952
16	16	0773681248	2,7852524925		20	18	0747505546	5,6062915972
16	18	0837976437	8,3797643701		20	20	0764221002	2,7511956062
16	20	0963058171	— 3,6596210491		20	22	0820950427	— 0,5746652988
16	22	1238146252	—10,0289846442		20	24	0937964069	— 5,0650059731
16	24	2398908364	—30,7060270587		20	26	1200796028	—12,6083582924
17	9	1455631631	20,9610954891		20	28	2319462139	—37,1113942193
17	11	0868658011	10,6844935385	9	9	0	0,0526315789	4,7368421053
17	13	0766030934	7,5071031529		9	2	0663423264	5,7717823972
17	15	0747556946	5,1581429277		9	4	0685345076	5,4827606100
17	17	0770661513	2,7743814468		9	6	0710728227	4,9040247678
17	19	0832799749	— 8,3279974859		9	8	0747520976	4,0366132723
17	21	0955594928	— 4,0134986981		9	10	0802409941	2,8084347919
17	23	1227196660	—10,6766109390		9	12	0887751470	1,0653017644
17	25	2375869628	—32,3118269431		9	14	1033230713	— 1,5498460697
18	10	1487104748	22,6039921626		9	16	1338460045	— 6,1569162060
18	12	0878165974	11,3283410605		9	18	2604814498	—21,0989974331
18	14	0769260212	7,8464541646		10	1	0751879699	7,4436090226
18	16	0747384390	5,3064291673		10	3	0666307713	6,2632925024
18	18	0768145067	2,7653222421		10	5	0670115186	5,6959790781
18	20	0828316957	— 0,2484950870		10	7	0690985776	4,9750975905
18	22	0949018693	— 4,3654859898		10	9	0725989946	3,9929447041
18	24	1217439077	—11,3221834202		10	11	7796072003	2,6506644810
18	26	2355164882	—33,9143743002		10	13	0863309482	0,7769785336
19	11	1514836305	24,2373808736		10	15	1005873003	— 2,0117460067
19	13	0886733447	11,9709015291		10	17	1304477849	— 6,9137325982
19	15	0772297203	8,1863503492		10	19	2541454145	—22,8730873074
19	17	0747384390	5,4559060453		11	2	0898986597	9,7090552468
19	19	0766024828	2,7576893803		11	4	0685345076	6,9219852701
19	21	0824400077	— 0,4122000386		11	6	0667294836	6,0056535200
19	23	0943180824	— 4,7159041187		11	8	0680501854	5,1037639075
19	25	1208688392	—11,9660150833		11	10	0711815270	3,9861655111

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n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
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	11	14	0844467410	0,5066804459
	11	16	0983949006	2,4598725147
	11	18	1276476969	— 7,6588618133
	11	20	2488136925	—24,6325555618
	12	3	1006864989	11,7803203661
	12	5	0705792615	7,6225602369
	12	7	0669064848	6,3561160596
	12	9	0674691117	5,2625907160
	12	11	0701983973	4,0013086436
	12	13	0750490629	2,4015700130
	12	15	0829514119	0,2488542358
	12	17	0965982571	— 2,8979477132
	12	19	1252992914	— 8,3950525216
	12	21	2442622226	—26,3803200363
	13	4	1090770404	13,7437070938
	13	6	0724820252	8,3354329027
	13	8	0672661971	6,7266197122
	13	10	0671462930	5,4388497352
	13	12	0694898731	4,0304126375
	13	14	0740724485	2,2962459027
	13	16	0817369868	— 0,0000000000
	13	18	0950989493	— 3,3284632271
	13	20	1233006294	— 9,1242465789
	13	22	2403295571	—28,1185581782
	14	5	1158473395	15,6393908309
	14	7	0741906586	9,0512603481
	14	9	0676944260	7,1079147333
	14	11	0669750015	5,6259001229
	14	13	0689640038	4,0688762213
	14	15	0732930832	2,1987924956
	14	17	0807319563	— 0,2421958689
9	17	26	0,2287874457	—35,0044791884
	18	9	1337607445	22,8730873074
	18	11	0793485683	11,9022852504
	18	13	0693965306	8,6745663254
	18	15	0669324242	6,4255127269
	18	17	0678182131	4,2725474281
	18	19	0713195529	1,8543083764
	18	21	0780077492	— 1,1701162385
	18	23	0902350485	— 5,4141029103
	18	25	1165520950	—12,7041783514
	18	27	2266290735	—36,7139099146
	19	10	1368475309	24,6325555618
	19	12	0803200586	12,6102491953
	19	14	0697701123	9,0701145952
	19	16	0669877860	6,6317908185
	19	18	0676621400	4,3303769579
	19	20	0709985963	1,7749649072
	19	22	0775342070	— 1,3956157256
	19	24	0895851397	— 5,8230340780
	19	26	1156171851	—13,4115934732
	19	28	2246772442	—38,4198087591
	20	11	1395784129	26,3803200363
	20	13	0811979755	13,3164679800
	20	15	0701196825	9,4661571423
	20	17	0670525711	6,8393622561
	20	19	0675352501	4,3897912584
	20	21	0707205949	1,6972942764
	20	23	0771154133	— 1,6194236785
	20	25	0890034180	— 6,2302392582
	20	27	1147732052	—14,1171042450
	20	29	2229034765	—40,1226257678
10	10	0	0,0476190476	5,2380952381
	10	2	0599324398	6,4127710581
	10	4	0616810569	6,1681056862
9	14	19	0,0938287717	— 3,7531508670
	14	21	1215784818	— 9,8478570271
	14	23	2368962777	—29,8489309891
	15	6	1214528559	17,4892112518
	15	8	0757108712	9,7667023873
	15	10	0681397841	7,4953762508
	15	12	0668962999	5,8199780943
	15	14	0685644972	4,1138698306
	15	16	0726587607	2,1071040596
	15	18	0798870870	— 0,4793225220
	15	20	0927389350	— 4,1732520735
	15	22	1200788391	—10,5669378435
	15	24	2338720699	—31,5727294353
	16	7	1261847854	19,3062721610
	16	9	0770628511	10,4805477446
	16	11	0685782640	7,8865003654
	16	13	0668757165	6,0188144812
	16	15	0682551994	4,1635671647
	16	17	0721339176	2,0197496933
	16	19	0791673835	— 0,7125064516
	16	21	0917936015	— 4,5896800734
	16	23	1187609671	—11,2822918710
	16	25	2311873140	—33,2909732141
	17	8	1302407249	21,0989974331
	17	10	0782686709	11,1924199120
	17	12	0689987551	8,2798506090
	17	14	0668921924	6,2209738901
	17	16	0680119795	4,2167427270
	17	18	0716935396	1,9357255695
	17	20	0785472976	— 0,9425675714
	17	22	0909658471	— 5,0031215912
	17	24	1175935473	—11,9945418198
10	10	6	0,0635518891	5,6561181299
	10	8	0661599025	4,8958327854
	10	10	0698873174	3,8438024571
	10	12	0753243672	2,4103797497
	10	14	0836424863	0,4182124315
	10	16	0976714087	— 2,5394566255
	10	18	1269012191	— 7,7409743669
	10	20	2476288654	—24,7628865436
	11	1	0683229814	8,1987577640
	11	3	0604118993	6,9473684211
	11	5	0604965098	6,4126300406
	11	7	0619504489	5,7613917499
	11	9	0644023339	4,8945773792
	11	11	0680384466	3,7421145614
	11	13	0733813060	2,2014391785
	11	15	0815572705	0,0815572705
	11	17	0953272274	— 3,0504712770
	11	19	1239733729	— 8,5541627328
	11	21	2421382032	—26,6352023554
	12	2	0819875776	10,6583850932
	12	4	0623297374	7,6665576986
	12	6	0604016877	6,7649890225
	12	8	0611510883	5,9316555644
	12	10	0632724684	4,9352525375
	12	12	0667102781	3,6690652976
	12	14	0718938470	2,0130277173
	12	16	0798932954	— 0,2396798861
	12	18	0934007538	— 3,5492286452
	12	20	1215136638	— 9,3565521126
	12	22	2374456024	—28,4934722872
	13	3	0921095008	12,8953301127
	13	5	0643596330	8,4311119294
	13	7	0607014479	7,1627708567
	13	9	0607514080	6,1358922057
	13	11	0625100014	5,0008001093

n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
13	13	0,0657186389	3,6145251377	10	16	14	0,0613178430	5,2733344983
13	15	0707213961	1,8387562975		16	16	0638721198	3,5129665917
13	17	0785351684	— 0,5497461786		16	18	0683478411	1,3669568219
13	19	0917890158	— 4,0387166938		16	20	0756381384	— 1,4371246305
13	21	1194170866	—10,1504523596		16	22	0882211879	— 5,4697136518
13	23	2333867032	—30,3402714169		16	24	1146431715	—12,4961056952
14	4	1000499750	15,0074962519		16	26	2239343786	—35,8295005834
14	6	0662470123	9,2083347121		17	7	1164505305	20,9610954891
14	8	0611510883	7,5827349483		17	9	0708145118	11,5427654214
14	10	0605686970	6,3597131825		17	11	0626436066	8,8953921345
14	12	0619774544	5,0821512568		17	13	0605842727	7,0883599116
14	14	0649558394	3,5725711686		17	15	0611122134	5,3778747823
14	16	0697754765	1,6746114365		17	17	0634778475	3,4912816127
14	18	0774061216	— 0,8514673372		17	19	0677975997	1,2203567947
14	20	0904204616	— 4,5210230796		17	21	0749335522	— 1,7234717010
14	22	1176080811	—10,9375515445		17	23	0873242405	— 5,9380483532
14	24	2298397928	—32,1775709954		17	25	1134126446	—13,2692794179
15	5	1065048121	17,0407699376		17	27	2214489804	—37,6463266743
15	7	0679456537	9,9880110878		18	8	1203846700	22,8730873074
15	9	0616502809	8,0145365105		18	10	0720250163	12,3162777809
15	11	0605102330	6,5956153951		18	12	0631086174	9,3400753821
15	13	0615960546	5,1740685891		19	14	0606696628	7,3410292042
15	15	0643549023	3,5395196271		18	16	0609588853	5,4862996771
15	17	0689976603	1,5179485273		18	18	0631513306	3,4733231850
15	19	0764530732	— 1,1467960982		18	20	0673262551	1,0772200816
15	21	0892437792	— 4,9976516355		18	22	0743185025	— 2,0065995687
15	23	1160308299	—11,7191138189		18	24	0865311008	— 6,4033014598
15	25	2267127208	—34,0060081219		18	26	1123138370	—14,0392296210
16	6	1118838430	19,0202533142		18	28	2192121220	—39,4581819688
16	8	0694617199	10,7665665914		19	9	1238144327	24,7628865436
16	10	0621545328	8,4530164597		19	11	0731125020	13,0871378562
16	12	0605252238	6,8393502838		19	13	0635462417	9,7861212197
19	15	0,0607703915	7,5962989414	11	12	7	0,0562050444	6,5197851489
19	17	0608439997	5,5976479758		12	9	0579899803	5,7410080538
19	19	0628776467	3,4582705670		12	11	0605866167	4,7257561033
19	21	0669184123	0,9368577727		12	13	0642582247	3,4056859076
19	23	0737770837	— 2,2870895942		12	15	0695328012	1,6687872280
19	25	0858247245	— 6,8659779580		12	17	0775018109	— 0,6975162981
19	27	1113265624	—14,8064328050		12	19	0908177034	— 4,1776143579
19	29	2171880027	—41,2657205190		12	21	1183786771	—10,2989449108
20	10	1268342969	26,6352023554		12	23	2316893454	—30,5829935883
20	12	0740936974	13,8555214211		13	2	0753623188	11,6057971014
20	14	0639559362	10,2329497997		13	4	0571714143	8,4041979010
20	16	0608794773	7,8534525701		13	6	0552058436	7,5079947293
20	18	0607578400	5,7112369572		13	8	0555918984	6,7266197122
20	20	0626458203	3,4455201140		13	10	0570743491	5,8215836055
20	22	0665623852	0,7987486228		13	12	0594983562	4,7003701380
20	24	0732969324	— 2,5653926357		13	14	0630453736	3,2783594253
20	26	0851916101	— 7,3264784672		13	16	0682015936	1,4322334654
20	28	1104345777	—15,5712754501		13	18	0760238694	— 1,0643341715
20	30	2153474264	—43,0694852874		13	20	0891100201	— 4,7228310664
					13	22	1161967842	—11,1548912784
11	0	0,0434782609	5,7391304348		13	24	2275181788	—32,5350995620
11	2	0546583851	7,0509316770		14	3	0848908879	14,0069965017
11	4	0560967636	6,8438051651		14	5	0591693401	9,2304170495
11	6	0575254169	6,3853212713		14	7	0555918984	7,9496414781
11	8	0594524470	5,7074349073		14	9	0553271751	6,9712240654
11	10	0621220599	4,7833986133		14	11	0564762175	5,9300028323
11	12	0658550182	3,5561709807		14	13	0586974168	4,6957933413
11	14	0712091437	1,9226468810		14	15	0620968356	3,1669386138
11	16	0793058446	— 0,3172233786		14	17	0671201730	1,2081631135
11	18	0928545506	— 3,6213274727		14	19	0747910499	— 1,4210299478
11	20	1209350273	— 9,4329321299		14	21	0876572231	— 5,2594333878
11	22	2365070822	—28,6173569493		14	23	1143118546	—12,0027447364
12	1	0626086957	8,9530434783		14	25	2238699281	—34,4759689236
12	3	0552657005	7,6266666667		15	4	0924215312	16,2661891859
12	5	0551653998	7,1163365686		15	6	0610275507	10,0695458722

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n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$	n_1	n_2	n	$b_n^{(n_1 0 n_2 0)}$	$b_n^{(n_1 1 n_2 1)}$
11	15	8	0,0561036969	8,4155545317	11	17	26	0,1099314222	—14,5109477363
	15	10	0552484736	7,2375500407		17	28	2152264471	—40,247345608
	15	12	0560748397	6,0560826823		18	7	1081530725	22,6039921626
	15	14	0580905881	4,7053376368		18	9	0655488173	12,5853729257
	15	16	0613375119	3,0668755959		18	11	0577203963	9,8701877687
	15	18	0662252373	0,9933785600		18	13	0554768777	8,0996241387
	15	20	0737471850	— 1,7699324397		18	15	0554860097	6,4918631301
	15	22	0864059372	— 5,7891977892		18	17	0569499838	4,7837986353
	15	24	1126665651	—12,8439884225		18	19	0597725777	2,8093111519
	15	26	2206508834	—36,4073957541		18	21	0642812335	0,3856874011
	16	5	0985829665	18,4350147507		18	23	0713971778	— 2,7844899325
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	14	2	0697318008	12,5517241379		16	22	0705930859	— 2,7531303504
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	17	5	0917760475	19,8236262514		19	19	0537171858	4,1899404957
	17	7	0582362698	11,8219627762		19	21	0567615421	2,1001770560 /574
	17	9	0524390539	9,7536640174		19	23	0613366663	— 0,4906933308
	17	11	0509297614	8,4034106390		19	25	0683731733	— 3,8972708780
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	18	29	1032183205	—17,8567694547
	18	31	2027956646	—47,4541855059
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	19	8	0558313652	13,6786844632
	19	10	0494248546	11,1700171497
	19	12	0474356476	9,6294364561
	19	14	0471227741	8,2936082498
	19	16	0477809363	6,9282357687
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	19	20	0514488051	3,6528651641
	19	22	0546764854	1,5309415914
	19	24	0593345944	— 1,1273572938
	19	26	0663576530	— 4,6450357124
	19	28	0780412652	— 9,7551581505
	19	30	1021060076	—18,7875054010
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	20	23	0542345260	1,3558631508
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	19	29	0761849509	—10,6658931232
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	20	14	0438530306	9,2091364333
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